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STATIONARY TOROIDAL EQUILIBRIA AT FINITE BETA

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Abstract

We investigate the effects of plasma flow on axisymmetric, self-consistent equilibria in toroidal geometry. The investigation is of considerable interest in relation to hot plasmas confined in toroidal systems with longitudinal current. On the basis of the one-fluid MHD plasma model, we use a concise formulation to elucidate important features of the equilibrium. In contrast to previous flow calculations which were treated almost exclusively in the low beta approximation, we retain, together with flow, all beta effects.

As in the treatment of the full flow problem at low beta, we find conditions for equilibrium. The form of our description allows a quite general discussion of its nature and existence for finite beta. This shows that there is a close relationship between the solvability conditions for equations arising from integrals of the system, and the nature of the characteristics of the partial differential equation describing the radial force balance. In the case of large aspect-ratio these considerations lead to a generalized Bennett relation and to an expression for plasma displacement exhibiting beta and flow effects.

Introduction

One of the central problems of the theoretical study of toroidal confinement is the calculation of the rate of plasma loss from a magnetic confinement region. Fundamental to this study is the investigation of plasma equilibrium.

In magnetohydrodynamic equilibrium the radial plasma flow, which is necessary for the occurrence of plasma loss, is in general coupled with velocity components in other directions. Thus, to understand plasma loss, a discussion of plasma flow and the nature of this flow coupling is unavoidable. In stationary equilibria, this coupling is accomplished linearly by Ohm's law, and nonlinearly by plasma inertia. As static toroidal equilibria already exhibit nonlinearity, the additional consideration of flow effects further complicates the problem.

Previous discussions of toroidal equilibrium concern limiting situations in which the simplifications involved ensure mathematical tractability. Most flow calculations profit by the restriction to axisymmetric solutions. Some tractable situations originate in the appropriate choice of physical properties. Here the plasma beta, the ratio of flow to sound speed, electrical conductivity and aspect ratio are convenient parameters.

Because of the complexity of the problem it can happen, that the calculation of particular cases results in a conflict with previous assumptions. The evaluation of the resistive mass losses for a low beta plasma in toroidal equilibrium /1/ gives parallel flows which can become very large. This casts doubts on the validity of the result, which was derived on the understanding that the equilibrium is always such that plasma flows are sufficiently small for inertial effects to be neglected.

Inclusion of plasma inertia in the same low beta situation with large electrical conductivity /2/ fortunately allows the resolution of this conflict. The somewhat surprising result is that the nonlinear coupling of flow components considerably restricts plasma losses in a stationary state.

In the above-mentioned investigations the plasma is placed in a toroidal axisymmetric model of a magnetic field to simulate a true low beta situation. This magnetic field is thought to be produced by a toroidal current which does not interact with the plasma. The justification of such an approach has awaited the solution of the appropriate finite beta problem.

Recently the increasing interest in toroidal systems with longitudinal current /3/ has encouraged renewed discussion of the small-flow equilibrium with self-consistent fields and its losses /4,5/. The results show that for large aspect-ratio the correct incorporation of the toroidal current does not drastically alter the results of the small-flow calculation with artificial current. Naturally the question arises how the nonlinear coupling of flow components affects the finite beta equilibrium.

The present paper discusses both the effects of plasma beta and flow on the behaviour of a highly-conducting axisymmetric toroidal plasma of arbitrary aspect-ratio. We can establish general properties of the equilibrium without restricting flows or plasma beta. For small toroidicity we solve explicitly and give analytical results, exhibiting the combined effects of the self-consistent magnetic field and plasma flow.

Basic Equations

As is well known, the determination of axisymmetric magnetohydrostatic toroidal equilibria reduces to the solution of a single elliptic differential equation in which the plasma pressure and poloidal current distributions appear. Prescription of these distributions as functions of the poloidal magnetic flux, and prescription of appropriate boundary conditions provide this problem with its most concise formulation.

The determination of stationary toroidal equilibria even with axisymmetry seems to be much more difficult to reduce. The introduction of flow not only converts the static force balance to an equation of motion, but also demands the consideration of other equations viz., Ohm's law, the mass continuity equation, an energy equation and an equation of state.

The MHD equations corresponding to all but the last two mentioned above are:

$$\rho \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{J}\times\mathbf{B} - \nabla p \tag{1}$$

$$\mathrm{rot}\,\mathbf{B} = \mu_0 \mathbf{J} \tag{2}$$

$$\mathrm{div}\,\mathbf{B} = 0 \tag{3}$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta \mathbf{J} \tag{4}$$

$$\mathrm{div}\,\rho\mathbf{v} = Q \tag{5}$$

where all symbols have their usual meaning. The source term Q is introduced to maintain a stationary equilibrium by restoring mass into the toroidal confining region. This is necessary as resistivity leads to a radial loss of plasma. A general energy equation would include an ohmic heating term which is due to the externally induced electric field. However, at high temperatures, this energy equation simplifies because of the tendency of the plasma to become isothermal. Also, for a given ohmic heating current and increasing temperature, the resistivity and the external electric field become small, if we can reach a source-free state of plasma. In such a situation we include the equation of state for an ideal gas $p = \rho c^2$, where the sound speed c is a constant to be specified. The full set of equations for the magnetic field, flow, electric potential ($E = -\nabla\Phi$) and mass density are then:

$$\rho \mathbf{v}\cdot\nabla\mathbf{v} = \frac{1}{\mu_0}\,\mathrm{rot}\,\mathbf{B}\times\mathbf{B} - c^2\nabla\rho \tag{6}$$

$$\mathbf{v}\times\mathbf{B} = \nabla\Phi \tag{7}$$

$$\mathrm{div}\,\mathbf{B} = 0 \tag{8}$$

$$\mathrm{div}\,\rho\mathbf{v} = 0 \tag{9}$$

Because of the existence of integrals of these equations it is possible to reduce this system /6,7,8/.

Exact Integrals and Radial Force Balance

To proceed we consider a set of nested toroidal magnetic surfaces. From equation (7), Φ is constant along magnetic as well as flow lines, so that the flow is everywhere tangent to magnetic surfaces. From equations (8) and (9) magnetic field and mass flow are divergence-free. Thus it is convenient to introduce

$$G = \int_{\mathbb{F}} \mathbf{B}\cdot d\mathbf{S} \quad , \quad \Gamma = \int_{\mathbb{F}} \rho\mathbf{v}\cdot d\mathbf{S} \tag{10}$$

where $\mathbb{F}$ is a surface bounded by the magnetic axis and any closed curve, not encircling this axis and lying on a magnetic surface. In this way G and Γ are the poloidal fluxes of magnetic field and mass flow respectively. Further, in the case of axisymmetry we have

$$\mathbf{B} = \frac{1}{2\pi}(\nabla\zeta\times\nabla G + \Lambda\nabla\zeta) \tag{11}$$

$$\rho\mathbf{v} = \frac{1}{2\pi}(\nabla\zeta\times\nabla\Gamma + L\nabla\zeta) \tag{12}$$

which, with arbitrary functions Λ and L, are the most general solutions of the equations (8) and (9) and so replace them. ζ is the angle about the axis of symmetry. Beside the equation of motion we have not as yet used the normal component of equation (7). The latter with equations (11) and (12) relates the function Λ and L as follows

$$L = \Lambda\dot{\Gamma} - 4\pi^2\rho\dot{\Phi}R^2 \tag{13}$$

We denote by a dot derivatives of surface quantities with respect to the poloidal magnetic flux G. R is the distance from the axis of symmetry.

The state of our considerations can be summarized as follows: specify any surface quantities Γ and Φ in terms of the poloidal magnetic flux G, then with any functions Λ, ρ and G magnetic and velocity fields (11),(12) satisfy all equations, except the equation of motion, from which we naturally expect to determine these last three functions. To treat this equation it is most convenient to investigate its components in toroidal direction, parallel to B and normal to a magnetic surface.

Using the equations (7), (8) and (11) one can prove that with axisymmetry the following two relations hold

$$\mathbf{B}\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) = \frac{1}{2}\,\mathbf{B}\cdot\nabla(v_{\parallel}^2 - v_{\perp}^2) + \mathrm{div}\left(\frac{\mathbf{v}\cdot\mathbf{B}}{B^2}\,\mathbf{B}\times(\mathbf{v}\times\mathbf{B})\right) \tag{14}$$

$$\mathrm{div}\,\gamma(\mathbf{B}\times\nabla\Phi) = \mathbf{B}\cdot\nabla(\gamma\dot{\Phi}\Lambda) \tag{15}$$

where γ is any scalar.

With the help of equations (12) - (15) the components of the equation of motion are:

$$\mathbf{B}\cdot\nabla\left\{\frac{\mu_0\dot{\Gamma}}{\rho}(\Lambda\dot{\Gamma} - 4\pi^2R^2\rho\dot{\Phi}) - \Lambda\right\} = 0 \tag{16}$$

$$\mathbf{B}\cdot\nabla\left\{\frac{B^2\dot{\Gamma}^2}{\rho^2} - 4\pi^2R^2\dot{\Phi}^2 + c^2\ln\rho^2\right\} = 0 \tag{17}$$

$$|\nabla G|^2 \operatorname{div}\frac{\nabla G}{R^2} + \frac{\Lambda}{R^2}\nabla G\cdot\nabla\Lambda + 4\pi^2\mu_0\nabla G\cdot\nabla p + 4\pi^2\mu_0\rho\left\{R^2\left[\frac{1}{R}\frac{\nabla G}{|\nabla G|}\cdot\nabla\frac{|\nabla G|}{R} - \operatorname{div}\frac{\nabla G}{R^2}\right]v_M^2 - \frac{1}{2}\frac{\nabla G\cdot\nabla R^2}{R^2}v_T^2\right\} = 0 \quad (18)$$

where

$$\mathbf{v} = \mathbf{v}_M + \mathbf{v}_T \qquad \mathbf{B} = \mathbf{B}_M + \mathbf{B}_T$$

$$\mathbf{v}_M = \frac{\dot{\Gamma}}{2\pi\rho}\nabla\zeta\times\nabla G \qquad \mathbf{B}_M = \frac{1}{2\pi}\nabla\zeta\times\nabla G$$

$$\mathbf{v}_T = \frac{1}{2\pi\rho}(\Lambda\dot{\Gamma} - 4\pi^2R^2\rho\dot{\Phi})\nabla\zeta \qquad \mathbf{B}_T = \frac{1}{2\pi}\Lambda\nabla\zeta$$

From equations (16) and (17) the expressions in brackets are surface quantities, so that the physical quantities which appear there are related. We can consider the first of these relations as an equation for Λ, and the second as an equation for the mass density ρ. Assuming that single-valued solutions exist, we see that they have a dependence on $|\nabla G|$ and on four arbitrary surface quantities viz., Γ, Φ and those arising from the integration of (16) and (17).

The dependence of these solutions on $|\nabla G|$ is important in relation to the radial force balance (18), for the derivatives of Λ and ρ, which appear there, provide second derivatives of G, and thus influence the nature of the second order differential operator. The nonlinear dependence of Λ and ρ on $|\nabla G|$ contributes to the quasi-linear character of this partial differential equation. Any solution of this equation for G will depend not only on the boundary conditions, but also on the previously mentioned surface functions, whose physical meaning we now briefly indicate.

By definition Γ is the poloidal mass flux, and Φ is the electric potential so that there is no difficulty in interpretation here. The remaining surface functions can be related to the density and poloidal current distributions on a reference line which will be introduced.

To do this let us fix in any meridional plane the straight line $R = R_o$ which passes through the magnetic axis and traverses all magnetic surfaces considered (see Fig. 1).

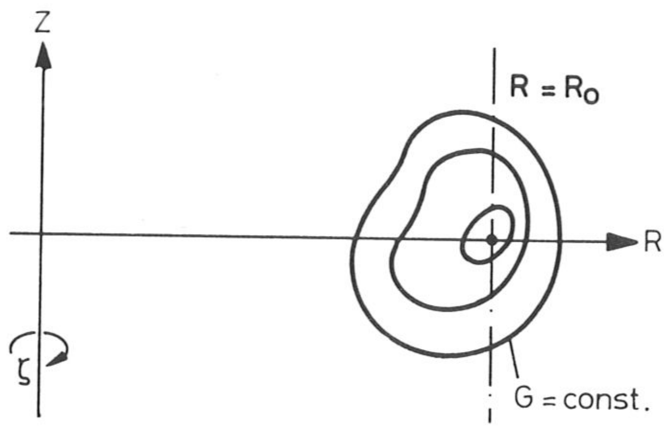

Fig. 1

Then the quantities G, Γ and Λ will be divided by $2\pi R_0$ which guarantees finite values for the resulting quantities in the limit $R_0 \rightarrow \infty$. Unless otherwise stated we now refer only to these specific quantities, in former notation. On each magnetic surface the values of all the dependent variables will be related to their values at the intersection of the particular surface and the chosen straight line. These latter reference values will be denoted by a subscript s.

$$\bar{\rho} = \frac{\rho}{\rho_0} \qquad \bar{\Lambda} = \frac{\Lambda}{\Lambda_0} \qquad \bar{R} = \frac{R}{R_0} \tag{19}$$

The reduced functions $\bar{\rho}$, $\bar{\Lambda}$ and $\bar{R}$ describe the variation on a magnetic surface of ρ, Λ and the distance R from the axis of symmetry. The specification of the directly identifiable quantities ρ_0 and Λ_0 as functions of G can then be seen to be equivalent to the specification of the surface functions arising from the integration of the magnetic differential equations (16), (17).

The introduction of ρ_0 and Λ_0 makes the relationship between stationary and static equilibria readily comprehensible. These quantities whose distribution on the reference line is arbitrary, in static equilibria retain their value following a magnetic field line, whereas in stationary equilibria flow causes ρ and Λ to vary on a magnetic surface. This variation is described by $\bar{\rho}$ and $\bar{\Lambda}$ and governed by equations (16) and (17). On the other hand flow is unrestricted to the extent inherent in the arbitrariness of the functions Φ and Γ.

For the subsequent discussion we wish to translate our freedom of choice for $\rho_0, \Lambda_0, \Phi, \Gamma$ into a convenient dimensionless form expressed by the four equivalent functions β_M, β_T, M, E . The first two are the obvious resolution of the local beta on the reference line into meridional and toroidal parts

$$\frac{B_0^2}{\mu_0 \rho_0 c^2} \equiv \frac{1}{\beta} = \frac{1}{\beta_M} + \frac{1}{\beta_T} \qquad (20)$$

The dimensionless functions M and E

$$M = \frac{\dot{\Gamma} B_0}{\rho_0 c} \qquad\qquad E = \frac{\dot{\Phi}}{c} \qquad (21)$$

are once again (see /2/) useful in the discussion of flow effects.

To summarise our previous discussion of the logical structure of the derived system of equations, and assuming for the moment the mathematical feasibility of each step we can state the following procedure for solution. Choose any betas and flows by prescription of β_M, β_T, M and E. Calculate $\bar{\rho}$ and $\bar{\Lambda}$ by (16) and (17) in terms of $|\nabla G|$ and use the results to solve, together with boundary conditions, equation (18). Without even considering the difficulties with respect to the solution of the partial differential equation (18), such a procedure encounters some problems.

Before we investigate in more detail the global properties of the expressions determining $\bar{\rho}$ and $\bar{\Lambda}$ we indicate some local characteristics. In studying the appropriate relations we naturally do not consider that spatial dependence of $\bar{\rho}$ and $\bar{\Lambda}$ which is of prescribable nature, but that which is determined by equations. As can be seen by (16) and (17), such dependence occurs via $\bar{R}$ and $B_M = \frac{|\nabla G|}{\bar{R}}$ only. On the reference line ($\bar{R} = 1$) we find with respect to these functions the following differentials for $\bar{\rho}$ and $\bar{\Lambda}$

$$d\bar{\rho} = \frac{D_0}{1-(1+\beta)M^2}\left\{\left(E^2 + \frac{\beta}{\beta_T}M^2\left(1 + \frac{2(\beta\beta_T)^{1/2}ME}{1-\beta M^2}\right)\right)d\bar{R} - \frac{\beta M^2}{\beta_M}d\bar{B}_M\right\} \tag{22}$$

$$-(1-\beta M^2)d\bar{\Lambda} = 2(\beta\beta_T)^{1/2}MEd\bar{R} + \beta M^2 d\bar{\rho} \tag{23}$$

Here D_o is

$$D = \frac{\left(1 - \frac{\beta M^2}{\bar{\rho}}\right)\left(1 - \frac{\beta M^2}{\bar{\rho}} - \frac{M^2\bar{B}^2}{\bar{\rho}^2}\right)}{\left(1 - \frac{\beta}{\beta_M}\frac{M^2\bar{B}_M^2}{\bar{\rho}^2}\right)\left(1 - \frac{\beta M^2}{\bar{\rho}}\right) - \frac{\beta}{\beta_T}\frac{M^2\bar{B}_T^2}{\bar{\rho}^2}} \tag{24}$$

taken on the reference line. Whereas zeros of denominators appearing in (22) have their counterparts in numerators of D_o and so are harmless, D_o itself has genuine poles for certain betas and flows. With the knowledge of the existence of these poles we can make the following points:

1. When we take, say for $\bar{\rho}$, the differential along the meridional projection of a field line starting somewhere on the reference line, equation (22) constitutes the complete variation of $\bar{\rho}$ whose rate of change becomes infinite at a pole of D_o. Closer investigation shows that for $ME > 0$, for example, this point in a $\bar{\rho}-\bar{R}$-plane is of parabolic type, so that for $\bar{R} < 1$ no solutions for $\bar{\rho}$ can be found. Thus, we must expect solvability conditions which will restrict the possible choices for flow and beta.

2. For differential variations of $\bar{\rho}$ and $\bar{\Lambda}$ normal to a magnetic surface, where $d\bar{B}_M$ involves normal derivatives of G, these poles will be important in the determination of the characteristic manifolds.

With reference to this latter point we write down the characteristic condition for (18)

$$\left(1-\frac{\beta M^2}{\bar{\rho}}\right)\nabla\psi\cdot\nabla\psi + \left\{D-\left(1-\frac{\beta M^2}{\bar{\rho}}\right)\right\}\frac{(\nabla G\cdot\nabla\psi)^2}{|\nabla G|^2} = 0 \quad (25)$$

where we once again encounter D. In Fig. 2 we have plotted the characteristic determinant on the reference line $D_o(1-\beta M^2)$ as a function of M.

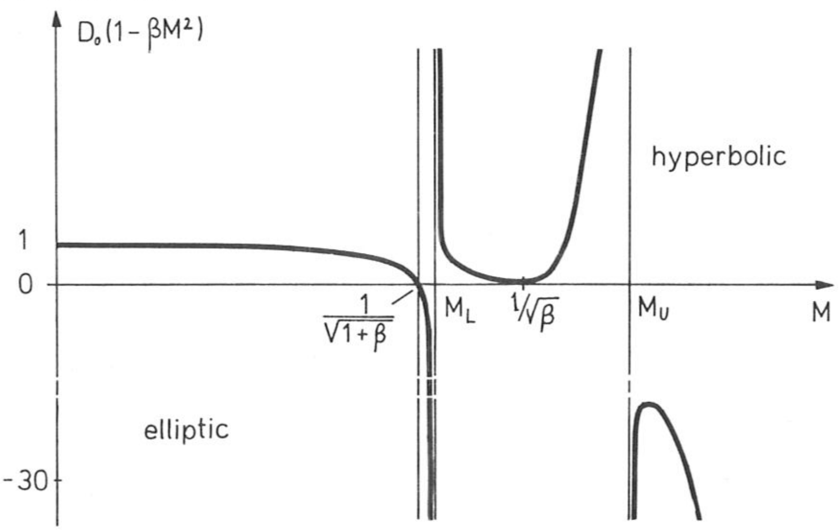

Fig. 2

Nature of the Partial Differential Equation

With vertical lines we have separated the regions where $D_o(1-\beta M^2)$ is alternately positive and negative i.e. where the partial differential equation (18) is correspondingly elliptic and hyperbolic. The particular M values of the poles and zeros of the characteristic determinant correspond to certain critical speeds. These are: the acoustic wave speed, and the three hydromagnetic wave speeds (slow, Alfven, fast), all affected by rotational transform and beta. Their significance to toroidal equilibrium recently has been noted by Taniuti et al /9, 10/. Characteristics for a different model of toroidal plasma flow have been discussed by Dobrott and Greene /11/.

We now investigate the analytical structure and the solvability range of equations (16) and (17), which in integrated form with reduced quantities are

$$\bar{\Lambda} = \frac{1-\beta M^2-(\beta\beta_T)^{1/2}ME(\bar{R}^2-1)}{1-\beta M^2/\bar{\rho}} \tag{26}$$

$$\ln\bar{\rho}^2 + \frac{M^2\bar{B}^2}{\bar{\rho}^2} = M^2 + E^2(\bar{R}^2-1) \tag{27}$$

where

$$\bar{B}^2 = \frac{\beta}{\beta_M}\bar{B}_M^2 + \frac{\beta}{\beta_T}\frac{\bar{\Lambda}^2}{\bar{R}^2} \tag{28}$$

The appropriate forms of these equations in the low beta approximation which we treated previously /2/, can be immediately recognized. In this limit $\bar{\Lambda} \to 1$, $\bar{B} \to 1/\bar{R}$ we have discussed quite generally the resulting Bernoulli-type equation for $\bar{\rho}$.

Here we have to try to find

$$\bar{\rho} = \bar{\rho}(|\nabla G|,\bar{R};\beta_M,\beta_T,M,E) \tag{29}$$

$$\bar{\Lambda} = \bar{\Lambda}(|\nabla G|,\bar{R};\beta_M,\beta_T,M,E) \tag{30}$$

For the discussion of such solutions we prefer to introduce new variables x, z to replace $\bar{\rho}$ and $\bar{\Lambda}$:

$$x = x(\bar{\rho}, \bar{\Lambda}) = MS\bar{B} \tag{31}$$

$$z = z(\bar{\rho}, \bar{\Lambda}) = \frac{M\bar{B}}{\bar{\rho}} \tag{32}$$

Here S is defined by

$$S = \exp\left\{-\tfrac{1}{2}\left(E^2(\bar{R}^2-1) + M^2-1\right)\right\} \tag{33}$$

This together with the following auxiliary functions

$$\alpha = \frac{1}{\beta M^2 S} \tag{34}$$

$$\mu = \left(\frac{\beta}{\beta_M}\right)^{1/2} MS\bar{B}_M \tag{35}$$

$$\nu = \frac{1}{(\beta\beta_T)^{1/2} M} \cdot \frac{K}{\bar{R}} \tag{36}$$

$$K = 1 - \beta M^2 - (\beta\beta_T)^{1/2} ME(\bar{R}^2-1) \tag{37}$$

transforms the equations (26) and (27) into

$$x = z \cdot \exp\left\{-\tfrac{1}{2}(z^2-1)\right\} \tag{38}$$

$$z = \alpha x \pm |\nu| \left(1 - \frac{\mu^2}{x^2}\right)^{-1/2} \tag{39}$$

For a particular choice of α, μ, ν the curves corresponding to these equations appear as shown in Fig. 3.

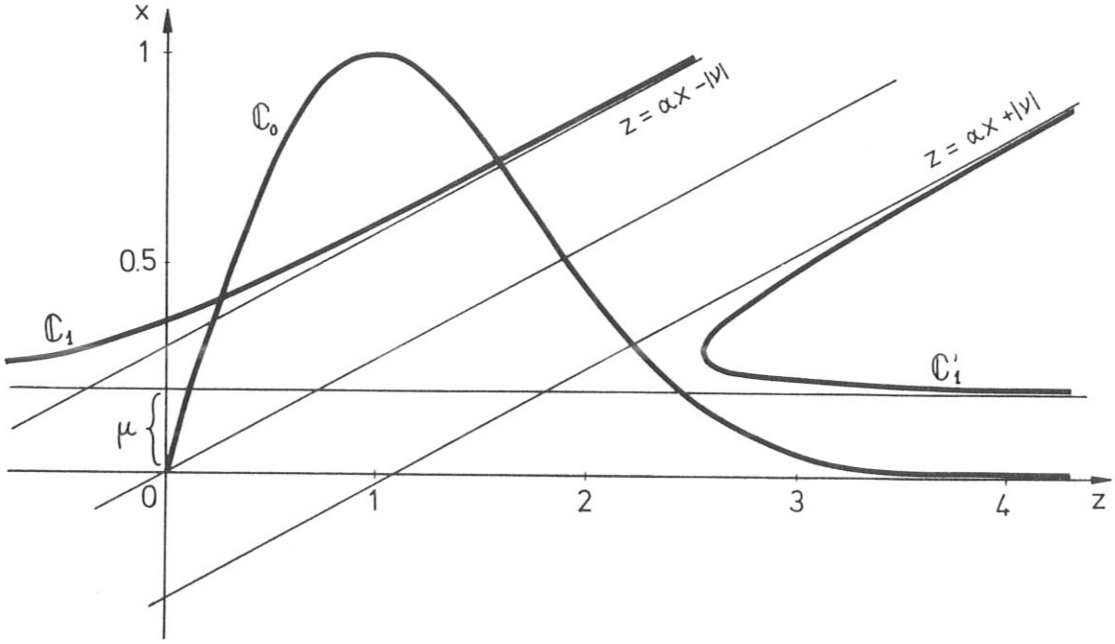

Fig. 3

While

$$\frac{\partial(x,z)}{\partial(\bar{\rho},\bar{\Lambda})} = \frac{\beta M^2 S}{\beta_T \bar{\rho}^2 \bar{R}} \cdot \bar{B}_T \neq 0 \qquad (40)$$

we can investigate the structure of (26) and (27) in the new variables x and z.

Analagous to the low beta case /2/ where to solve for variations of $\bar{\rho}$ on a magnetic surface we considered the intersection of a straight line and a fixed curve, intersections of the curves (38), (39) represent possible solutions. Here "possible" has the restricted sense that we can find functions as indicated by (29) and (30). This does not mean that these solutions guarantee a solution of the partial differential equation (18). Indeed limitations stemming from the solvability of this equation are additional to any which come from the solvability of the equations for x, z (or equivalently $\bar{\rho}, \bar{\Lambda}$) which we now discuss.

The manner in which we introduced the reduced quantities ensures that on the reference line ($\bar{R}$ = 1) there is always a solution. Although in general there will be several intersections of these curves for $\bar{R}$ = 1, the particular value of M chosen determines which point x_o, z_o is to be considered:

$$x_o = M \cdot \exp\left\{-\tfrac{1}{2}(M^2-1)\right\} \tag{41}$$

$$z_o = M \tag{42}$$

We choose this point and try, proceeding along the meridional projection of a magnetic field line passing through this point, to find solutions everywhere on this closed curve. The resulting variation in $\bar{R}$ depends sensitively on x_o and z_o, that is on M.

On the curve $\mathbb{C}_0$ which is not affected by this alteration in $\bar{R}$, we recognize three special points whose abscissae are M_L , $1/\sqrt{\beta}$ and M_U . The lower and upper values for M are characterised by the fact, that for these the variable curve $\mathbb{C}_1$ for $\bar{R} = 1$ is tangent to the invariant curve $\mathbb{C}_0$. At the intermediate point $1/\sqrt{\beta}$, ν , as can be seen by (36) and (37), vanishes as we reach the reference line and curve $\mathbb{C}_1$ degenerates to a straight line which separates solutions of different character. $\nu = 0$ is implied by the vanishing of B_T , thus condition (40) represents a somewhat intricate case. Here we exclude this point from consideration but remark that it might well be that transitions and reflections at the $\nu = 0$ line lead to interesting branched solutions. According to these critical points we distinguish for $\bar{R}$-variations of intersections P, four regions of possible initial points

$$P_o = \left\{M, M \cdot \exp\left\{-\tfrac{1}{2}(M^2-1)\right\}\right\} \tag{43}$$

We investigate these region separately:

$\boxed{0 < M < M_L}$: For this case we can derive a sufficient condition for no solution to exist. Consider the straight line tangent to $\mathbb{C}_1$ at $z = 0$ which is given by

$$x = \frac{z}{\alpha\gamma^2} + \frac{\nu\gamma}{\alpha} \tag{44}$$

$$\gamma^2 := 1 + \frac{\alpha^2\mu^2}{\nu^2} \tag{45}$$

If this line has at most one point in common with $\mathbb{C}_0$ because of the convexity of $\mathbb{C}_1$ no intersections of $\mathbb{C}_0$ and $\mathbb{C}_1$ will exist. This is the case, if

$$\gamma^2\alpha < \left(1 + \frac{v\gamma^3}{z^*}\right) \exp\left\{\frac{1}{2}(z^{*2}-1)\right\} \tag{46}$$

where z^* is the positive root of

$$z^3 + v\gamma^3 z^2 - v\gamma^3 = 0 \tag{47}$$

Using the further inequality that

$$z^* > \frac{v\gamma^3}{1+v\gamma^3} \tag{48}$$

we obtain the simpler condition

$$\frac{1}{2}\gamma^2\alpha < 1 + \frac{1}{2}v\gamma^3 \tag{49}$$

which reexpressed in terms of flow is

$$\frac{1}{2}\exp\left\{\frac{1}{2}E^2(\bar{R}^2-1)\right\} < MS_0\left\{\frac{\beta M}{\gamma^2} + \frac{1}{2}\frac{\beta^{1/2}}{\beta_T^{1/2}}\gamma\frac{K}{\bar{R}}\right\} \tag{50}$$

$$\gamma^2 = 1 + \frac{\beta_T}{\beta_M}\frac{\bar{R}^2}{K^2}\bar{B}_M^2 \tag{51}$$

This indicates that for given M there are possibly severe restrictions on E for a solution to exist. This is seen by the fact, that for $\bar{R} < 1$ the left hand side exponentially decreases with E^2, whereas the right hand side at least for not strongly variable γ^2 is only linear in E. Indeed, at low β this inequality exactly goes over to that given elsewhere /2/.

For $ME > 0$ it is possible to show that for sufficiently large E^2 points $P = \{x, z\}$ of intersection always go upward in the x - z plane if we vary $\bar{R}$ in the direction of $\bar{R}_{min}$. This has the consequence that limitations on continuous flow solutions are to be expected on the inside of the torus, and then discontinuous solutions, as discussed in /12/, become possible. In the considered flow range we see from the differential of $\bar{\rho}$ (22) that for increasing $\bar{R}$ the mass density increases if E^2 is large enough to dominate effects arising from the $\bar{B}_m$ variation.

$\boxed{M_L < M < 1/\sqrt{\beta}}$: Naturally, the sufficient condition for no solution to exist is unaltered from the previous case, because in both cases just the variable curve $\mathbb{C}_1$ is important. Whereas in low beta the modified sound speed ($M = 1$) separated this region from the former, at finite beta this function is assumed by $M_L < 1$.

With the same comment as above on the largeness of E and for $ME > 0$ the variation of $\bar{\rho}$ here is such that for increasing $\bar{R}$ the density diminishes, as in the low beta supersonic situation.

$M > 1/\sqrt{\beta}$: For super-Alfvenic flows we can derive a corresponding sufficiency condition for no solution. We do not present this result here. In this flow range with ME < 0, for the case $M < M_U$ the density variation on a magnetic surface is as for $M < M_L$, and for $M > M_U$ as for $1/\sqrt{\beta} > M > M_L$.

Large Aspect-Ratio Limit

To exemplify our considerations outlined above we now give some results for the case of large aspect-ratio. As in the finite beta, quasi-stationary problem /5/ a coordinate system is introduced which anticipates the form of the magnetic surface and which has sufficient freedom in the form of a built in function, to satisfy the conditions of the problem up to first order in inverse aspect ratio .

The appropriate coordinate system is shown in Fig. 4

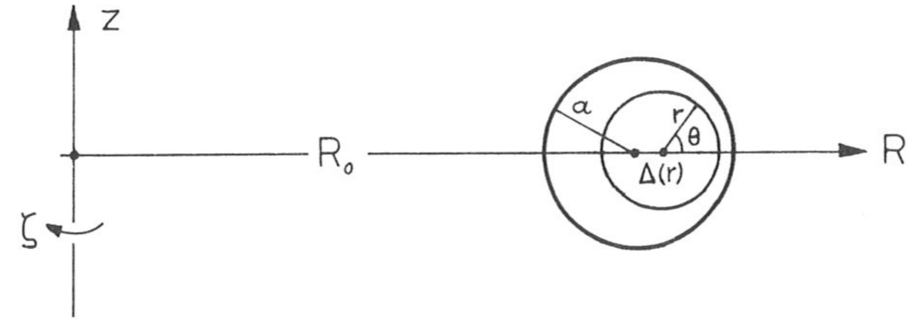

Fig. 4

For convenience we have altered slightly the notation which was used in the general discussion. For instance, R_o is no longer the distance from the symmetry axis to the magnetic axis, but is the major radius. We will introduce the coordinate system given by $\mathbf{x} = (x,y,z)$ and

$$\mathbf{x} = R(r,\theta)(\cos\zeta\,\mathbf{e}_x - \sin\zeta\,\mathbf{e}_y) + r\sin\theta\,\mathbf{e}_z \qquad (52)$$

$$R(r,\theta) = R_0 + \Delta(r) + r\cos\theta \qquad (53)$$

$\Delta=\Delta(r)$, r and θ are defined in fig. 4 . For completeness we give the metric tensor

$$\mathbf{g} = \begin{pmatrix} 1 + 2\Delta'\cos\theta + \Delta'^2 & -r\Delta'\sin\theta & 0 \\ -r\Delta'\sin\theta & r^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix} \qquad (54)$$

Plasma quantities taken at $\theta = \pi/2$ will be identified by the subscript o which previously indicated the reference line values. These, together with the new R_o lead to a redefinition of the reduced quantities. Assuming in zeroth order a cylindrical axisymmetric equilibrium then in this order all reduced quantities are identically one. To find $\bar{\rho}$ and $\bar{\Lambda}$ in first order we replace in (22) and (23) the differentials by the corresponding first-order quantities (denoted by the superscript 1). Up to first order G is independent of θ. Then

$$\bar{B}_M^{(1)} = -(\Delta' + \frac{r}{R_0})\cos\theta \qquad (55)$$

$$\bar{R}^{(1)} = \frac{r}{R_0}\cos\theta \qquad (56)$$

From (22) and (23), we now find

$$\bar{\rho}^{(1)} = \frac{D_0}{1-(1+\beta)M^2}\left\{\frac{r}{R_0}\left(E^2+M^2\left(1+\frac{\beta}{\beta_T}\frac{2(\beta\beta_T)^{1/2}ME}{1-\beta M^2}\right)\right)\Delta'\,\frac{\beta M^2}{\beta_M}\right\}\cos\theta$$

$$\bar{\Lambda}^{(1)} = \frac{D_0}{(1-(1+\beta)M^2)(1-\beta M^2)}\left\{\frac{r}{R_0}\left(\beta M^2(E^2+M^2)+2(\beta\beta_T)^{1/2}ME(1-M^2)\right)\right.$$

$$\left. +\Delta'\,\frac{\beta^2 M^4}{\beta_M}\right\}\cos\theta$$

As is to be expected from the general considerations $\bar{\rho}$ and $\bar{\Lambda}$ depend on $|\nabla G|$, i.e., Δ' and the characteristic coefficient D_0 appears in these expressions.

All that remains is the partial differential equation (18) which reduces in zeroth and first orders to two ordinary differential equations. These must be solved with the appropriate boundary conditions. It is interesting to note that even with flow they can be directly integrated. Because it will occur frequently we introduce the average for functions A (r):

$$\langle A \rangle = \frac{1}{V}\int_0^V A\,d\bar{V} \quad , \quad V = 2\pi R_0 \cdot \pi r^2$$

In lowest order, assuming the absence of line currents on the axis, integration leads to the Bennett Pinch Relation generalized to include flow effects

$$B_M^2 = \left(\frac{\mu_0 I}{2\pi r}\right)^2 = \left(\langle B_T^2\rangle - B_T^2\right) + 2\mu_0\left(\langle p\rangle - p\right) + \mu_0\,\langle \rho v_M^2\rangle$$

where $I(r)$ is the total longitudinal enclosed in the surface of radius r.

In passing we recover the well-known results (see for example /13, 14/) in the quasi-stationary case concerning plasma diamagnetism.

$\beta_M < 1$ implies $dB_T^2/dr < 0$, which is a paramagnetic situation,

$\beta_M > 1$ implies $dB_T^2/dr > 0$, which is a diamagnetic situation.

The effect of flow can now be clearly seen. For given pressure and toroidal current profiles the poloidal mass rotation ρv_M tends to make the plasma more diamagnetic. It acts as an effective kinetic pressure to push toroidal magnetic field outside of the plasma region. We note that toroidal mass flows in no way affect the zeroth order radial force balance.

In first order we obtain a second-order differential equation for Δ which, as noted above, can be directly integrated. After elimination of B_T, using the zeroth order relation, we obtain

$$\Delta' = \frac{1}{D_0}\left\{ \Delta'_S - \frac{r}{R_0}\left(\frac{\mu_0}{B_M^2}\left(\langle \rho v^2\rangle - \rho v^2\right) - \frac{\mu_0 \langle \rho v_M^2\rangle}{2B_M^2} - \frac{D_0}{\beta}\left(\beta_M \frac{v_M}{c}\right)^2 \left(\frac{v_\perp^2}{c^2} + \frac{v_M^2}{c^2}\left(1 - \beta\frac{v^2}{c^2}\right)\right)\right)\right\}$$

where Δ'_S is the result obtained in the static case (see /3/):

$$\Delta'_S = -\frac{r}{R_0}\left\{ \frac{2\mu_0}{B_M^2}\left(\langle p\rangle - p\right) + \frac{1}{2}\frac{\langle B_M^2\rangle}{B_M^2}\right\} .$$

This represents a generalisation of the result for the plasma displacement with negligible nonlinear flow coupling /3/. The last three terms in brackets and the departure of D_o from 1 represent the effects of flow. D_o as a function of M is shown in Fig. 5.

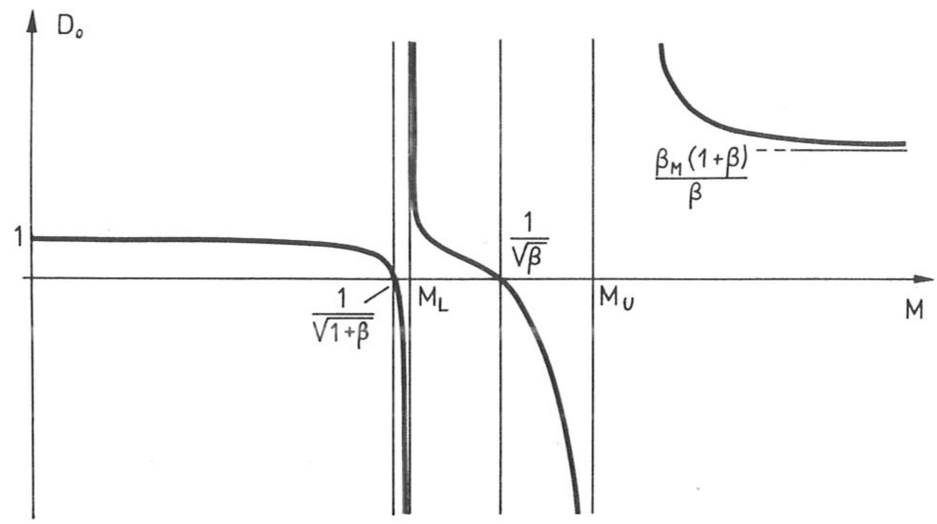

Fig. 5

Not only do the local flow values appear in the expression for Δ' but also the flow profiles. An investigation of the radial behavior is difficult because of the freedom involved in the choice of the functions of pressure, meridional magnetic field, poloidal and toroidal mass fluxes. However, a direct result of this freedom is the interesting possibility of having $\Delta' \equiv 0$ which with the condition $\Delta(a) = 0$ imply $\Delta \equiv 0$. The four functions above can be chosen to satisfy this condition with the additional constraint, that they also satisfy the generalised Bennett relation.

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References

1. D.Pfirsch, A.Schlüter, MPI/PA/7/62, Max-Planck Institute Report (1962), unpublished.
2. H.P.Zehrfeld, B.J.Green, Nuclear Fusion 10 (1970) 251
3. V.D.Shafranov, Nuclear Fusion 3 (1963) 183
4. E.K.Maschke, Plasma Physics 13 (1971) 905
5. H.P.Zehrfeld, B.J.Green, IPP Report (1970) III/1
6. L.Woltjer, Astrophys. J. 130 (1959), 405
7. D.Dobrott, J.M.Greene, Bull.Am.Phys.Soc. 14 (1969) 1016
8. L.S.Soloviev, Reviews of Plasma Physics (Editor M.A.Leontovich), Consultants Bureau (1967), Vol.3 ,p.277
9. T.Taniuti, Phys.Rev.Letters 25 (1970) 1478
10. N.Asano, T.Taniuti, Phys.Fluids 15 (1972) 423
11. D.Dobrott, J.M. Greene, Phys.Fluids 13 (1971) 313
12. R.D.Hazeltine, E.P.Lee, M.N.Rosenbluth, Phys.Fluids 14 (1971) 361
13. J.Andreoletti, Plasma Physics 13 (1971), 313
14. J.D.Callen, R.A.Dory, ORNL-TM-3430, (1971)