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Macroscopic Equilibria of Relativistic Electron  
Beams in Plasmas

G. Küppers, A. Salat, H.K. Wimmel

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### Abstract

Macroscopic equilibria of relativistic electron beams in plasmas, with return current and with or without an external magnetic field parallel to the beam, are investigated in cylindrical geometry. BENNET-type identities and general solutions are derived, and special examples are considered, in particular low-net-current, quasineutral configurations. A general expression for the current screening factor is obtained. Because of screening relaxed versions of the ALFVÉN-LAWSON conditions determine whether beam propagation is possible. These conditions are easily satisfied for low perpendicular temperatures and sufficiently diffuse density profiles of the beam; they do not limit the beam current. Only equilibria with zero total charge, nonzero total net current, and beam profiles without sharp boundary, can be obtained from the equations and boundary conditions used.

## 1. Introduction

Properties of straight, relativistic electron beam equilibria partially neutralized by ions, but not surrounded by plasma, have been extensively studied by many authors (among others BENNETT, 1934, 1955, 1958; LINHART, 1960; BENFORD et al., 1971; YOSHIKAWA, 1971). As far as relativistic electron beams in plasmas are concerned, theories of charge neutralization and return currents exist (COX et al., 1970; LEE et al., 1971; HAMMER et al., 1970; RUKHADZE et al., 1972; KÜPPERS et al., 1973); but, to our knowledge, equilibria with return current have not been studied so far, except in planar geometry by IVANOV et al. (1970).

Information about beam plasma equilibria may be of interest for diagnostics and for studies of stability and heating. As the most simple case, we consider relativistic electron beams of cylindrical symmetry in infinite, nonrelativistic plasmas. We take into account the plasma return current, but omit questions of stability and heating. An external magnetic field  $\underline{B}_0$  parallel to the beam may be present; it does not influence the beam plasma equilibria considered. Only macroscopic equilibria are investigated, by using collisionless, isothermal fluid equations for the beam, the plasma electrons, and the plasma ions. From calculations without return currents it is well known that a large class of beam equilibria is described equally well by macroscopic and microscopic theory.

In view of the difference in the mass of the ions and electrons we investigate two different classes of equilibria, viz.

- A) two-component equilibria, i.e. equilibrium configurations of plasma electrons and beam electrons only, with a fixed, homogeneous plasma ion background;
- B) three-component equilibria including the plasma ions as well.

It turns out that the systems of equations permit two functions to be chosen arbitrarily. From the physical point of view one would like to specify the density and velocity profiles of the beam, viz.  $n'(r)$  and  $v'(r)$ , with  $v' \approx \text{const}$ . Mathematically, it is advantageous to formulate the general equilibrium solutions in terms of given beam and ion density profiles (in the case of three components) or in terms of given beam density and radial electric field (in the case of two components). Among these general solutions we find special equilibria that satisfy the experimental requirement  $v'(r) \approx \text{const}$ , either as special cases of the general solutions or, alternatively, by using the approximation of low-net-current, quasineutral equilibrium.

Section 2 concerns general properties of the equilibria studied; in particular, BENNETT-like relations are obtained from the virial theorem and from asymptotics at large radii. Equilibria without and with electric field are considered in Sections 3 and 4 respectively. In section 5 low-net-current equilibria are obtained by assuming electric and magnetic

quasineutrality; this approximation may possibly become invalid in some cases at the beam edge. In Section 6 the results are summarized, and modified ALFVÉN-LAWSON conditions for beam propagation are derived.

## 2. General Properties of Equilibria

The equilibria studied have currents, velocities and the external B-field  $B_0$  parallel to the z-axis, they have radial gradients and E-fields, and azimuthal, selfconsistent, B-fields. We shall omit the pertinent indices, i.e. write B for  $B_\phi$ , E for  $E_r$ , etc.

Cylindrical coordinates  $r, \phi, z$  are used.

The following equilibrium equations are used:

$$-E + \frac{v'}{c} B = \frac{T'}{e} \partial_r \ln n' , \quad (2.1)$$

$$-E + \frac{v_e}{c} B = \frac{T_e}{e} \partial_r \ln n_e , \quad (2.2)$$

$$E = \frac{T_i}{e} \partial_r \ln n_i , \quad (2.3)$$

$$\frac{1}{r} \partial_r (r B) = \frac{4\pi}{c} j , \quad (2.4)$$

$$\frac{1}{r} \partial_r (r E) = 4\pi \rho , \quad (2.5)$$

with

$$j = -e(n_e v_e + n' v') , \quad (2.6)$$

$$\rho = e(n_i - n_e - n') . \quad (2.7)$$

The total kinetic pressure is

$$p = T_i n_i + T_e n_e + T' n' . \quad (2.8)$$

The external  $\underline{B}$ -field  $\underline{B}_0$  parallel to the beam does not enter the equations. The beam quantities are primed, otherwise the notation is standard. While  $\mathcal{V}'$  and  $v_z$  are functions of the radius, the temperatures are taken as constants. Equations (2.1) to (2.8) describe three-component equilibria. For two-component equilibria eq. (2.3) must be dropped and  $n_z$  put equal to a constant. One notices that the time-independent relativistic equation of motion that describes the beam equilibrium, eq. (2.1), does not contain relativistic factors and is, in fact, identical to the nonrelativistic equation. All quantities are measured in the laboratory system. The temperatures entering the equations are kinetic temperatures that refer to components of the pressure tensors perpendicular to the z-axis. Maxwellian distribution functions have not been assumed, merely isotropy of "perpendicular" pressures. The assumption  $T' = \text{const}$  in the laboratory system is not equivalent to  $T'_0 = \text{const}$  in the beam rest system if  $\mathcal{V}'(r) \neq \text{const}$ . It appears, however, that there is no special reason for assuming  $T'_0 = \text{const}$ ; hence we prefer the assumption  $T' = \text{const}$ , which is mathematically much simpler.

The equations obey the following rules of similarity:

$$\left. \begin{aligned}
 v', v_e &\sim 1 \\
 \Delta N_i, \Delta N_e, T, Q, I &\sim N' \\
 p', p_i, p_e, P &\sim N'^2/R^2 \\
 E, B &\sim N'/R \\
 n, g, j &\sim N'/R^2
 \end{aligned} \right\} \quad (2.9)$$

Here  $N', Q, \Delta N_i, \Delta N_e$  are the line densities of beam particles, electric charge, and variation of plasma particles with respect to the unperturbed, homogeneous background density, e.g.

$$\Delta N_{i,e} = 2\pi \int_0^\infty (n_{i,e} - n_{e\infty}) r dr ; \quad (2.10)$$

$I$  is the total net current, and  $R$  is the "beam radius".

We postulate the following properties of our equilibria:

- a)  $N', \Delta N_i, \Delta N_e, Q, I$  finite. It follows that  $n', (n_{i,e} - n_{e\infty}), g, j, v_e$  all go to zero for  $r \rightarrow \infty$ .
- b) All fields, densities, velocities etc. finite.
- c)  $|v'| \approx c, |v_e| \ll c$ .
- d)  $n' \ll n_{e\infty}$ .
- e) All temperatures nonzero.



The following additional properties then follow:

a') The beam density  $n'(r)$  cannot drop to zero at finite radius  $r$ .

This is seen in the following way. When  $n'$  goes to zero at finite

$r = r_0$  then the logarithmic derivative  $\partial_r \ln n'$  must

diverge at  $r = r_0$ . This means that one of the quantities

$E, B, v'$  of the left-hand side of eq. (2.1) must also diverge,

in contradiction to assumption b).

b') The line density  $Q$  of electric charge is zero. Otherwise the ion density could not go to a nonvanishing constant, since [eq. (2.3)]

$$n_i = n_{i0} \exp \left[ -\frac{e}{T_i} (\phi - \phi_0) \right], \quad (2.11)$$

and  $\phi \sim \ln r$  for large radii. From  $Q = 0$  it immediately

follows that

$$\Delta N_i = \Delta N_e + N'. \quad (2.12)$$

Because  $Q$  and  $E$  are related by

$$Q = \frac{1}{2} \lim_{r \rightarrow \infty} (rE), \quad (2.13)$$

the result  $Q = 0$  also means that  $E(r)$  drops to zero faster than

$r^{-1}$  for large radii. This result holds for two-component equilibria,

too, because it can also be derived from eqs. (2.1), (2.2), (2.5),

and  $n_i = \text{const.}$

c') The total net current  $I$  is nonzero. Otherwise the line density of beam particles,  $N'$ , would diverge. This follows from considering

eq. (2.1) at large radii and observing  $Q = 0$ . Because

$$I = \frac{c}{2} \lim_{r \rightarrow \infty} (rB) \quad (2.14)$$

it follows that  $B \sim r^{-1}$  for large  $r$ . From this consideration one also proves that

$$v'_\infty =: \lim_{r \rightarrow \infty} v'(r) \neq 0. \quad (2.15)$$

Of course, these properties hinge on the special form of the equations used and will not necessarily follow from a microscopic theory.

(11.5)

Now various exact relations will be derived that are satisfied by the equilibria. These relations contain the total net current  $I$ , the total beam current  $I'$ , the line densities  $N'$ ,  $\Delta N_i$ ,  $\Delta N_e$ , the temperatures, and the beam velocity  $v'_\infty = c \beta'_\infty$  at  $r = \infty$ .

They are generalizations of relations derived by BENNETT (1934) for a beam in vacuo. Some of the relations hold only for two-component or three-component equilibria. When nothing is said a relation holds for both cases. For some of the relations the derivation makes use of an asymptotic profile of the beam density, viz.

$$n'(r) \sim r^{-\alpha'} \quad \text{for } r \rightarrow \infty, \quad (2.15)$$

where  $\alpha' > 2$  to guarantee finite  $N'$ .

The asymptotics of the beam equilibrium [eq. (2.1)] for large  $r$  together with  $Q = 0$  and eq. (2.15) yields an expression for the net current:

$$I = - \frac{\alpha' c T'}{2 e \beta'_{\infty}} . \quad (2.16)$$

If the beam velocity  $v' = c \beta'$  is a constant, then the beam current is given by

$$I' = - N' e v' \quad (2.17)$$

and the shielding factor, i.e. the ratio of the beam current and the net current can be written down exactly:

$$\frac{I'}{I} = \frac{2 N' e^2 \beta'^2}{\alpha' T'} . \quad (2.18)$$

For three-component equilibria the following additional relations can be derived. Total pressure balance, viz.

$$\partial_r p = \frac{1}{8\pi r^2} \partial_r (r^2 E^2 - r^2 B^2) , \quad (2.19)$$

yields on account of eqs. (2.13) and (2.14) the virial theorem in the form

$$I^2 = 2 c^2 \Delta P \quad (2.20)$$

with

$$\Delta P = 2\pi \int_0^{\infty} (p - p_{\infty}) r dr . \quad (2.21)$$

Because of eq. (2.12) and

$$\Delta P = T_i \Delta N_i + T_e \Delta N_e + T' N' \quad (2.22)$$

[compare eq. (2.8)] , eq. (2.20) can be written

$$I^2 = 2c^2 (T' - T_e) N' + 2c^2 (T_i + T_e) \Delta N_i \quad (2.23)$$

or

$$I^2 = 2c^2 (T' + T_i) N' + 2c^2 (T_i + T_e) \Delta N_e \quad (2.24)$$

Eliminating the net current by means of eq. (2.16) yields

$$\Delta N_i = \frac{\alpha^{1/2} T^{1/2}}{8e^2 \beta_\infty^{1/2} (T_i + T_e)} - \frac{T' - T_e}{T_i + T_e} N' \quad (2.25)$$

and

$$\Delta N_e = \frac{\alpha^{1/2} T^{1/2}}{8e^2 \beta_\infty^{1/2} (T_i + T_e)} - \frac{T' + T_i}{T_i + T_e} N' \quad (2.26)$$

The formulas simplify for electrically neutral equilibria, because

for them  $E = \Delta N_i = 0$ .

For two-component equilibria the E-field must be considered partially

due to "external sources" (i.e. the ions). Then one has for  $p = T'n' + T_e n_e$  :

$$\partial_r p + n_{e\infty} e E = \frac{1}{8\pi r^2} \partial_r (r^2 E^2 - r^2 B^2), \quad (2.27)$$

and the virial theorem reads

$$I^2 = 2c^2 (\Delta P - \Psi), \quad (2.28)$$

with

$$\Psi = \pi e n_{e\infty} \int_0^{\infty} E(r) r^2 dr = -2\pi^2 e n_{e\infty} \int_0^{\infty} \rho(r) r^3 dr, \quad (2.29)$$

where the integrals are required to converge, and

$$\Delta P = T' N' + T_e \Delta N_e = (T' - T_e) N'. \quad (2.30)$$

The homogeneous ion background yields, of course,  $\Delta N_i = 0$ ,

$\Delta N_e = -N'$ . Eliminating the net current again by means of

eq. (2.16) one obtains

$$\Psi = (T' - T_e) N' - \frac{\alpha'^2 T'^2}{8 e^2 \beta_{\infty}'^2}. \quad (2.31)$$

For electrically neutral equilibria  $\Psi = 0$ , and the formulas agree with the three-component case.

In the case of constant beam velocity  $v'$  eqs. (2.16), (2.17), (2.25)

or (2.31) can be combined to give alternative formulae for the

screening factor: For three-component equilibria one has

$$\frac{I'}{I} = \frac{\alpha' T'}{4 (T' - T_e)} - \frac{2 e^2 \beta'^2 \Delta N_i}{\alpha' T'} \cdot \frac{T_i + T_e}{T' - T_e}, \quad (2.32)$$

and for two-component equilibria:

$$\frac{I'}{I} = \frac{\alpha' T'}{4 (T' - T_e)} + \frac{2 e^2 \beta'^2 \Psi}{\alpha' T' (T' - T_e)}. \quad (2.33)$$

If, in addition, the equilibrium is neutral ( $\Delta N_i = 0$ ,  $\Psi = 0$ , respectively) then the screening factor is

$$\frac{I'}{I} = \frac{\alpha' T'}{4(T' - T_e)} \quad (2.34)$$

The above relations almost all suffer from the difficulty that the asymptotic exponent  $\alpha'$  of the beam density profile is not known a priori. An exception is the case of neutral equilibrium ( $\underline{\Xi} = 0$ ) with  $v' = \text{const}$ , where it can be shown that  $\alpha' = 4$  (Section 3).

Nevertheless the relations may be of some use in discussions (Section 6) and for checking special solutions.

### 3. Equilibria with Vanishing E-Field

Electrically neutral equilibria,  $E = 0$ ,  $g = 0$ , with  $\Delta N_i = 0$ ,  $n_i = \text{const}$ , are treated first since this case is especially simple, and the distinction between two-component equilibria and three-component equilibria no longer obtains. Equations (2.1) to (2.7) then immediately yield

$$n' + n_e = n_i = n_{e\infty} \quad (3.1)$$

and

$$\frac{I'}{I} = \frac{j'}{j} = \frac{n'v'}{n'v' + n_e v_e} = \frac{T'}{T' - T_e}, \quad (3.2)$$

i.e. the ratios of the current densities  $j', j_e, j$  are all constants.

The general solution for equilibria with  $E = 0$  can easily be obtained in terms of a given beam density profile:

$$B^2 = - \frac{8\pi(T' - T_e)}{r^2} \int_0^r s^2 \partial_s n'(s) ds, \quad (3.3)$$

$$n_e = n_{e\infty} - n', \quad (3.4)$$

$$v' = \frac{cT'}{eB} \partial_r \ln n', \quad (3.5)$$

$$v_e = - \frac{c T_e}{e B} \partial_r \ln n_e . \quad (3.6)$$

In keeping with the conditions of Section 2 singular B-fields are excluded from consideration. In order for B to be real-valued the condition

$$\int_0^r s^2 \partial_s n'(s) ds \leq 0 \quad (3.7)$$

or

$$N'(r) \geq \pi r^2 n'(r) \quad (3.8)$$

must be satisfied, where  $N'(r)$  is the line density of beam particles inside a cylinder of radius  $r$ .

The solution given can yield unphysical values (profiles) of  $v'$  and/or  $v_e$ , however, if  $n'(r)$  is chosen inappropriately. (We should characterize profiles  $v'(r)$  as unphysical if, for instance,  $v' > c$  or  $v'$  were to change sign.) This difficulty is avoided by specializing to  $v'(r) = \text{const}$ . Then, comparing eqs. (3.2) and (2.34) shows that when  $v'$  is a constant the exponent of the beam density profile satisfies  $\alpha' = 4$ ; hence eqs. (2.16) to (2.26) are readily evaluated. Further consequences then are, for instance, that  $T' > T_e$  and

$$N' = \frac{2 T'^2}{e^2 \beta'^2 (T' - T_e)} . \quad (3.9)$$



On comparing eqs. (3.3) and (3.5) the following differential equation for  $n'(r)$  is derived:

$$n'_{rr} + \frac{1}{r} n'_r - \frac{n'^2_r}{n'} + K n'^2 = 0 \quad (3.10)$$

with

$$K = 4\pi(T' - T_e) \left( \frac{e\beta'}{T'} \right)^2. \quad (3.11)$$

According to KAMKE (1944) the solution satisfying the boundary conditions  $n'_r(0) = 0$ ,  $n'(\infty) = 0$  is of the BENNETT type, viz.

$$n' = \frac{N' d^2}{\pi(r^2 + d^2)^2}, \quad (3.12)$$

with  $N'$  given by eq. (3.9), and  $d$  arbitrary, characterizing the radial extent of the beam. The other quantities then are

$$n_e = n_{e\infty} - \frac{N' d^2}{\pi(r^2 + d^2)^2}, \quad (3.13)$$

$$B = \frac{T'}{e\beta'} \partial_r \ln n' = - \frac{4T'}{e\beta'} \cdot \frac{r}{r^2 + d^2}, \quad (3.14)$$

$$v_e = - \frac{T_e n' v'}{T' n_e} \approx - \frac{T_e n' v'}{T' n_{e\infty}}. \quad (3.15)$$

The condition  $|v_e| \ll c$  is satisfied automatically since

$$n' \ll n_{e\infty}, \quad T' > T_e.$$

#### 4. General Equilibria

From eqs. (2.1) to (2.7) the following general equilibrium solutions subject to conditions a) to e) of Section 2 can be derived.

##### a) Three-component equilibria

The general solution for given  $n'(r)$  and  $n_i(r)$  reads:

$$B^2 = E^2 + \frac{2T_e}{e} \left( \partial_r E - \frac{E}{r} \right) - \frac{8\pi(T' - T_e)}{r^2} \int_0^r s^2 \partial_s n'(s) ds - \frac{8\pi(T_i + T_e)}{r^2} \int_0^r s^2 \partial_s n_i(s) ds, \quad (4.1)$$

$$E = \frac{T_i}{e} \partial_r \ln n_i, \quad (4.2)$$

$$n_e = n_i - n' - \frac{\partial_r(rE)}{4\pi e r}, \quad (4.3)$$

$$v' = \frac{c}{B} \left( \frac{T'}{e} \partial_r \ln n' + E \right), \quad (4.4)$$

$$v_e = \frac{c}{B} \left( \frac{T_e}{e} \partial_r \ln n_e + E \right). \quad (4.5)$$

The functions  $n'$  and  $n_i$  must be chosen such that  $B^2 \geq 0$ ,

$v' < c$ ,  $n_e > 0$ , and  $v_e \ll c$ .

Special solutions with  $v' \approx \text{const}$  and  $E$  small are obtained with the ansatz:

$$n' = \frac{N' d^2}{\pi (d^2 + r^2)^2} \quad , \quad (4.6)$$

$$n_i = n_{e\infty} + \frac{d^2 \Delta N_i}{\pi (d^2 + r^2)^2} \quad , \quad (4.7)$$

with  $\Delta N_i$  given by eq. (2.25). For  $E$  small, viz.

$$|\Delta N_i| \ll \pi d^2 n_{e\infty} \quad (4.8)$$

and

$$|\Delta N_i| \ll \pi d^2 n_{e\infty} \frac{T_e}{T_i} \quad (4.9)$$

one obtains

$$E \approx - \frac{4 T_i \Delta N_i}{\pi e n_{e\infty}} \cdot \frac{d^2 r}{(d^2 + r^2)^3} \quad (4.10)$$

and

$$B^2 \approx \frac{16 T^{1/2}}{e^2 \beta^{1/2}} \cdot \frac{r^2}{(d^2 + r^2)^2} + \frac{48 T_i T_e \Delta N_i}{\pi e^2 n_{e\infty}} \cdot \frac{d^2 r^2}{(d^2 + r^2)^4} \quad . \quad (4.11)$$

In deriving eq. (4.11) the term  $E^2$  has been neglected in eq. (4.1) by virtue of eq. (4.9). The second term on the r.h.s. of eq. (4.11) will also now be neglected, assuming

$$|\Delta N_i| \ll \pi d^2 n_{e\infty} \frac{T'^2}{3 T_i T_e} . \quad (4.12)$$

In eq. (4.4) for  $v'$  the  $E/B$  term can be neglected if Debye lengths are small compared to  $d$  and if

$$|\Delta N_i| \ll d^2 n_{e\infty} \frac{T'}{T_i} . \quad (4.13)$$

Then

$$\beta' \approx -\sqrt{2} \frac{T'}{e} [N'(T' - T_e) + \Delta N_i (T_i + T_e)]^{-\frac{1}{2}} , \quad (4.14)$$

which agrees with the exact relation (2.25).

For the plasma electrons one derives

$$n_e \approx n_{e\infty} + (\Delta N_i - N') \frac{d^2}{\pi(d^2 + r^2)^2} \quad (4.15)$$

and

$$\frac{v_e}{c} \approx -\frac{N'\beta'}{n_{e\infty}} \left(1 - \frac{2T'}{N'e^2\beta'^2}\right) \frac{d^2}{\pi(d^2 + r^2)^2} . \quad (4.16)$$

The screening factor is nearly constant and given by

$$\frac{j'}{j} \approx \frac{N'T'}{N'(T' - T_e) + \Delta N_i (T_i + T_e)} = \frac{N'e^2\beta'^2}{2T'} . \quad (4.17)$$

b) Two-component equilibria

The general solution for given  $n'(r)$  and  $E(r)$  is given by

$$B^2 = E^2 + \frac{2T_e}{e} \left( \partial_r E - \frac{E}{r} \right) - \frac{8\pi(T' - T_e)}{r^2} \int_0^r s^2 \partial_s n' ds - \frac{8\pi e n_{e\infty}}{r^2} \int_0^r s^2 E(s) ds, \quad (4.18)$$

$$n_i = n_{e\infty}, \quad (4.19)$$

and by eqs. (4.3) to (4.5). The same restrictions must be observed as above.

Again special solutions with  $v' \approx \text{const}$  and  $E$  small are obtained, this time using eq. (4.6) for  $n'(r)$  and

$$E = \frac{E_0 r}{(d^2 + r^2)^3}, \quad E_0 = \frac{4 d^2 \psi}{\pi e n_{e\infty}}. \quad (4.20)$$

For small  $E_0$  [compare eqs. (4.27) to (4.29)] the  $\underline{B}$ -field is given by

$$B^2 \approx \left[ 8N'(T' - T_e) - \frac{2\pi}{d^2} e n_{e\infty} E_0 \right] \frac{r^2}{(d^2 + r^2)^2} \quad (4.21)$$

and  $v'$  by

$$\beta' \approx - \frac{4T'd}{e} \left[ 8N'd^2(T' - T_e) - 2\pi e n_{e\infty} E_0 \right]^{-\frac{1}{2}}. \quad (4.22)$$

The latter equation agrees with eqs. (2.29) and (2.31). Equation (4.22) can be used to write

$$B \approx - \frac{4T'}{e\beta'} \frac{r}{d^2 + r^2} , \quad (4.23)$$

where eq. (2.16) and sign  $B = \text{sign } I$  have been observed. The electron quantities are given by

$$n_e \approx n_{e\infty} - \frac{N'd^2}{\pi(d^2 + r^2)^2} , \quad (4.24)$$

$$\frac{v_e}{c} \approx - \frac{e\beta'}{4T'} \left[ \frac{4N'd^2T_e}{\pi en_{e\infty}} + E_0 \right] \frac{1}{(d^2 + r^2)^2} , \quad (4.25)$$

where  $n_e \approx n_{e\infty}$  was used. Again the screening factor is approximately constant:

$$\frac{j'}{j} \approx \frac{4N'd^2T'}{4N'd^2(T' - T_e) - \pi en_{e\infty} E_0} = \frac{N'e^2\beta'^2}{2T'} . \quad (4.26)$$

The conditions for eqs. (4.21) to (4.26) to be valid are:  $n'$  small compared to  $n_{e\infty}$ , and

$$|E_0| \ll \frac{12T_e}{e} d^4 , \quad (4.27)$$

$$|E_0| \ll \frac{4T'^2}{3eT_e\beta'^2} d^4 , \quad (4.28)$$

$$|E_0| \ll \frac{4T'}{e} d^4 . \quad (4.29)$$

### 5. Low Net Current Equilibria

We consider equilibria with near compensation of the beam current by the plasma current and with nearly vanishing charge density, i.e. low-net-current, quasineutral equilibria. Equations (2.4) to (2.7) are replaced by

$$j = -e(n_e v_e + n' v') \approx 0, \quad (5.1)$$

$$\rho = e(n_i - n_e - n') \approx 0, \quad (5.2)$$

while  $E$  and  $B$  are kept nonzero. The equilibria are then given, essentially, by algebraic formulae.

#### a) Three-component equilibria

If  $n'(r)$  and  $v'(r) \approx \text{const}$  are given, the other quantities are given by the following expressions:

$$n_i \approx n_{e0} - \frac{T' - T_e}{T_i + T_e} n', \quad (5.3)$$

$$n_e \approx n_{e0} - \frac{T' + T_i}{T_i + T_e} n', \quad (5.4)$$

$$E = \frac{T_i}{e} \partial_r \ln n_i, \quad (5.5)$$

$$B \approx \frac{cT'}{ev'} \left[ 1 - \frac{(T' - T_e)(T' + T_i)n'}{(T_i + T_e)T' n_{e\infty}} \right] \cdot \left[ 1 - \frac{(T' - T_e)n'}{(T_i + T_e)n_{e\infty}} \right]^{-1} \cdot \partial_r \ln n' \quad (5.6)$$

$$v_e \approx - \frac{n' v'}{n_e} . \quad (5.7)$$

As to the general relations of Section 2, eqs. (2.16) to (2.18) remain

valid,  $\Delta P$  and  $I^2$  become small and hence, for  $T'$  not nearly equal

to  $T_e$

$$\Delta N_i \approx - \frac{T' - T_e}{T_i + T_e} N' , \quad (5.8)$$

$$\Delta N_e \approx - \frac{T' + T_i}{T_i + T_e} N' , \quad (5.9)$$

$$\frac{I'}{I} \approx - \frac{2e^2 \beta'^2 \Delta N_i}{\alpha' T'} \cdot \frac{T_i + T_e}{T' - T_e} . \quad (5.10)$$

It follows that  $E \neq 0$  for  $\partial_r n' \neq 0$ .

The approximation will be valid if the net current and the charge density are sufficiently low and if the Debye lengths are small compared to the radial length scale. The approximation may possibly break down at the beam edge, i.e. for small  $n'(r)$ , but this is not necessarily so. In fact, when eqs. (5.8) to (5.10) are satisfied and the square brackets in eq. (5.6) are nearly equal to



each other ( $T_i \ll T'$ ), the special solutions given by eqs. (4.6) to (4.17) are of the low-current, quasineutral type, i.e. satisfy eqs. (5.1) to (5.7) for arbitrary radii.

#### b) Two-component equilibria

For given  $n'(r)$  and  $v' \approx \text{const}$  the low-net-current, quasineutral solution is found to be

$$n_i = n_{e\infty} , \quad (5.11)$$

$$n_e \approx n_{e\infty} - n' \quad (5.12)$$

$$E \approx - \frac{T' - T_e}{en_{e\infty}} \partial_r n' , \quad (5.13)$$

$$B \approx \frac{cT'}{ev'} \left[ 1 + \frac{(T' - T_e)n'}{T'n_{e\infty}} \right] \partial_r \ln n' , \quad (5.14)$$

$$v_e \approx - \frac{n'v'}{n_e} . \quad (5.15)$$

As to the general relations of Section 2, eqs. (2.16) to (2.18) remain valid again, and for  $T'$  again not nearly equal to  $T_e$  the other relations become:

$$\Delta N_e = - N' , \quad (5.16)$$

$$\psi \approx (T' - T_e) N' = \Delta P , \quad (5.17)$$

$$\frac{I'}{I} \approx \frac{2 e^2 \beta'^2 \psi}{\alpha' T' (T' - T_e)} . \quad (5.18)$$

Again it follows that  $E \neq 0$  for  $\partial_r n' \neq 0$ . When eqs. (5.16) to (5.18) are satisfied and the square bracket in eq. (5.14) is nearly equal to 1, then the special solutions given by eqs. (4.6), (4.20) to (4.26) are of the low-net-current, quasineutral type and satisfy eqs. (5.11) to (5.15).

## 6. Summary and Discussion

The isothermal equations used describe a restricted class of beam plasma equilibria of cylindrical symmetry with return current, viz. equilibria with total charge  $Q = 0$ , net current  $I \neq 0$ , and a diffuse density profile of the beam. An external B-field  $B_0$  parallel to the beam may be present; it does not influence the equilibria.

In order to see how critical the assumption of isothermality is, an adiabatic set of equations was also investigated, viz. with  $p \sim n^2$ ,  $T \sim n$ .

Then for  $E \equiv 0$  spatially oscillating solutions were obtained, with the oscillation length of the order of the beam's Debye length.

Hence the choice of different equations of state may have a strong influence on the type of equilibria derived. Whether such oscillating solutions should be taken seriously remains an open question.

In Sections 3 and 4 we have derived exact and approximate equilibrium solutions for given density profile of the beam. In the case  $E \equiv 0$ ,  $v' = \text{const}$ , a BENNETT-like solution with return current obtains. When  $E \neq 0$ , the condition  $v' = \text{const}$  can be taken into account by employing the low-net-current, quasineutral approximation (Section 5). Special solutions with  $E \neq 0$ ,  $v' \approx \text{const}$  were also found by choosing appropriate profiles for  $n'$  and  $n_i$  or  $n'$  and  $E$ .

Generalized BENNETT relations have been derived in Section 2. They

relate particle line densities, currents, temperatures, the beam velocity, and the asymptotic exponent of the density profile at large radii. The screening factor for the beam current is always  $I'/I = (2 N' e^2 \beta'^2 / \alpha' T')$ . In order for it to be large, the beam's perpendicular temperature ought to be small ("parallel motion of beam particles") and the beam density exponent  $\alpha'$  ought to be small ("diffuse beam").

On account of screening the ALFVÉN (1939) criterion for propagation of an unscreened beam,

$$\frac{N' e^2}{m c^2} < \gamma' , \quad (6.1)$$

is modified to become

$$\frac{N' e^2}{m c^2} < \gamma' \frac{I'}{I} = \frac{2 N' e^2 \beta'^2 \gamma'}{\alpha' T'} . \quad (6.2)$$

This is no longer a restriction of  $N'$  and the beam current  $I'$ , but rather for  $\alpha' T' / \gamma'$ , viz.

$$\frac{\alpha' T'}{2 m c^2 \gamma'} < \beta'^2 \approx 1 . \quad (6.3)$$

If a divergence angle  $\theta$  of beam particle velocities is introduced:

$$T' \approx m c^2 \gamma' \langle \theta^2 \rangle , \quad (6.4)$$

then eq. (6.3) becomes

$$\frac{\alpha'}{2} \langle \theta^2 \rangle < 1 . \quad (6.5)$$

This is easily satisfied if the beam particles move nearly parallel and the beam is sufficiently diffuse. These two conditions are quite different

from the one condition resulting from the theory of return current (KÜPPERS et al., 1973) for the initial phase of beam injection.

There  $R_e' > R$  yields instead the condition  $n_e \gamma' / n' > 1$ , which is satisfied whenever the plasma density is larger than the beam density.

ALFVÉN'S condition and eqs. (6.2), (6.3), (6.5) originate from the requirement that the gyroradius of the beam electrons be greater than the beam radius, viz.  $R_e' > R$ . These conditions can be refined by taking into account the electric field also (LAWSON, 1957, 1958) and requiring that the radius of curvature  $R_c'$  of the orbits of the beam electrons, be greater than the beam radius  $R$ . When  $\underline{v}'$  is chosen perpendicular to  $\underline{E}$ ,  $R_c'$  is given by

$$\frac{v'^2}{R_c'} = \frac{e}{m \gamma'} \left| \frac{v'}{c} B - E \right|. \quad (6.6)$$

Then eqs. (6.3) and (6.5) are replaced by

$$\frac{\alpha' T'}{2 m c^2 \gamma'} < \frac{\beta'^3}{|\beta' - g|} \lesssim \frac{1}{|\beta' - g|} \quad (6.7)$$

and

$$\frac{\alpha'}{2} \langle \theta^2 \rangle < \frac{1}{|\beta' - g|}, \quad (6.8)$$

with  $g = E/B$ . Comparing eqs. (6.5) and (6.8) one observes that eq. (6.8) could be less restrictive than eq. (6.5) only for  $\frac{E}{B} \approx \text{const.}$

Because  $E/B \neq \text{const}$  for the equilibria derived above and since  $E/B$  is usually small, eq. (6.5) remains, in our case, the relevant condition for beam propagation.

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