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Current and Fields Induced in Plasmas by Relativistic
Electron Beams with Arbitrary Radial and Axial
Density Profiles

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Abstract

The return current and the electric and magnetic fields are investigated that result from injection of a relativistic electron beam of arbitrary radial and axial density profile into a plasma. It is found that, in general, the radial step function profile used in previous calculations overestimates the resulting B_φ -field by a factor $\tau\omega_p/c$. Radial density profiles with slow variation over one skin depth c/ω_p yield a B_φ field screening of order $(c/\tau\omega_p)^2$, where τ is the scale length of the radial inhomogeneity. The results apply to the first (inertial) stages of beam injection, where collisions can be ignored.

1. Introduction

Without special measures the propagation of high current relativistic electron beams is possible only as long as the current does not exceed the Alfvén - Lawson limit (ALFVEN, 1939; LAWSON, 1959), where trapping in the beam's own magnetic field becomes dominant. It is nevertheless possible to overcome this limit by special techniques such as hollowing out the beam, external guiding fields, or injection into a plasma where reverse currents lowering the effective azimuthal magnetic field are induced. Beam propagation in plasmas with or without external magnetic fields has been investigated theoretically by (COX and BENNET, 1970; HAMMER and ROSTOKER, 1970; LEE and SUDAN, 1971; RUKHADZE and RUKHLIN, 1972). These authors used a radial and - apart from COX and BENNET (1970) - an axial step function profile of the beam. Under this condition the net current and the magnetic field were found to exist in a small radial boundary layer of width c / ω_p only. The return current is generated by axial electric fields that originate locally from variations of the beam density in time and extend radially beyond the beam volume owing to the skin effect. For smooth profiles the skin effect will act wherever the radial density gradient is non-negligible and consequently, compared to a step function profile, the net current density and the distribution and amplitude of the magnetic field should be changed considerably. In this paper the effects of general radial and axial beam profiles will be

investigated. The existence of beam plasma equilibria with inclusion of the return current to which the present solutions may possibly develop will be investigated in another paper (KÜPPERS, SALAT and WIMMEL, to be published).

2. The beam-plasma model

We consider the interaction of a tenuous relativistic electron beam of cylindrical symmetry with a plasma in the first stages during and after injection of the beam into the plasma. It is assumed that during this interval the radial beam profile does not appreciably change, so that the beam density n' may be written in the laboratory frame as

$$n'(r, z, t) = n'(r) f(t - z/v') \quad (1)$$

where v' is the velocity of the beam moving in the z -direction.

The dimensionless function f can be normalized by, for example, putting $f = 1$ at the point where n' has its maximum value n'_{max} .

The beam velocity v' is assumed to be constant across the beam and constant in time. The functions $n'(r)$ and $f(z, t)$ are continuous and non-negative, with

$$n'(r \rightarrow \infty) = 0; \quad f(t - z/v' \leq 0) = 0; \quad (2)$$

they are determined by the experimental set-up. Results will be derived for general n' and f , while several specific examples will be discussed in detail.

The ions are assumed to be unaffected by the beam. The equation of motion for the plasma electron fluid is

$$n_e \left(\frac{\partial}{\partial t} + \underline{v}_e \cdot \nabla \right) \underline{v}_e = n_e \frac{e}{m} \left(\underline{E} + \frac{1}{c} [\underline{v}_e \times \underline{B}] \right) - \frac{\nabla p_e}{m_e} - \gamma n_e \underline{v}_e \quad (3)$$

m_e , n_e , $\underline{v}_e$ and p_e are the electron mass, number density, velocity and pressure respectively; $\theta \ll 0$, ν is an effective collision frequency, and $|\underline{v}_e| \ll c$. We are interested in the early stages of beam plasma interaction. Therefore, we neglect the nonlinear terms $\underline{v}_e \cdot \nabla \underline{v}_e$, $[\underline{v}_e \times \underline{B}]$ (see Appendix A for justification), the pressure term and the collisional term. The latter is justified for times that are small compared to $(\tau_b \omega_p / c)^2 / \nu$ (LEE and SUDAN, 1971; RUKHADZE and RUKHLIN, 1972), provided $\nu \ll \omega_p$ where $\omega_p = (4\pi n_e e^2 / m_e)^{1/2}$ is the plasma frequency of the background electrons, and τ_b is the radius of the beam. It will be assumed throughout that the beam density is much smaller than the plasma density,

$$n' \ll n_e \quad (4)$$

and n_e will be treated as a constant.

With these simplifications the equation for the time evolution of the plasma electric current $\underline{j}_e = e n_e \underline{v}_e$ reduces to

$$\frac{\partial}{\partial t} \underline{j}_e = \frac{\omega_p^2}{4\pi} \underline{E}. \quad (5)$$

Together with Maxwell's equations

$$\text{rot } \underline{B} = \frac{1}{c} \frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c} (\underline{j}' + \underline{j}_e) \quad (6)$$

$$\text{rot } \underline{E} = - \frac{1}{c} \frac{\partial \underline{B}}{\partial t} \quad (7)$$

the time evolution of $\underline{E}$, $\underline{B}$ and $\underline{j}_e$ is determined as a function of the beam current $\underline{j}' = e n' \underline{v}'$.

3. Induced fields and plasma current

If we introduce a cylindrical coordinate system (r, φ, z) the φ -component of equation (6) as well as the r - and z -components of equation (7) are identically satisfied with

$$B_z = B_r = E_\varphi = j'_\varphi = j_{e\varphi} = \frac{\partial}{\partial \varphi} \equiv \sigma \quad (8)$$

The remaining equations are

$$-\frac{\partial B_\varphi}{\partial z} = \frac{1}{c} \frac{\partial E_r}{\partial t} + \frac{4\pi}{c} j_{er} \quad (9)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_\varphi) = \frac{1}{c} \frac{\partial E_z}{\partial t} + \frac{4\pi}{c} (j'_z + j_{ez}) \quad (10)$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{1}{c} \frac{\partial B_\varphi}{\partial t} \quad (11)$$

If we take the time derivative of equation (9) and use equation (5) we obtain

$$-\frac{\partial^2 B_\varphi}{\partial z \partial t} = \frac{1}{c} \left(\omega_p^2 + \frac{\partial^2}{\partial t^2} \right) E_r \quad (12)$$

When equation (10) is treated in the same way and the operator

$\frac{1}{c} \left(\omega_p^2 + \frac{\partial^2}{\partial t^2} \right)$ is applied to equation (11) a single equation

for $\dot{B}_\varphi \equiv \frac{\partial B_\varphi}{\partial t}$ results:

$$\begin{aligned} \frac{1}{c^2} \left(\omega_p^2 + \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial z^2} \right) \dot{B}_\varphi - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \dot{B}_\varphi) \right) \\ + \frac{4\pi}{c} \frac{\partial^2}{\partial r \partial t} j'_z = \sigma \end{aligned} \quad (13)$$

The general solution may be given by means of a Green's function technique when boundary and initial conditions are specified.

Instead, we make the ansatz

$$B_{\varphi}(r, z, t) = b(r) g(t, z) \quad (14)$$

which should be reasonable in the neighbourhood of the beam since the source term there also factorizes into a radial and a z, t -dependence. The function $g(t, z)$ may be normalized similarly to $f(t - z/v')$. Equation (13) may then be written as follows

$$U(t, z) - S(t, z) T(r) + W(r) = 0 \quad (15)$$

where

$$U(t, z) = \frac{1}{c^2 f} \left(\omega_p^2 + \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial z^2} \right) \dot{g} \quad (16)$$

$$S(t, z) = \dot{g} / \dot{f} ; \quad T(r) = \frac{1}{b(r)} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (r b) \right) \quad (17)$$

$$W(r) = \frac{4\pi e v'}{c} \frac{dn'}{dr} \frac{1}{b(r)} \quad (18)$$

If we apply the operators $\frac{\partial^2}{\partial r \partial z}$ and $\frac{\partial^2}{\partial r \partial t}$ to equation (15) it follows that

$$\frac{\partial S}{\partial z} \cdot \frac{dT}{dr} = 0 ; \quad \frac{\partial S}{\partial t} \cdot \frac{dT}{dr} = 0. \quad (19)$$

For $T(r) = \text{const}$ $W(r)$ would also have to be constant, and hence $b(r)$ and $n'(r)$ would have to be Bessel functions, of order one and zero respectively. Since n' is in general, not a Bessel

function, we consider conversely

$$S(t, z) = c_1 = \text{const} \quad (20)$$

and hence

$$g(t, z) = c_1 f(t - z/v') + g_1(z) \quad (21)$$

If B_φ is to be zero far ahead of the beam for all t , it follows that $g_1(z) \equiv 0$. From equations (15) (16) (20) it follows that

$$\begin{aligned} U(t, z) &= \frac{c_1}{c^2} \frac{1}{f} \left(\omega_p^2 + \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial z^2} \right) \dot{f}(t - z/v') \\ &= \frac{c_1}{c^2} \left[\omega_p^2 + \left(1 - \frac{c^2}{v'^2} \right) \ddot{f} / \dot{f} \right] = c_2 = \text{const} \end{aligned} \quad (22)$$

Equation (22) shows that the product ansatz for B_φ is compatible with the basic equations (for the cases $\ddot{f}/\dot{f} = 0$ or $\ddot{f}/\dot{f} = \text{const}$ it is even exact) provided the time scale for changes of the beam current is longer than $(\omega_p \beta' \gamma')^{-1}$ where $\beta' = v'/c$, $\gamma' = (1 - \beta'^2)^{-1/2}$. Under this condition $c_2 = c_1 \omega_p^2 / c^2$ and

$$c_1 T(\tau) - W(\tau) = c_1 \omega_p^2 / c^2 \quad (23)$$

that is

$$\frac{d^2 b}{dx^2} + \frac{1}{x} \frac{db}{dx} + \left(1 + \frac{1}{x^2} \right) b = \frac{4\pi}{\omega_p} \frac{d\dot{j}_z}{dx} \quad (24)$$

where

$$x = \tau \omega_p / c \quad (25)$$

For the boundary conditions $B_\varphi(\tau \rightarrow \infty) = B_\varphi(\tau \rightarrow 0) = 0$ the solution is (KAMKE, 1944)

$$B_{\varphi}(x, t, z) = b(x) \tilde{f}(t - z/v')$$

$$b(x) = -\frac{4\pi e v'}{\omega_p} \left\{ K_1(x) \int_0^x dx' x' I_1(x') \frac{dn'}{dx'} + I_1(x) \int_x^{\infty} dx' x' K_1(x') \frac{dn'}{dx'} \right\} \quad (26)$$

where $I_1(x)$, $K_1(x)$ are modified Bessel functions of order one.

The electric field E_r can be determined from equation (12) and

$$(16) \quad \text{Green's function } G(t, t') = \omega_p^{-1} \sin[\omega_p(t-t')] H(t-t')$$

of the operator $(\omega_p^2 + \frac{\partial^2}{\partial t^2})$, where $H(t)$ is the Heaviside step function with $H(t < 0) = 0$, $H(t > 0) = 1$:

$$(18) \quad E_r(r, t - z/v') = \frac{b(r)}{\omega_p} H(t - z/v') \int_0^{t-z/v'} dt' \sin(\omega_p t') \ddot{f}(t - z/v' - t') \quad (27)$$

Similarly,

$$(19) \quad E_z(r, t - z/v') = \frac{c}{\omega_p} \left[\frac{1}{r} \frac{d}{dr} (r b) - \frac{4\pi e v' n'(r)}{c} \right] \cdot H(t - z/v') \int_0^{t-z/v'} dt' \sin(\omega_p t') \dot{f}(t - z/v' - t') \quad (28)$$

From Maxwell's equations

$$\dot{j}_{ez}(r, t - z/v') = -\dot{j}_z + \frac{c}{4\pi} \left[\frac{1}{r} \frac{\partial}{\partial r} (r B_{\varphi}) - \frac{1}{c} \dot{E}_z \right] \quad (29)$$

Equations (26) to (28) represent the general solution of eqs. (5) to (7)

that satisfies the ansatz of eqs. (14) and the boundary conditions mentioned.

While the magnetic field B_φ is proportional to the beam current at any time, equation (26), E_r and E_z depend on the time derivatives of j_z' only and, in general, have strong oscillatory behaviour superimposed which stems from induced plasma oscillations. A simple example with a beam current that increases linearly during a time of order $\tau \gg \omega_p^{-1}$ and then saturates might be illustrative. For

$$f(t) = 1 - e^{-t/\tau} \quad (30)$$

the fields in the two regimes are approximately

$$B_\varphi \approx b(r) \cdot \begin{cases} \tilde{t}/\tau & ; \quad \tilde{t} \ll \tau \\ 1 & ; \quad \tilde{t} \gg \tau \end{cases} \quad (31)$$

$$E_r \approx \frac{-1}{(\omega_p \tau)^2} b(r) \cdot \begin{cases} 1 - \cos(\omega_p \tilde{t}) & ; \quad \tilde{t} \ll \tau \\ -\cos(\omega_p \tilde{t}) & ; \quad \tilde{t} \gg \tau \end{cases} \quad (32)$$

$$E_z \approx \frac{c}{\omega_p^2 \tau} \left[\frac{1}{r} \frac{d}{dr} (r b) - \frac{4\pi e v' n'}{c} \right] \cdot \begin{cases} 1 - \cos(\omega_p \tilde{t}) & ; \quad \tilde{t} \ll \tau \\ -\cos(\omega_p \tilde{t}) & ; \quad \tilde{t} \gg \tau \end{cases} \quad (33)$$

where $\tilde{t} = t - z/v'$. Equations (29), (31) and (33) show that the displacement current in Maxwell's equations can be neglected after a few plasma periods.

4. The screening of the beam current

In order to investigate the effects of various radial beam density profiles on the amount of current compensation, the results for several such profiles will be compared with each other.

a) Step-like density profile

When the beam density $n'(r)$ has a constant value n'_1 up to a point $r = r_1 = x_1 c / \omega_p$ where it drops to zero (see Fig. 1a), one has

$\frac{dn'}{dx} = -n'_1 \delta(x - x_1)$. The radial profile of the magnetic field, equation (26), is then

$$b(x) = \frac{4\pi j'}{\omega_p} \begin{cases} x_1 K_1(x_1) I_1(x) ; & x \leq x_1 \\ x_1 I_1(x_1) K_1(x) ; & x \geq x_1 \end{cases} \quad (34)$$

where

$$j' = e n'_1 v' \quad (35)$$

is the current density of the beam at $r = 0$.

After a few plasma periods the displacement current can be neglected

and the net current density j_z , equation (29), is given by

$$\begin{aligned}
 j_z(x) &= j_{ez} + j'_z = \frac{\omega_p}{4\pi} \frac{1}{x} \frac{d}{dx}(xb) \\
 &= j'_z \cdot \begin{cases} x_1 K_1(x_1) I_0(x) ; & x < x_1 \\ x_1 I_1(x_1) K_0(x) ; & x > x_1 \end{cases} \quad (36)
 \end{aligned}$$

The total net current $J_z(x)$ is

$$J_z(x) = 2\pi \int_0^x d\tau' \tau' j_z(\tau') = \frac{c^2}{2\omega_p} x b(x) \quad (37)$$

For a thick beam, $x_1 \gg 1$, the asymptotic expansions of the Bessel functions give

$$\begin{aligned}
 b(x) &\approx \frac{2\pi j'_z}{\omega_p} \sqrt{\frac{x_1}{x}} e^{-|x-x_1|} ; \quad x \gg 1 \\
 j_z(x) &\approx j'_z \frac{1}{2} \sqrt{\frac{x_1}{x}} e^{-|x-x_1|} \text{sign}(x_1-x) ; \quad x \gg 1 \quad (38)
 \end{aligned}$$

Both the magnetic field and the net current density are confined to a small sheath of thickness c/ω_p at the edge of the beam; see Fig. 1a.

The field b and the total net current have their maximum value approximately at the boundary, where

$$b_{\max} \approx b(x_1) = \frac{2\pi j'_z}{\omega_p} \quad (39)$$

$$J_{z\max} \approx J_z(x_1) = \frac{\pi \tau_1 e n'_1 v' c}{\omega_p} = J'_z(x_1) \frac{c}{\tau_1 \omega_p} \quad (40)$$

The "screening factor", i.e. the ratio of the total beam current over the maximum of the net current,

$$S = \frac{j_z'(\tau \rightarrow \infty)}{j_{z \max}} \quad (41)$$

is found to be

$$S = \frac{\tau_1 \omega_p}{c} = x_1 \gg 1 \quad (42)$$

The equations (34) - (42) agree with the results of HAMMER and ROSTOKER (1970), LEE and SUDAN (1971), RUKHADZE and RUKHLIN (1972), who used a sharp boundary from the beginning.

b) Parabolic density profile

When the beam density drops to zero towards the edge as

$$n'(x) = n_1' \cdot \left(1 - \frac{x^2}{x_1^2}\right) \quad (43)$$

(see Fig. 1b), the integrals in equation (26) can be done analytically.

The result for $x \leq x_1$ with $j' = en_1'v'$ is

$$b(x) = \frac{4\pi j'}{\omega_p} \cdot \frac{2}{x_1^2} \left[x - x_1^2 K_2(x_1) I_1(x) \right] \quad (44)$$

which for $1 \ll x \leq x_1$ (thick beam!) reduces to

$$b(x) \approx \frac{4\pi j'}{\omega_p} \cdot \frac{2}{x_1^2} \left[x - \frac{x_1}{2} \sqrt{\frac{x_1}{x}} e^{-(x_1-x)} \right] \quad (45)$$

For $x \geq x_1$, $b(x)$ drops to zero as $K_1(x)$ i.e. exponentially.

For the current density we find

$$j_z(x) = j' \frac{2}{x_1^2} \left[2 - x_1^2 K_2(x_1) I_0(x) \right] \quad (46)$$

which for $1 \ll x \leq x_1$ reduces to

$$j_z(x) \approx j' \cdot 4 \left(\frac{c}{r_1 \omega_p} \right)^2 \left[1 - \frac{x_1}{4} \sqrt{\frac{x_1}{x}} e^{-(x_1 - x)} \right] \quad (47)$$

see Fig. 1b. The field $b(x)$ in contrast to the previous case rises linearly from the origin almost up to the edge, where it quickly drops to zero. The maxima of $b(x)$ and $j_z(x)$ close to the edge are

$$b_{max} \approx \frac{8\pi j'}{\omega_p} \cdot \frac{c}{r_1 \omega_p} \quad (48)$$

$$j_{z\ max} \approx \frac{2\pi c^2 e n'_1 v'}{\omega_p^2} = j'_z(x_1) \cdot 4 \left(\frac{c}{r_1 \omega_p} \right)^2 \quad (49)$$

where b_{max} is reduced by a factor of order $c/r_1 \omega_p$ compared to the step function profile, equation (39). This is due to the improved screening: The screening factor s has changed from x_1 , equation (42), to $x_1^2/4$.

c) Plateau-like density profile with exponential decrease

A more realistic beam profile than the previous examples should be given by a constant density n' out to some radius r_1 and a subsequent rapid decay, see Fig. 1c.

$$n'(x) = \begin{cases} n'_1 & ; \quad 0 \leq x \leq x_1 \\ n'_1 e^{-\left(\frac{x-x_1}{x_0}\right)^2} & ; \quad x_1 \leq x \end{cases} \quad (50)$$

where $x_1 \gg 1$ will be assumed henceforth. From equations (26) and (50) the magnetic field is found to be approximately

$$b(x) = \frac{2\pi j}{\omega_p} \sqrt{2\pi x_1} e^{-x_1} I_1(x) \left[1 - \sqrt{\pi} \frac{x_0}{2} e^{\left(\frac{x_0}{2}\right)^2} \operatorname{erfc}\left(\frac{x_0}{2}\right) \right]; \quad x \leq x_1 \quad (51)$$

$$b(x) = \frac{2\pi j}{\omega_p} \left\{ e^{-\delta x} - \sqrt{\pi} \frac{x_0}{2} e^{\left(\frac{x_0}{2}\right)^2} e^{-\delta x} \left[\operatorname{erf}\left(\frac{x_0}{2} - \frac{\delta x}{x_0}\right) - \operatorname{erf}\left(\frac{x_0}{2}\right) \right] - \sqrt{\pi} \frac{x_0}{2} e^{\left(\frac{x_0}{2}\right)^2} e^{\delta x} \operatorname{erfc}\left(\frac{x_0}{2} + \frac{\delta x}{x_0}\right) \right\}; \quad x \geq x_1 \quad (52)$$

where

$$\delta x = x - x_1; \quad \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}; \quad \operatorname{erfc} x = 1 - \operatorname{erf} x \quad (53)$$

For $1 \ll x \leq x_1$ equation (51) simplifies to

$$b(x) \approx \frac{2\pi j}{\omega_p} \sqrt{\frac{x_1}{x}} e^{-(x_1-x)} \begin{cases} 1 & ; \quad x_0 \ll 1 \\ 2/x_0^2 & ; \quad x_0 \gg 1 \end{cases} \quad (54)$$

For a sharp boundary, $x_0 \ll 1$, the magnetic field falls off as

$$b(x) \approx \frac{2\pi j}{\omega_p} e^{-(x-x_1)}; \quad x - x_1 \gg x_0 \quad (55)$$

In the small intermediate regime, $0 < \delta x \leq x_0$, $b(x)$ increases insignificantly above the value at $x = x_1$, equation (54), and goes down smoothly to expression (55). Thus, when the boundary region is smaller than c/ω_p the results are a "rounded off" version of the step function case, equation (38), as was to be expected.

For a smooth boundary, $x_0 \gg 1$, we get approximately

$$b(x \gg x_1) \approx \frac{2\pi j^1}{\omega_p} \cdot \frac{2}{x_0^2} \left[e^{-\delta x} + 2 \delta x e^{-\left(\frac{\delta x}{x_0}\right)^2} \right]; \quad 0 \leq \delta x \ll \frac{x_0^2}{2} \quad (56)$$

For $x - x_1 \gg 1$ this is an almost linear increase with δx up to $\delta x = x_0/\sqrt{2}$ and a subsequent rapid decrease proportional to the beam density, see Fig. 1c. For $\delta x \gg x_0^2/2$, $b(x)$ changes its x -dependence into $e^{-(x-x_1-x_0^2/4)}$, but this is practically irrelevant since $b(x \gg x_1 + x_0)$ is exceedingly small.

At $x = x_1 + x_0/\sqrt{2}$ $b(x)$ has its maximum

$$b_{max} = \frac{4\pi j^1}{\omega_p} \cdot \frac{1}{x_0} \sqrt{\frac{2}{e}} \quad (57)$$

The total beam current is

$$j_z^1(r \rightarrow \infty) = \pi \left[r_1^2 + r_0^2 \left(1 + \sqrt{\pi} \frac{r_1}{r_0} \right) \right] e n_1^1 v^1 \quad (58)$$

If we compare this with the maximum of the net current

$$j_z = \frac{c^2}{2\omega_p} x b(x), \quad \text{we find the following screening factors}$$

$$S \approx \sqrt{\frac{e}{8}} \cdot \frac{r_1 \omega_p}{c} \cdot \frac{r_0 \omega_p}{c}; \quad r_1 \gg r_0 \quad (59)$$

$$S \approx \sqrt{\frac{e}{8}} \cdot \left(\frac{r_0 \omega_p}{c} \right)^2; \quad r_0 \gg r_1 \quad (60)$$

When the boundary region is larger than the central section of the beam, equation (60), the screening factor essentially agrees with example b), which also has a slow decrease of the beam density.

In the opposite case, equation (59), the screening factor $s \approx x_0 x_1$ is intermediate between the cases of instantaneous cut-off and slow decrease.

In the examples considered above the magnetic field B_φ decreases exponentially to zero outside the beam. Equation (26) shows that this holds for any radial beam profile if it has a well-defined edge. If, instead, the beam density falls off radially with some power of the distance, $n'(r) \sim r^{-\alpha}$ ($\alpha > 2$ in order to have a finite number of particles in the beam), it may be shown with equation (26) that asymptotically $B_\varphi \sim r^{-\alpha-1}$. Hence the magnetic field falls off more strongly than r^{-1} in practically all cases. The same holds for the radial electric field, equation (27). This implies that in the limit $r \rightarrow \infty$, apart from residual plasma oscillations, there is complete current and charge neutralization when the beam current has reached its steady state, $\frac{\partial j_z}{\partial z} \sim \frac{\partial j_z}{\partial t} = \sigma$. At earlier stages there is a non-vanishing space charge from $\frac{\partial E_z}{\partial z}$ and a finite displacement current.

5. Results and discussion

The amplitude and the profile of the induced return current, the net current and the electric and magnetic fields have been investigated for arbitrary radial and axial density profiles of a relativistic beam injected into a plasma. A detailed analysis has been made for three specific radial density profiles, see Fig. 1a-1c. It is found that for $\tau\omega_p/c \gg 1$, where τ is the length scale of the radial beam density inhomogeneity, the net current within the beam cross section and the amplitude of the magnetic field B_φ are reduced by a factor $S^{-1} \approx (c/\tau\omega_p)^2$ compared to the values of a beam streaming freely in a vacuum. For smooth profiles $\tau \approx r_b$ per order of magnitude, where r_b is the effective radius of the beam. For $\tau\omega_p/c \lesssim 1$ the results of HAMMER and ROSTOKER (1970), LEE and SUDAN (1971), RUKHADZE and RUKHLIN (1972) are recovered. They used a radial step function profile and found $S^{-1} = c/r_b\omega_p$. The magnetic field B_φ exists in a sheath of thickness $\delta\tau$ only, where $\delta\tau \approx c/\omega_p$ for steep profiles ($\tau\omega_p/c \lesssim 1$), while $\delta\tau \approx \tau$ for smooth profiles ($\tau\omega_p/c \gg 1$).

As a consequence of magnetic shielding, the gyroradius $R' = \left| \frac{v'}{\Omega'} \right| = \left| \frac{v'm'c}{eB_\varphi} \right|$ of the beam electrons may be shown to be

$$R' \approx \delta\tau \frac{n_e}{n'} \gamma' \quad (61)$$

which for $n_e \gg n'$, $\gamma' = (1 - v'^2/c^2)^{-1/2} \gtrsim 1$ is much larger

than δr . Therefore, beam electrons are not trapped in the beam
 but are reflected back into the beam's interior. Hence, beam
 currents above the Alfvén - Lawson limit are possible. The ratio
 of the gyroradius over the width δr of the radial magnetic field
 as given by equation (61) is the same for all kinds of profiles. The
 different screening factors for smooth and steep profiles should,
 however, be relevant to the diagnostics of the initial phases of beam
 plasma injection experiments and to the subsequent existence of stable
 or unstable beam plasma equilibria.

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Appendix A

The nonlinear terms $\underline{v}_e \cdot \nabla \underline{v}_e$ and $[\underline{v}_e \times \underline{B}]$ in the equation of motion of the plasma electrons have been neglected throughout. It is readily seen that this is consistent with the results obtained, provided $n' \ll n_e$, except for one term. If we take the condition

$$|E_r| \gg \frac{1}{c} |v_{ez} B_\varphi| \quad \text{A.1}$$

with example (30) for the time dependence of the beam current and using equations (31) and (32) for E_r , B_φ in the long time limit we get

$$\frac{1}{(\omega_p \tau)^2} |\cos(\omega_p \tilde{t})| \gg \frac{|v_{ez}|}{c} = \left| \frac{e n_e v_{ez}}{e n_e c} \right| \approx \left| \frac{n' v_z'}{n_e c} \right| \quad \text{A.2}$$

where $|j_e| = |e n_e v_{ez}| \approx |j_z'|$ has been used. Hence, with $v_z' \approx c$ the necessary condition for A.1 to be valid is

$$\frac{n'}{n_e} \ll \frac{1}{(\omega_p \tau)^2} \quad \text{A.3}$$

It should be noted that $\omega_p \tau \gg 1$ has been assumed.

The gyro-radius of the plasma electrons in the B_φ field per order of magnitude is

$$R = \left| \frac{v_{ez}}{\Omega_\varphi} \right| = \left| \frac{v_{ez} m c}{e B_\varphi} \right| \approx \left| \frac{v_{ez} n_e}{v' n'} \right| \delta r \quad \text{A.4}$$

where δr is the effective radial scale for which $B_y \neq 0$;
 $\delta r \approx c/\omega_p$, r_1 , r_0 in examples 1 - 3. Since
 $|n_e v_{ez}| \approx |v_z' n'|$ however, the gyroradius of the plasma
 electrons is approximately equal to δr . Hence, the results obtained
 can be valid only for times t that are small compared to one gyro-
 period:

$$t \ll t_\Omega = \frac{1}{|\Omega_y|} \approx \frac{\delta r}{|v_{ez}|} \approx \frac{1}{\omega_p} \cdot \frac{n_e}{n'} \cdot \frac{\delta r \omega_p}{c} \quad \text{A.5}$$

The gyroperiod under the assumed conditions, equation A.3 and
 $\omega_p \tau \gg 1$, is, however, still long compared to the time τ of
 beam current saturation :

$$t_\Omega = \tau \frac{1}{(\omega_p \tau)^2} \cdot \frac{n_e}{n'} \cdot \omega_p \tau \cdot \frac{\delta r \omega_p}{c} \gg \tau \quad \text{A.6}$$

so that inequality A.5 is only a weak restriction of the validity of
 the results obtained.

FIG. 1. - A schematic plot of the plasma current density j_{ez} , the net current density j_z and the azimuthal magnetic field b induced by an electron beam j'_z with (a) a step function profile of the density, (b) a parabolic profile, (c) a plateau-like profile with exponential decrease. Note the difference in

the scales: $b_a = \frac{2\pi}{\omega_p} j'_1$; $b_b = \frac{8\pi}{\omega_p} \frac{1}{x_1} j'_1$;

$b_c = \frac{4\pi}{\omega_p} \sqrt{\frac{2}{e}} \frac{1}{x_0} j'_1$ and $j_a = \frac{1}{2} j'_1$;

$j_{b1} = \frac{4}{x_1^2} j'_1$; $j_{b2} = \frac{1}{x_1} j'_1$; $j_c = \frac{2}{x_0^2} j'_1$, where

$x_1 = \tau_1 \omega_p / c \gg 1$; $x_0 = \tau_0 \omega_p / c \gg 1$.

Fig. 1

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