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Determination of the Magnetic Field B in Vacuum for General Two-dimensional MHD Equilibria

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Abstract

For general 2-dimensional MHD equilibria it is possible using complex functions to obtain the solution of the vacuum magnetic field which fits the boundary conditions at the plasma surface. The solution allows to distinguish between the geometry governed and plasma magnetic field governed singularities and stagnation points. Several examples, particularly the elliptic plasma boundary case, are discussed.

It has been proposed for toroidal plasmas with circular cross sections that more general cross sections be considered in order to achieve more favourable stability behaviour. The investigation of linear configurations with appropriate cross sections will yield valuable information in this respect. This paper therefore deals with the MHD equilibrium of such cylindrical plasmas by determining and discussing the magnetic field outside the plasma for arbitrary plasma cross sections and current distributions in the plasma. There already exist very elegant methods in the case of surface currents [1)].

If the cylinder axis is in the z direction, the vacuum field can be represented by

$$\vec{B} = \nabla\psi \times \nabla z + B_z \nabla z$$

where B_z is space independent and ψ is the flux function for the azimuthal field components.

At the plasma boundary it holds that $\psi = \psi_0 = \text{const}$.

ψ satisfies the Laplace equation (in Cartesian coordinates x, y, z):

$$\Delta\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = 0$$

with the general solution:

$$\psi = \operatorname{Im} \varphi(\zeta) \qquad\qquad \zeta = x + iy$$

where $\varphi(\zeta)$ is an analytic function and Im denotes the imaginary part of the function. Thus, it also holds that

$$(1) \qquad \frac{\partial\varphi}{\partial\zeta} = \frac{\partial}{\partial y}\left(\operatorname{Im}\varphi(\zeta)\right) + i\frac{\partial}{\partial x}\left(\operatorname{Im}\varphi(\zeta)\right) = B_x - iB_y$$

In order to satisfy the boundary condition $\psi = \psi_o = \text{const}$ on the plasma surface in a suitable form, orthogonal coordinates u and v such that $v = v_o$ corresponds to the plasma boundary and $v > v_o$ to the vacuum are introduced instead of x and y. This transition can be performed by a mapping that is analytic in a non-vanishing region $v \geqslant v_o$ of the w plane:

$$\zeta = h(w) \qquad\qquad w = u + iv$$

which otherwise has only a finite number of singularities in any finite region of the w plane. Since the intersection of the plasma boundary with the x,y plane forms a closed line, $h(u+iv)$ must be periodic in w.

From the definitions

$$\phi(w) = \varphi(h(w)) = \varphi(\zeta)$$

and

$$g(u,v) = \left|\frac{dh}{dw}\right| = |h'(w)|$$

it then follows that

$$\frac{d\phi}{dw} = g(u,v)\left(B_u - iB_v\right) \tag{2}$$

where B_u is the u component and B_v the v component of the magnetic field.

As the plasma boundary is a magnetic surface, it follows that $B_v = 0$ there. The u component $B_u^o(u) = B_u(u, v_o)$ depends at the boundary on the particular equilibrium and is treated here as a given function. At the boundary one thus has

$\phi'(u + iv_o) = g(u, v_o)\, B_u^o(u)$ and the analytic continuation of ϕ' for $v > v_o$ is:

$$\phi'(u+iv) = g(u+i(v-v_0), v_0)\, B_u^{\circ}(u+i(v-v_0)) = g(u,v)(B_u - iB_v) \tag{3}$$

After rearranging ($*$ denotes the transition to the complex conjugate):

$$\frac{g(u+i(v-v_0), v_0)}{g(u,v)} = \left[\left(\frac{h'(u-i(v-2v_0))}{h'(u+iv)}\right)^*\right]^{1/2}$$ one gets :

$$B_u + iB_v = \left[\frac{h'(u-i(v-2v_0))}{h'(u+iv)}\right]^{1/2} B_u^{\circ}(u-i(v-v_0)) \tag{4}$$

With eqs. (1) - (3) the flux function ψ in vacuum is :

$$\psi = \psi_0 + \mathrm{Im}\left\{\int_0^{u+i(v-v_0)} g(w', v_0)\, B_u^{\circ}(w')\, dw'\right\} = \psi_0 + \frac{1}{2i}\int_{u-i(v-v_0)}^{u+i(v-v_0)} g(w', v_0)\, B_u^{\circ}(w')\, dw'$$

Formula (4) shows that the magnetic field is given by a product of two factors one of which is determined by the shape of the plasma boundary and the other by the distribution of the magnetic field on the plasma surface.

In particular, it can be seen that the singular points of the magnetic field in vacuum that are governed by the shape of the plasma boundary are due to odd zeros of $h'(u-i(v-2v_0))$ or to zeros of $h'(u+iv)$ or to singularities of both. Singularities due to the magnetic field distribution on the plasma surface are given by singularities of $B_u^{\circ}(u-i(v-v_0))$. Singularities of individual factors however may again disappear when the product is taken. Depending on the type of singularity, there must be line or surface currents present. Surface currents flow along branch cuts according to the magnetic field jumps present there. The paths of the branch cuts are restricted by the condition $\mathrm{Im}\left(\phi^{+}(w)-\phi^{-}(w)\right)=\mathrm{const}$. ϕ^{+} and ϕ^{-} denote the values of ϕ in adjacent sheets of the Riemann surface. This condition is equivalent to continuity of the field component normal to the branch cut.

Pure stagnation points of the magnetic field in vacuum, i.e. singularity free points at which this field vanishes, are given by even zeros of $h'(u-i(v-2v_0))$ or by even poles of $h'(u+iv)$ in so far as they are governed by the geometry of the boundary. Odd zeros of $h'(u-i(v-2v_0))$ and odd poles of $h'(u+iv)$ are stagnation points and, at the same time, branch points where the surface current density vanishes. The magnetic field at the boundary yields stagnation points at the zeros of $\overset{\circ}{B}_u(u-i(v-v_0))$. The zeros of individual factors may also be cancelled when the product is taken. The paths of the separatrices are often of interest. These are given by the magnetic surfaces connecting stagnation points.

A few examples are given as illustrations. Gajewski [2] investigated the magnetic field for elliptic plasma cylinders a) with surface currents and b) with constant volume currents in the z direction. Using infinite series he obtained the result that with the same boundary in both cases the stagnation points are the same despite the different current distributions, which at first glance is surprising. This situation can readily be explained by formula (4), which contains a factor determined solely by the geometry of the surface. The elliptic coordinates u, v $(0 \leq u \leq 2\pi \; ; \; 0 < v < \infty)$ that have to be used here are obtained with $h(w) = \ell \sin(w)$ where 2ℓ denotes the distance between the foci. The field at the boundary in case a) is $B_u^o = b = \text{const}$ and in the case b) $B_u^o = c \cdot g(u, v_o)$; $c = \text{const}$. The zeros P_1, P_2 of $h'(u - i(v - 2v_o))$ that are solely governed by the geometry of the plasma surface are located at $v = 2v_o$; $u = \frac{\pi}{2}, \frac{3\pi}{2}$ symmetric to the origin on the extension of the major axis of the ellipse $v = v_o$. In case a) the stagnation points P_1, P_2 are at the same time branch points whereas in case b) they are pure stagnation points since $B_u^o(u - i(v-v_o))$ also contains the factor $\left[h'(u-i(v-2v_o))\right]^{1/2}$.

The fact that stagnation points lie on the curve $v = 2v_o$ is not a matter of chance either. Quite generally if the function h, which is given by the shape of the plasma boundary, is real for real arguments, apart from a possibly complex constant factor, then zeros of the magnetic field on the curve $v = 2v_o$ are given by $h'(u) = 0$. Because $h(u)$ is real and periodic $h'(u)$ must always have zeros.

The presence of a pure stagnation point in case b) is due to the coincidence of zeros both of the geometric and magnetic factors. An example of pure stagnation points due solely to the geometry with the property that $h'(u-i(v-2v_o))$ has quadratic zeros for $v = 2v_o$ and $u = 0, \pi$ is given by the function

$$h(w) = e^{iw} - \frac{1}{3} e^{-iw} - e^{i3w} + \frac{1}{3} e^{i5w}$$

For $v \geqslant \frac{1}{2} \ln 5$ this function yields singly closed curves $v = v_o$. The curve $v = \frac{1}{2} \ln 5$ is shown in Fig. 1. Much simpler maps do not seem to lead to magnetic fields with such stagnation points. Other examples can easily be constructed for surface currents using Merkel's and Gorenflo's [1] method.

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References

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R. Gorenflo, Funktionentheoretische Bestimmung des Außenfeldes zu einer zweidimensionalen magnetohydrostatischen Konfiguration, ZAMP 16, 279 (1965).

2) R. Gajewski, MHD Equilibrium of an Elliptical Plasma Cylinder, MIT PRR-713 (1971).

Fig. 1

The curve $v = const = \frac{1}{2}\ln 5$