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ON THE ROLE OF AZIMUTHAL ELECTRIC FIELDS IN TOROIDAL TRANSPORT

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Abstract

The role of azimuthal electric fields in toroidal transport is considered. We give a consistent kinetic theory for the collision dominated regime and compare with a fluid calculation. The weakly collisional and intermediate regimes are also discussed.

Enhancement of diffusion in toroidal systems over diffusion in a cylindrical system may be caused either by electric fields or gradients of the pressure tensor [1, 2]. In the collision dominated regime $\frac{2\pi\lambda}{L} \ll 1$ (λ mean free path, L connection length) Pfirsch and Schlüter have shown that an electric potential

$$\frac{e\Phi_{PS}}{T_e} = \epsilon \frac{r_L}{\Theta a} \frac{L}{2\pi\lambda} \sin\delta \tag{1}$$

arises which is responsible for the enhanced diffusion [3]. In (1) $\epsilon = r/R \ll 1$, r_L electron Larmor radius, $\Theta = B_\delta/B \ll 1$ $a = \left(\frac{1}{p}\frac{\partial p}{\partial r}\right)^{-1}$ plasma radius, $L = 2\pi a/\Theta$ and δ is the small azimuth in the usual toroidal coordinates r, δ, ξ. Yet in a number of papers, dealing with the kinetic theory of transport, nonradial electric fields have been ignored. This may be justified in the low collision frequency regime, as discussed below. However, as one approaches larger collision frequencies, this treatment must be inconsistent, even if it will give the correct result for the diffusion flux. We wish to demonstrate this point by comparing the Pfirsch-Schlüter calculation with a consistent kinetic theory for the collision dominated regime.

We assume the following ordering:

$$1 \gg \frac{r_L}{\Theta a} \gtrsim \frac{2\pi\lambda}{L} \gg \epsilon\left(\frac{r_L}{\Theta a}\right)^2 . \tag{2}$$

The first inequality allows the neglect of plasma rotation $v_E\delta/\Theta v_\parallel \ll 1$ due to radial electric fields $e\Phi_0/T_e \approx 1$. The second inequality makes the ExB drift due to the field (1) comparable or larger than the toroidal drifts and the last inequality ensures that the drift velocities are small compared to free streaming along field lines, $v_E r/\Theta v_\parallel \ll 1$. To lowest order in $\frac{r_L}{\Theta a}$ only motion along the magnetic field lines is allowed. To this order the guiding center distribution function $f(\mathcal{E}, \mu, r, \delta)$ +) may be assumed to have relaxed to a local Maxwellian

$$f_0 = \frac{N_0}{(2\pi T/m)^{3/2}} \exp(-\mathcal{E}/T) \tag{3}$$

where $N_0 = N_0(r)$ and $T = T(r)$ are constant on the magnetic surfaces, but the electric potential in $\mathcal{E} = mv_\parallel^2/2 + \mu B + e\Phi$, $\mu = mv_\perp^2/2B$, is at this point arbitrary, $\Phi = \Phi(r, \delta)$. To first order in $r_L/\Theta a$ we obtain

$$\Theta v_\parallel \frac{1}{r}\frac{\partial}{\partial \delta} f_1 + v_{dr} \frac{\partial f_0}{\partial r} = C(f_0, f_1) \tag{4}$$

where the drift velocity $\underline{v}_d$ is given by (neglecting small contributions from nonpotential electric and magnetic fields)

$$\underline{v}_d = v_\parallel \underline{\xi}_0 \times \nabla(v_\parallel/\Omega) - \frac{v_\parallel^2 + v_\perp^2/2}{B\,\Omega}(\nabla B \times \underline{\xi}_0) + \frac{c}{B^2}(\underline{B} \times \nabla\Phi). \tag{5}$$

+) The discussion applies to both electrons and singly ionized ions of a fully ionized plasma. For brevity, the subscript $j = e,i$ has been omitted on occasion. Weakly ionized plasmas are considered in Appendix I.

In (5), $\Omega_j = -e_j B/m_j c$ and

ξ_0 is the unit vector along $\underline{B}$. Note that the spatial derivatives in (4) and (5) have to be taken at constant ε , μ . We further expand (4) in the small parameter $2\pi\lambda/L$ [4].

$$0 = C(f_0, f_1^0) \tag{6}$$

$$\Theta v_\parallel \frac{1}{r}\frac{\partial}{\partial\delta} f_1^0 = C(f_0, f_1^1) \tag{7}$$

$$\Theta v_{\parallel} \frac{1}{r}\frac{\partial}{\partial\delta} f_1^1 + v_{dr}\frac{\partial f_0}{\partial r} = C(f_0, f_1^2) \quad . \tag{8}$$

To lowest order in $2\pi\lambda/L$ we obtain a Maxwellian distribution $f_1^0 = \frac{N_1}{N_0} f_0$ with $N_1(r,\delta)$ to be determined from (7) and (8). From (3) we find a density

$$\hat{n}_j(r,\delta) = N_{oj}(r)\exp[-e_j\phi(r,\delta)/T_j] \quad , \; j = e, i \; . \tag{9}$$

Quasineutrality to lowest order in $\frac{r_L}{\Theta a}$ requires

$$n_0 = n_{oj}(r) = N_{oj}(r)\exp(-e_j\phi_0/T_j) \; ; \; \phi_0 = \phi_0(r) \; . \tag{10}$$

The first order density becomes, including f_1^0

$$n_{1j} = n_{oj}\left(-e_j/T_j\,\phi_1 + \frac{N_1}{N_0}\Big|_j\right) \tag{11}$$

from which we may determine ϕ_1 , assuming quasi-neutrality. The flux of guiding centers across the magnetic surface r = const is given by

$$\Gamma_c = \int \frac{d\delta}{2\pi} \left[1 + r/R \cos\delta \right] n(r,\delta)\, v_{dr}(r,\delta) \tag{12}$$

which may using (5) and $B = B_0/(1 + r/R \cos\delta)$ be written as

$$\Gamma_c = \int \frac{d\delta}{2\pi} (1 + r/R \cos\delta)^2 \int d\underline{v} \frac{v_\parallel^2}{\Omega_0} \frac{1}{r} \frac{\partial f}{\partial \delta}\Big|_{\varepsilon,\mu} . \tag{13}$$

The flux due to f_1^0 becomes

$$\Gamma_{cj}^0 = \frac{c}{B_0} \int \frac{d\delta}{2\pi} (1 + r/R \cos\delta)^2 A_j(r,\delta)\, \tilde{n}_j(r,\delta) \tag{14}$$

where

$$-A_j(r,\delta) \equiv \frac{T_j}{e_j} \frac{1}{r} \frac{\partial}{\partial\delta} \frac{N_1}{N_0}\Big|_j = \frac{T_j}{e_j n_{0j}} \frac{1}{r} \frac{\partial n_{1j}}{\partial\delta} + \frac{1}{r} \frac{\partial \phi_1}{\partial\delta} \tag{15}$$

using (11).

With (9), (15) and $\epsilon = r/R \ll 1$, $e\phi_1/T_j \ll 1$ (14) becomes

$$\Gamma_{cj}^0 = \frac{2crn_0}{B_0 R} \left\langle \cos\delta\, A_j(r,\delta) \right\rangle_\delta + \frac{c}{B_0 r} \left\langle \phi_1 \frac{\partial n_{1j}}{\partial\delta} \right\rangle_\delta \tag{16}$$

where the bracket indicates an average over δ .

The expression (16) for the guiding center flux is equal to the neoclassical flux obtained from the fluid equations

$$\Gamma_{fl} = \int \frac{d\varsigma}{2\pi} \left(1 + r/R \cos\varsigma\right) n(r,\varsigma)\, u_r(r,\varsigma) \tag{17}$$

where $u_r = -\frac{c}{Br}\left[\frac{\partial\phi}{\partial\varsigma} + \frac{1}{ne_j}\frac{\partial p_j}{\partial\varsigma}\right]$ is the radial drift velocity due to electric fields and pressure gradients. The equality of the two fluxes follows also from the more general relation [5],

$$\langle n\underline{v}\rangle_v = \langle n\underline{v}_c\rangle_v - \mathrm{curl}\left(\frac{c\,p_j}{e_j B^2}\underline{B}\right) \tag{18}$$

between fluid drift flux (neglecting collisions) and guiding center flux.

Integrating (8) over velocity space we obtain the continuity equation for the guiding center flux $\langle n v_c\rangle_v = \int d\underline{v}\,[v_\parallel \underline{\tau}_0 + \underline{v}_d]\, f$. Writing the parallel component as $e_j\langle n v_{\parallel c}\rangle_v = \alpha_j \underline{B}$ the continuity equation becomes

$$e_j\nabla\cdot\langle n\underline{v}_{\parallel c}\rangle_{v,j} = \underline{B}\cdot\nabla\alpha_j = -e_j\int d\underline{v}\,\underline{v}_{d_j}\cdot\nabla f_{0j} = -e_j\nabla\cdot\langle n\underline{v}_{\perp c}\rangle_{v,j}. \tag{19}$$

The right hand side may be evaluated, using (5), and results in the guiding center flux

$$\langle n\underline{v}_{\perp c}\rangle_{v,j} = \frac{2c\,p_j}{e_j B^3}\left(\underline{B}\times\nabla B\right) + n_j\underline{v}_E \tag{20}$$

and the magnetic differential equation for α_j

$$\underline{B}\cdot\nabla\alpha_j = -\frac{2c}{B^3}(\underline{B}\times\nabla B)\cdot\nabla p_j - \nabla(n_j e_j \underline{v}_E) \tag{21}$$

where $\underline{v}_E = \frac{c}{B^2}(\underline{B}\times\nabla\phi)$.

The guiding center current $\sum_j e_j \langle n_j \underline{v}_{cj}\rangle_v$ has the perpendicular component $\underline{j}_{\perp c} = \frac{2cp}{B^3}(\underline{B}\times\nabla B)$ which differs from the diamagnetic current $\underline{j}_\perp = \frac{c}{B^2}(\underline{B}\times\nabla p)$ but has the same divergence, c.f. (18). Therefore the same magnetic differential equation for the longitudinal current is obtained from the fluid equations and drift kinetic theory:

$$\underline{B}\cdot\nabla\alpha = \frac{2c}{B^3}[\nabla B\times\underline{B}]\cdot\nabla p \tag{22}$$

where $\underline{j}_\parallel = \alpha\underline{B}$, $p = p_e + p_i$.

In the fluid theory the longitudinal current is related to the driving electric fields and pressure gradients by Ohm's law.

$$j_\parallel = \kappa_\parallel \hat{E}_\parallel \quad ; \hat{E}_\parallel = -\Theta\frac{1}{r}\left[\frac{\partial\phi}{\partial s} - \frac{1}{n|e|}\frac{\partial p_e}{\partial s}\right] \tag{23}$$

If we neglect inertia (plasma rotation) viscosity and longitudinal temperature gradients we find from the longitudinal component of the equation of motion of a fully ionized plasma that the longitudinal density and pressure gradients vanish:

$$\nabla_\parallel p = \nabla_\parallel(p_e + p_i) = (T_e + T_i)\,\Theta\frac{1}{r}\frac{\partial n}{\partial s} = 0 \tag{24}$$

Thus the current $J_{\parallel}$ and diffusion are indeed driven by the electric field (1). The same conclusions can be drawn from kinetic theory. Equ.(7) relates the current to the driving force A_j. Taking the first moment of (7) we obtain with (15),

$$-n_j e_j \Theta A_j = \int dv\, m_j v_{\parallel} C_j \equiv R_j \tag{25}$$

For a fully ionized plasma we have

$$R_e = -R_i = J_{\parallel} n e / \varkappa_{\parallel} \tag{26}$$

From (25), (26) and quasi-neutrality we conclude $A_e = A_i = A$ which with (15) also implies $\frac{\partial n_1}{\partial s} = 0$ and that A is just the azimuthal electric field. Equ.(7) will then be recognized as the Spitzer Harm problem for the electrical conductivity with the effective electric field $\hat{E}_{\parallel} = \Theta A$, resulting in a relation equivalent to (23)

$$J_{\parallel} = \varkappa_{\parallel} \Theta A \tag{27}$$

Equ.(26) does not hold for a weakly ionized plasma in which collisions between charged particles and neutrals dominate. Generally we have then $\frac{\partial n_1}{\partial s} \neq 0$, as shown in Appendix I. The role of electric fields on transport in a weakly ionized plasma has been analyzed by Kovrizhnykh [6].

We conclude that the calculations using fluid equations and drift kinetic equations are rather similar and result in the same electric field (1) and diffusion flux.

$$\Gamma = -\frac{n c^2}{B^2 \varkappa_\perp} \frac{dp}{dr} \left(1 + \frac{2\varkappa_\perp}{\varkappa_\parallel} \frac{r^2}{R^2 \Theta^2} \right) \tag{28}$$

where the classical flux (first term) has been added to the guiding center flux.

Had we simply neglected azimuthal electric fields in the kinetic equations, as is frequently done, then we would still obtain the same diffusion flux. This may be seen from (16), (22) and (27). The same conclusion can be drawn from the energy principle which relates the neo-classical diffusion to the dissipation by the longitudinal current [1, 2]. The nature of the driving force for $J_\parallel$ (electric fields or pressure gradients), however, does not enter this relation.

If we neglect the azimuthal electric field then according to (15) the driving force A must be interpreted as a longitudinal pressure gradient. From Poisson's equation, we find that the corresponding charge density perturbation would result in an electric field which is even a factor $a^2/\lambda_D^2 = \frac{a^2}{r_L^2}\left(\frac{\omega_e}{\Omega_e}\right)^2 \gg 1$ bigger than (1), thus showing the inconsistency of this approach for the collision dominated plasma. The azimuthal electric field is essential for the diffusion process in this case.

We now consider briefly the low collision frequency (banana) regime $\frac{L}{2\pi\lambda} \ll \epsilon^{3/2}$. The usual kinetic calculation [4] can easily be modified to include the azimuthal electric field. In the velocity coordinates $\mathcal{E}$, μ, this field enters the calculation only through the velocity dependence of the collision frequency $\nu = \nu(v)$, $v^2 = \frac{2}{m}(\mathcal{E} - e\phi)$ and the parallel velocity $v_\parallel^2 = \frac{2}{m}\left[\mathcal{E} - \mu B_0 (1 - r/R \cos S) - e\phi\right]$ On performing the required S averages the velocity dependence of ν produces corrections of order $\epsilon \frac{e\phi_1}{T_e}$, $\left(\frac{e\phi_1}{T_e}\right)^2$.. The effect on $v_\parallel$ becomes important for fields $e\phi_1/T_e \gtrsim \epsilon$. Then the magnetic trapping is dominated by the electric trapping and the toroidal drift is dominated by the E x B drift. The case $e\phi_1/T_e \gg \epsilon$ has been considered by Kovrizhnykh assuming that low frequency instabilities produce azimuthal potential variations of that magnitude [7, 8]. The self consistent electric field due to toroidal effects is,however,much smaller. The distribution function is written as an expansion in $r_L/\Theta a$ and $L/2\pi\lambda$

$$f = f_0 + \frac{v_\parallel}{\Theta\Omega}\frac{\partial f_0}{\partial r} + g(\mathcal{E}, \mu, r) + f_1' + \ldots + f_2^{\frown} + \ldots \tag{29}$$

The non collisional term g is obtained from the solubility condition for f_1' and is an odd function of velocity. Thus we obtain the estimate [9]

$$\frac{e\phi_1}{T_e} = O\left(\frac{r_L}{\Theta a}\frac{L}{2\pi\lambda}\right) + O\left(\left[\frac{r_L}{\Theta a}\right]^2\right) \tag{30}$$

for the potential. It may be shown from the equation for ξ_i that the estimate for the collisional contribution to $e\phi_1/T_e$ (first term) may be improved by the small factor ϵ for transiting particles and $\epsilon^{1/2}$ for trapped particles.

Finally, consider the intermediate regime $\epsilon^{3/2} \ll \frac{L}{2\pi\lambda} \ll 1$ in which trapped particles are collision dominated and transiting particles are noncollisional. The toroidal drifts must produce charge separations of order $\Delta r/r$ where Δr is the radial particle excursion from the magnetic surface. The electric field becomes thus

$$\frac{e\phi_1}{T_e} \sim \frac{\Delta r}{r} \sim \epsilon \frac{r_L}{\Theta a} \tag{31}$$

as may also be shown from the solution for f_1, which neglects azimuthal electric fields [10, 4]. From a self consistent calculation we expect that (31) is enhanced by the factor $^1/_D$ where D is the dielectric constant for a mode with $k_\parallel = \Theta/r$, $k_\perp = 1/r$, $m = 1$, $\omega = v_E/r$. However $D \propto 1$ unless a natural plasma mode is approached [11]. For $\frac{r_L}{\Theta a} \ll 1$ electric fields may be neglected [12].

Concluding, we find that azimuthal electric fields due to toroidal effects play a significant role in diffusion only as the collision dominated regime is approached. Low frequency modes affect diffusion only for amplitudes $\frac{e\phi_1}{T_e} \gtrsim \epsilon$. We did not consider the effects of turbulence and high frequency electric fields.

We wish to acknowledge a discussion with H. Tasso. We also wish to thank the referee, for bringing to our attention the study of the role of electric fields on transport in a weakly ionized plasma contained in Ref.[6], which lead us to include the Appendix.

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Appendix Transport in a weakly ionized plasma

Kovrizhniykh [6] has analyzed transport processes in a weakly ionized plasma. For the collision dominated and intermediate regime he has obtained expressions for the particle-flux, energy flux and charge density which all consist of two kinds of terms; terms which only depend on toroidicity and terms which explicitely depend on the azimuthal electric field. In the regime of very low collision frequencies the azimuthal electric field has been neglected. All expressions depend also on the radial electric field.

The radial electric field has to be determined from the condition of ambipolarity $\sum_j e_j \Gamma_j = 0$ and the azimuthal electric field is obtained from the quasi-neutrality condition $\sum_j e_j n_j = \sum_j q_j = 0$. The charge density q_j consists, as mentioned above, of a source term due to the toroidal drifts and a term proportional to the potential Φ_1. The proportionality factor is essentially the partial dielectric constant ϵ_j. Just as discussed in the main text above, the azimuthal electric field may become large if for certain rotation speeds a natural plasma mode is approached. We can conclude that qualitatively speaking the effects of the azimuthal electric field are similar for the fully ionized and the weakly ionized plasma. Thus we shall contend ourselves with a consideration of the collision dominated case and contrast our development for the fully ionized plasma with a corresponding calculation for the weakly ionized plasma.

As in Reference [6] we choose a model collision term

$$C_j = \nu_{jn}(f_M - f) \tag{A1}$$

where ν_{jn} is the effective collision frequency of species j with the neutrals and f_M is a Maxwellian distribution of the same density n as f. Equ.(3)-(22) also hold for the electron and ion components of a weakly ionized plasma. The essential difference is that (26) no longer holds, i.e. $R_e + R_i = -R_n \neq 0$.

From (A1) we obtain

$$R_j = -\langle n v_{\parallel} \rangle_{N,j} m_j \nu_{jn} \tag{A2}$$

assuming as in reference [6] that ν_{jn} is independent of velocity. We can no longer conclude that $A_e = A_i = -1/r \frac{\partial \phi_1}{\partial s}$ and $\frac{\partial n_1}{\partial s} = 0$ as for the fully ionized plasma, but may obtain a system of equations for $\frac{\partial \phi_1}{\partial s}$, $\frac{\partial n_1}{\partial s}$.

From (15), (25) and (A2) we obtain

$$\alpha_j = -\varkappa_j \frac{\Theta}{B}\left[\frac{T_j}{n_j e_j}\frac{1}{r}\frac{\partial n_{1,j}}{\partial \delta} + \frac{1}{r}\frac{\partial \Phi_1}{\partial \delta}\right] \qquad j = e, i \tag{A3}$$

where

$$\varkappa_j = \frac{n_j e_j^2}{m_j \nu_{jn}}$$

The α_j also have to satisfy equ.(21). In (A3) and (21) we assume quasineutrality $n = n_e = n_i$ and may then solve for n_1, Φ_1, in terms of the pressure gradients $\frac{\partial p_j}{\partial r}$

. Taking the sum and difference of the α_j,

the system of equations may be written as

$$\alpha_e^{(1)} + \alpha_i^{(1)} = (\varkappa_e T_e - \varkappa_i T_i)\frac{\Theta}{B n|e|}\frac{1}{r}\frac{\partial n_1}{\partial \delta} - (\varkappa_e + \varkappa_i)\frac{\Theta}{B}\frac{1}{r}\frac{\partial \Phi_1}{\partial \delta} \tag{A4}$$

and

$$2\alpha^{(2)} + \left[\alpha_e^{(1)} - \alpha_i^{(1)}\right] = (\varkappa_e T_e + \varkappa_i T_i)\frac{\Theta}{B n|e|}\frac{1}{r}\frac{\partial n_1}{\partial \delta} - (\varkappa_e - \varkappa_i)\frac{\Theta}{B}\frac{1}{r}\frac{\partial \Phi_1}{\partial \delta} \tag{A5}$$

where

$$\underline{B}\cdot\nabla\alpha_j^{(1)} = \frac{2c}{B^3}(\nabla B \times \underline{B})\cdot\nabla p_j \tag{A6}$$

and $\underline{B}\cdot\nabla\alpha^{(2)} = \nabla\cdot(n|e|\underline{v}_E)$; $\quad v_E = \frac{c}{B^2}(\underline{B}\times\nabla\Phi)$

(A7)

Generally we will find that $\frac{\partial n_1}{\partial \delta} \neq 0$ and $A_e \neq A_i$. From (16) it follows that ambipolarity is not automatically satisfied, as for the fully ionized plasma, but determines the radial electric field. It is interesting to note that (15), (16) and the conditions of quasi-neutrality and ambipolarity imply $A_e^c = A_i^c$ and $\left(\frac{\partial n_1}{\partial \delta}\right)^c = 0$ where the superscript c indicates the cos δ component. For simplicity, in the above treatment, as for the fully ionized plasma, we did not include inertia effects due to plasma rotation.

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