Vacuum Magnetic Fields of Linear Stellarators with Shaped Coils.

G. Lehner and F. Pohl

IPP 1/119 July 1971

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Vacuum Magnetic Fields of Linear
Stellarators with Shaped Coils.

G. Lehner and F. Pohl

IPP 1/119 July 1971

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

IPP 1/119	G. Lehner F. Pohl	Vacuum Magnetic Fields of Linear Stellarators with Shaped Coils. (in English) July 1971

Abstract

The vacuum magnetic fields of linear stellarators with shaped coils are discussed. Two types of expansions are given. The fast one is valid for sufficiently slender coils and the second one for coils with a sufficiently small helical deviation. Some numerical results are described also and compared with the results of the approximations just mentioned.

1. Introduction

Linear high-ß stellarators have been investigated both with helical windings in θ-pinch coils and with shaped coils /1/. The magnetohydrodynamic equilibrium configuration in a linear stellarator can be described by the well known elliptic differential equation for the flux function ψ , which may be written as follows (/2/, for instance):

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{m^2+k^2r^2}\frac{\partial\psi}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\xi^2}=$$

$$\frac{2mk\ J(\psi)}{(m^2+r^2k^2)^2}-4\pi\frac{dp(\psi)}{d\psi}-\frac{J(\psi)}{m^2+r^2k^2}\frac{dJ(\psi)}{d\psi} \qquad (1)$$

Here ξ is the so-called helical coordinate

$$\xi = m\varphi - kz \qquad (2)$$

and it is assumed that all quantities depend on r and ξ only, this being the definition of helical symmetry.
r, φ, z are cylindrical coordinates.
m and k are the azimuthal and longitudinal wave numbers of the configuration. The two functions $p(\psi)$ and $J(\psi)$ can be prescribed arbitrarily. p is the pressure of the plasma.
J , p and ψ define the magnetic fields and currents in the following manner:

$$B_r = \frac{1}{r}\frac{\partial\psi}{\partial\xi} \qquad (3)$$

$$B_\varphi = \frac{rkJ - m\frac{\partial\psi}{\partial r}}{m^2+r^2k^2} \qquad (4)$$

$$B_z = \frac{mJ + rk\frac{\partial\psi}{\partial r}}{m^2+r^2k^2} \qquad (5)$$

$$j_r = \frac{1}{4\pi r}\frac{\partial J}{\partial\xi} \qquad (6)$$

$$j_\varphi = -\frac{k}{4\pi r}\frac{\partial^2\psi}{\partial\xi^2} - \frac{1}{4\pi}\frac{\partial}{\partial r}\left(\frac{mJ + rk\frac{\partial\psi}{\partial r}}{m^2+r^2k^2}\right) \qquad (7)$$

$$j_z = -\frac{m}{4\pi r^2}\frac{\partial^2\psi}{\partial\xi^2} + \frac{1}{4\pi r}\frac{\partial}{\partial r}\left(r\left[\frac{rkJ - m\frac{\partial\psi}{\partial r}}{m^2+r^2k^2}\right]\right) \qquad (8)$$

so that

$$m j_\varphi - r k j_z = - \frac{1}{4\pi} \frac{\partial J}{\partial r} \tag{9}$$

$$j_\varphi = rk \frac{dp}{d\psi} + \frac{rkJ \frac{dJ}{d\psi} - m \frac{dJ}{d\psi}\frac{\partial \psi}{\partial r}}{4\pi(m^2+r^2k^2)} \tag{10}$$

Unfortunately, it is not clear how one has to choose $p(\psi)$ and $J(\psi)$ in order to obtain configurations of experimental relevance. But even the vacuum configuration in a linear stellarator device is of interest in connection with the experiments, and we shall therefore derive some of the properties of the vacuum field.

For the vacuum field we have to choose

$$p = \text{const} \tag{11}$$

and

$$J = \text{const} \tag{12}$$

as can be seen from the preceding formulas, so that eq.(1) now reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{m^2+k^2r^2}\frac{\partial \psi}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \xi^2} = \frac{2mkJ}{(m^2+r^2k^2)^2} \tag{13}$$

If the configuration is to be produced by a shaped coil with a "rotating" circular cross section of radius c, as described by fig.1, we have to solve eq.(13) with the boundary conditions

$$\psi = \text{const on the surface of the shaped coil} \tag{14}$$

$$\psi = \text{finite on the axis } r=0 \tag{15}$$

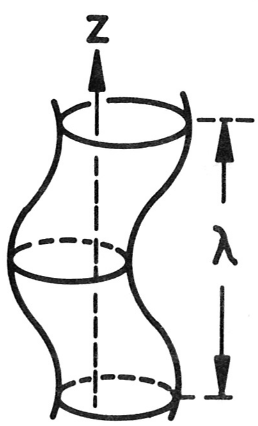

Figure 1a
The wavelength of the configuration is $\lambda = 2\pi/k$

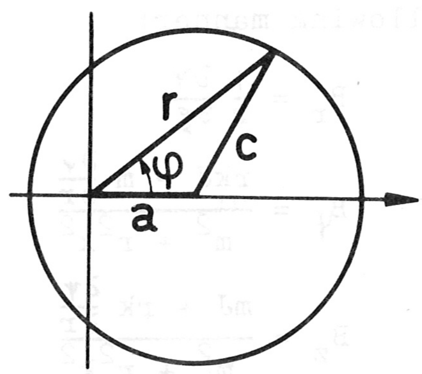

Figure 1b
The cross section is circular and it rotates around the axis with the center of the circle at a distance a from the axis ($a < c$)

It is sufficient to consider one plane corresponding to z=0 e.g. $\xi = \varphi$. The boundary condition is then

$$\psi(r,\varphi) = \text{const} \tag{16}$$

on the circle

$$r^2 - 2r\, a \cos\varphi = c^2 - a^2 \tag{17}$$

where $\psi(r,\varphi)$ is the most general solution of eq.(13)for p and J constant fulfilling the non-singularity condition (15):

$$\psi = \frac{J}{2k}\cdot(rk)^2 + rk\sum_{n=1}^{\infty} \tilde{a}_n\, I_n'(nkr)\cos(n\varphi) \tag{18}$$

where I_n' denotes the modified BESSEL function derived with respect to its argument. The constant J can be chosen arbitrarily and is connected with the strength of the fields obtained. The coefficients $\tilde{a}_n$ have to be derived from the boundary condition (16). To simplify the notation we introduce dimensionless quantities

$$R = rk \tag{19}$$

$$A = ak \tag{20}$$

$$C = ck \tag{21}$$

$$F = \frac{2k}{J}\psi \tag{22}$$

$$\vec{b} = \frac{\vec{B}}{J} \tag{23}$$

$$a_n = \tilde{a}_n \frac{k}{J} \tag{24}$$

so that - in the plane z=0 -

$$F = R^2 + 2R\sum_{n=1}^{\infty} a_n I_n'(nR)\cos(n\varphi) \tag{25}$$

$$b_z = 1 + \sum_{n=1}^{\infty} n a_n I_n(nR)\cos(n\varphi) \tag{26}$$

$$b_\varphi = -\frac{1}{R}\sum_{n=1}^{\infty} n a_n I_n(nR)\cos(n\varphi) \tag{27}$$

$$b_r = -\sum_{n=1}^{\infty} n a_n I_n'(nR)\sin(n\varphi) \tag{28}$$

where the coefficients are defined by the boundary condition

$$F=\text{const for } R^2 - 2RA\cos\varphi = C^2 - A^2 \tag{29}$$

In the following sections we shall first derive some approximate results by expanding F for sufficiently small R=kr and then discuss some numerical results

2. Expansion for a slender helical coil

If the coil is sufficiently slender, i.e. if $R \ll 1$, $(r \ll \frac{\lambda}{2\pi})$ we can expand F. If R is small and of first order, the deviation A and radius C are also small. We therefore assume that the three quantities R, A, C are all small of first order. We then expand F as given by eq.(25) keeping all quantities up to the 6^{th} order. We use

$$\cos\varphi = \frac{R^2 - C^2 + A^2}{2RA}$$

$$\cos(2\varphi) = 2\cos^2\varphi - 1 \tag{30}$$

$$\cos(3\varphi) = 4\cos^3\varphi - 3\cos\varphi$$

$$I_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left(\frac{z}{2}\right)^{n+2k} \tag{31}$$

so that

$$I_1'(R) = \frac{1}{2} + \frac{3}{16}R^2 + \frac{5}{24*16} R^4$$

$$I_2'(2R) = \frac{1}{2}R + \frac{1}{3}R^3 \tag{32}$$

$$I_3'(3R) = \frac{9}{16}R^2$$

More terms than given in eq.(30) and (32) are not needed because, as we shall anticipate, the coefficients a_n turn out to be small of order n. Expanding along the circle and assuming that F is constant on it, we obtain the following results by putting equal to zero the coefficients of R^2, R^4 and R^6 :

$$a_1 = -2A\left[1 - \frac{3}{8}(A^2+C^2) + \frac{22C^4 + 77A^2C^2 + 75A^4}{192}\right]$$

$$a_2 = \frac{3}{4}A^2\left[1 - \frac{22A^2 + 65C^2}{72}\right] \tag{33}$$

$$a_3 = -\frac{43}{108} A^3$$

Here a_1, a_2, a_3 are given up to 5^{th}, 4^{th} and 3^{rd} order respectively, so that the corresponding terms in eq.(25) are are of at least 6^{th} order. The magnetic axis of the configuration may be obtained by looking for the stationary value of F as given by eq.(25) for $\varphi=0$. For a=0 the magnetic axis would coincide with the geometric axis r=0. For $a\neq 0$ the magnetic axis is shifted to r=s (R=S=ks) and, taking into account terms up to the same order as above, one finds

$$S = A\left[1 - \frac{3}{8}C^2 + \frac{11C^4 + 9C^2A^2}{96}\right] \tag{34}$$

The coordinates of the magnetic axis in the plane z=0 are thus R=S, $\varphi=0$. In general, its coordinates are R=S, $\varphi=kz$. The magnetic axis has to be a magnetic field line itself. Indeed, on the axis the radial magnetic field is zero, while its azimuthal and longitudinal components B_φ and B_z are related by eqs. (4) and (5). For m=1 and because on the magnetic axis we have $\frac{\partial\psi}{\partial r} = 0$ eqs. (4) and (5) yield

$$\frac{B_\varphi}{B_z} = rk \tag{35}$$

so that the field line is defined by

$$\frac{r\,d\varphi}{dz} = \frac{B_\varphi}{B_z} = \mathbf{rk}$$

and $\qquad dr \sim B_r = 0 \quad$ which again gives $\varphi=kz$.

The cross sections of the magnetic surfaces Ψ=const in the planes z=const are approximately (up to terms of 4^{th} order) circles. Expanding as before one can show that F is constant (except for terms of higher than 4^{th} order) on circles of radius d (D=kd), centered at r=b (B=kb), where B is the following function of D:

$$B = A(1 - \frac{3}{8}(C^2 - D^2)) \tag{36}$$

For C=D we have B=A, corresponding to our boundary condition. For D=0 we find $B=A(1-\frac{3}{8}C^2)$ which except for higher order terms reproduces the above given result (34).

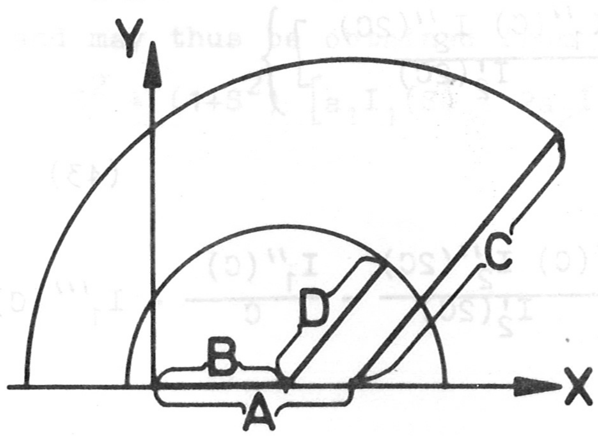

Figure 2

3. Expansion for small helical deviation

The results of the preceding section can be generalized, by allowing the radius C to be large, but still keeping the helical deviation A small $(A < 1)$. We may write F from eq.(25) in the following manner:

$$F = R^2 + 2R\Big[a_1 \frac{X}{R} I_1'(R) + a_2\Big(2\frac{X^2}{R^2} - 1\Big)I_2'(2R) + a_3\Big(4\frac{X^3}{R^3} - 3\frac{X}{R}\Big)I_3'(3R) + \ldots\Big] \tag{37}$$

where
$$X = R\cos\varphi \tag{38}$$

Now F has to be constant along the circle

$$R^2 = 2AX + C^2 - A^2 \tag{39}$$

or
$$R = C + \frac{X}{C}A - \Big(\frac{1}{2C} + \frac{X^2}{2C^3}\Big)A^2 + \Big(\frac{X}{2C^3} + \frac{X^3}{2C^5}\Big)A^3 - \ldots \tag{40}$$

up to 3rd order in A. Introducing this into eq.(37) together with the expansions of I_1', I_2', I_3' we obtain F in the following form:

$$F = c_0 + c_1 X + c_2 X^2 + c_3 X^3 + \ldots \tag{41}$$

We require
$$c_1 = c_2 = c_3 = 0 \tag{42}$$

thus obtaining three equations for a_1, a_2, a_3 which yield, up to 3rd order in A

$$a_1 = -A\left\{\frac{1}{I_1'(C)} - \frac{3}{8}\frac{A^2}{I_1'(C)^2}\Big[\frac{I_1''(C)}{C} + I_1'''(C) + - \frac{4}{3}\frac{I_1''(C)\, I_2''(2C)}{I_2'(2C)}\Big]\right\}$$

$$a_2 = A^2 \frac{I_1''(C)}{2\, I_1'(C)\, I_2'(2C)} \tag{43}$$

$$a_3 = -A^3 \frac{1}{8\, I_3'(3C)\, I_1'(C)}\left\{\frac{4I_1''(C)\, I_2''(2C)}{I_2'(2C)} - \frac{I_1''(C)}{C} - I_1'''(C)\right\}$$

If, in addition, we assume C to be small also, i.e. if we expand the derivatives of the modified BESSEL functions in eq.(43), we recover our formulae (33) except for the terms of higher than 3^{rd} order in A.

I_1', I_1'', I_1''' ... denote the corresponding derivatives of I_1,.. with respect to their argument, for example

$$I_2''(2C) = \frac{d^2}{d(2C)^2} I_2(2C)$$

If we eliminate the higher order derivatives by means of the differential equation of the modified BESSEL functions eq.(43) may be rewritten in terms of $I_1(C)$, $I_1'(C)$, $I_2(2C)$, $I_2'(2C)$, $I_3(3C)$, $I_3'(3C)$:

$$a_1 = -A\left\{\frac{1}{I_1'(C)} - \frac{3}{8}\frac{A^2}{I_1'(C)^2}\left[(1+\frac{4}{3C^2})I_1'(C) + (1-\frac{2}{C^2})\frac{2I_1(C)}{3C}\right.\right.$$
$$\left.\left. - \frac{4}{3}\cdot\left((1+\frac{1}{C^2})I_1(C) - \frac{1}{C}I_1'(C)\right)\cdot(1+\frac{1}{C^2})\frac{I_2(2C)}{I_2'(2C)}\right]\right\}$$

$$a_2 = \frac{A^2}{2I_2'(2C)}\left[(1+\frac{1}{C^2})\frac{I_1(C)}{I_1'(C)} - \frac{1}{C}\right] \tag{44}$$

$$a_3 = \frac{-A^3}{8I_3'(3C)\ I_1'(C)}\left\{4\left[(1+\frac{1}{C^2})I_1(C) - \frac{I_1'(C)}{C}\right](1+\frac{1}{C^2})\frac{I_2(2C)}{I_2'(2C)}\right.$$
$$\left. - \frac{2}{C}I_1(C) - I_1'(C)\right\}.$$

The magnetic axis R=S is located at the extremum of

$$F_{X=R} = R^2 + 2R\left[a_1I_1'(R) + a_2I_2'(2R) + a_3I_3'(3R) + ..\right] \tag{45}$$

and may thus be obtained from the following equation:

$$S^2 + (1+S^2)\left[a_1I_1(S) + 2a_2I_2(2S) + 3a_3I_3(3S) + ..\right] = 0 \tag{46}$$

One can show that to first order in A the magnetic surfaces are of circular cross section. Considering a circle centered at r=b (B=bk) which has the radius d (D=kd), we can conclude from the first of eq.(43) that - to first order -

$$a_1 = - \frac{A}{I_1'(C)} = - \frac{B}{I_1'(D)} \tag{47}$$

or

$$B = A \frac{I_1'(D)}{I_1'(C)} \tag{48}$$

If we expand for $D \ll 1$ and $C \ll 1$, we again find the previous formula (36). The fact that we do not obtain circles for more than first order in A is also apparent from eq.(48). Considering second order, for instance, we would have to postulate both

$$a_1 = - \frac{A}{I_1'(C)} = - \frac{B}{I_1'(D)} \tag{47}$$

$$a_2 = \frac{A^2 I_1''(C)}{2 I_1'(C)\, I_2'(2C)} = \frac{B^2 I_1''(D)}{2 I_1'(D)\, I_2'(2D)} \tag{49}$$

Eq.(47) and (49), however, cannot be satisfied simultaneously. If we put D=0 in eq.(48), we get the shift S of the magnetic axis from the geometric axis to first order in A

$$ks = S = A \frac{I_1'(0)}{I_1'(C)} = \frac{A}{2\, I_1'(C)} \tag{50}$$

which is the lowest order solution of eq.(46).
We may solve eq.(46) up to terms of 3^{rd} order in A, to get

$$S = \frac{A}{2\, I_1'(C)} + \frac{A^3}{4\, I_1'(C)^2} \Bigg[\frac{9}{16\, I_1'(C)} - \frac{3 I_1'(C)}{4} \left(1 + \frac{2}{C^2}\right) + \frac{3\, I_1(C)}{2\, C^3} + \frac{I_1''(C)}{I_2'(2C)} \Big(I_2''(2C) - 1 \Big) \Bigg] \tag{51}$$

Expanding the BESSEL functions in this equation (51) for $C < 1$ we again get eq.(34).

4. Numerical results

The boundary problem posed by eq.(25) and (29) will be solved with the help of the least mean square method. We restrict ourselves to the computation of coefficients a_n with $n = N_{max}$. Let be R_J, φ_J points on the boundary circle (29) with $J=1,2,..J_{max}$ where $J_{max} \geq N_{max}$. Let us, furthermore, define

$$c_{Jn} = 2R_J \, I_n'(nR_J) \cos(n\varphi_J) \tag{52}$$

$$F_J = R_J^2 + \sum_{n=1}^{N_{max}} a_n c_{Jn} \tag{53}$$

$$\overline{F} = \frac{1}{J_{max}} \sum_{J=1}^{J_{max}} F_J \tag{54}$$

$$\overline{c}_n = \frac{1}{J_{max}} \sum_{J=1}^{J_{max}} c_{Jn} \tag{55}$$

$$\overline{R^2} = \frac{1}{J_{max}} \sum_{J=1}^{J_{max}} R_J^2 \tag{56}$$

$$S_q = \sum_{J=1} (F_J - F)^2 \tag{57}$$

We now calculate the set of coefficients $a_1,..a_{Nmax}$ that minimalizes the function S_q by solving the N_{max} equations

$$\frac{\partial S_q}{\partial a_n} = 0 \qquad n=1,2,..N_{max} \tag{58}$$

This set of equations (58) may be written in the following form:

$$\sum_{k=1}^{N_{max}} g_{nk} a_k = b_n \qquad n=1,2...N_{max} \tag{59}$$

with

$$g_{nk} = \sum_{J=1}^{J_{max}} (c_{Jn} - \overline{c}_n)(c_{Jk} - \overline{c}_k)$$

$$b_n = -\sum_{J=1}^{J_{max}} (c_{Jn} - \overline{c}_n)(R_J^2 - \overline{R^2}) \tag{60}$$

Table 1 shows some examples for several sets of parameters A and C, giving S (the location of the magnetic axis, i.e. the minimum of F), S/A , and the six coefficients $a_1, \dots a_6$. $N_{max}=6$ and $J_{max}=34$ has been used for all cases.
Figure 3a shows a plot of the F=const lines for the case A=0.5, C=1 as an example. Within the plotting precision the F=const lines are circles as described by eq.(48).

Table 2 gives more details concerning this case (A=0.5, C=1) for different values of N_{max} and J_{max}. As a figure of merit we introduce the maximum relative fluctuation of F along the boundary circle,

$$T = \frac{F_{J,max} - F_{J,min}}{F} \tag{61}$$

where $F_{J,max}$ and $F_{J,min}$ are the largest and smallest values of F_J on the circle. This allows us to compare the precision of the different cases in table 2.
T as defined by eq.(60) is of importance because according to the mean value theorem for elliptic differential equations the error F inside the circle is never larger than on the boundary circle /3/ ,

$$|\Delta F| \leq |F_{max} - F_{min}| \tag{62}$$

Figure 3b gives F(X), again for A=0.5, C=1 .
Table 3 gives a comparison of the two approximations introduced in the preceding sections 2 and 3 with the numerical results. We consider some of the cases introduced by table 1 and give the shift S of the magnetic axis as computed

a) numerically and already contained in table 1,
b) from eq.(50), which is valid for A small and C arbitrary,
c) from eq.(34), which is valid for both A and C small,

and the maximum relative fluctuation T as computed

a) numerically,
b) from the set of coefficients (44) , which is valid for A small and C arbitrary,
c) from the set of coefficients (33), which is valid for both A and C small.

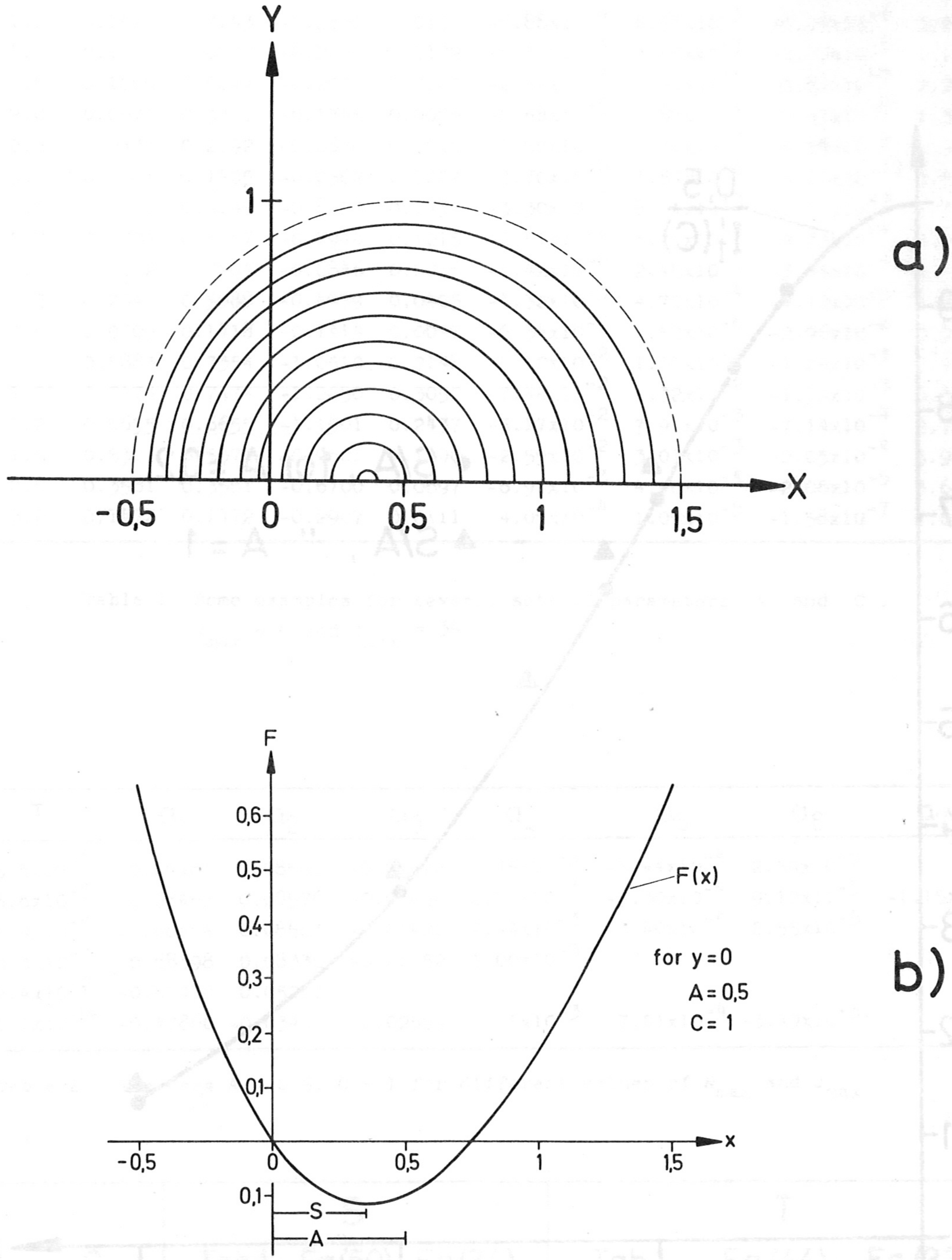

Fig. 3 F = const. lines (a) and magnetic flux function (b) for the case A = 0.5, C = 1.

Finally we want to give S as a function of A and C. According to eq. (51) S/A is constant to first order in A; in Fig. 5 we plot S/A as a function of C with A as a parameter.

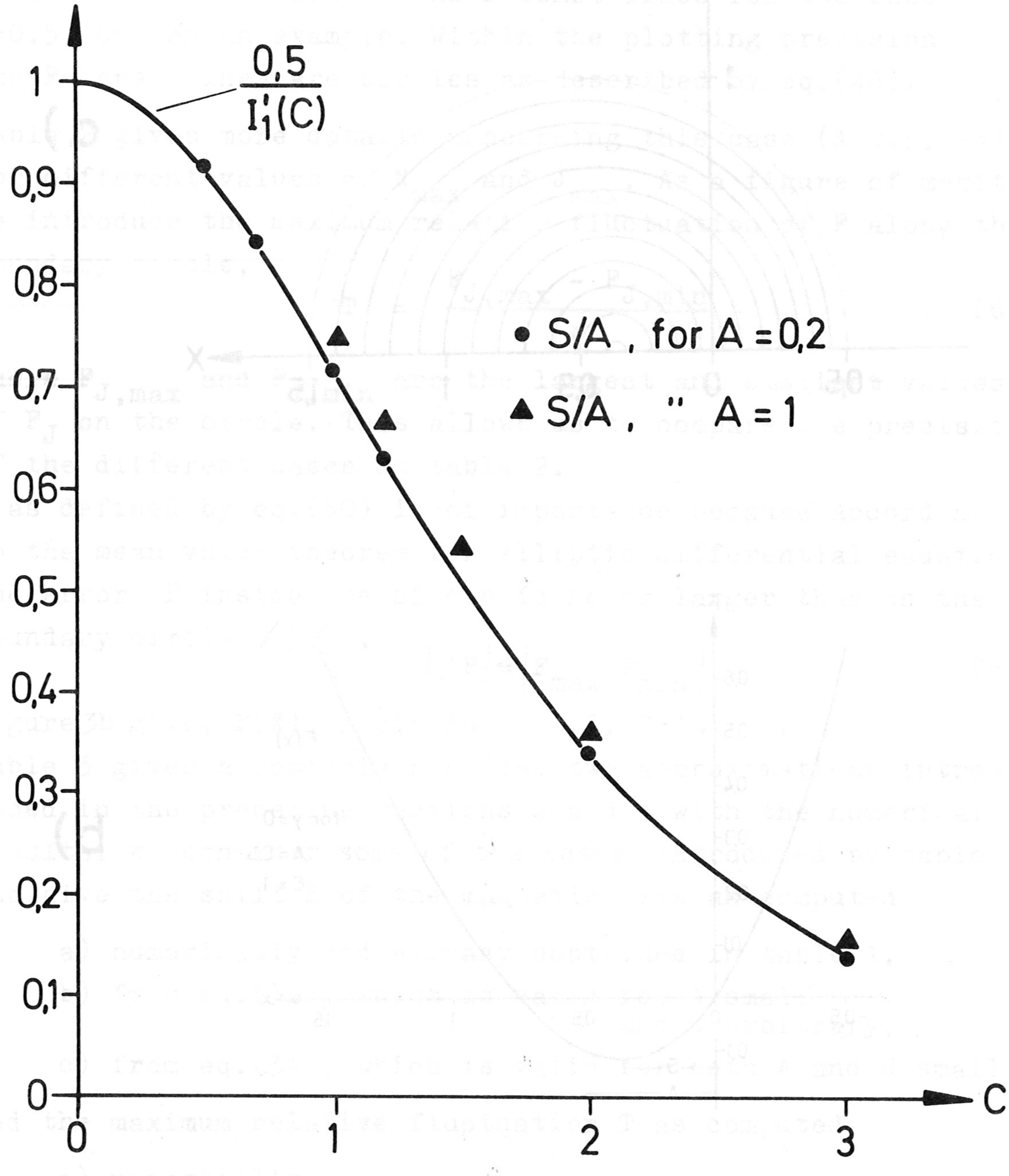

Fig. 5 Location S of the magnetic axis for some cases of A, C.

A	C	S	S/A	a_1	a_2	a_3	a_4	a_5	a_6
0.2	0.21	0.1968	0.9840	-0.3880	0.0285	-2.95×10^{-3}	3.55×10^{-4}	-4.56×10^{-5}	5.14×10^{-6}
0.2	0.5	0.1827	0.9137	-0.3607	0.0240	-2.22×10^{-3}	2.36×10^{-4}	-2.67×10^{-5}	2.55×10^{-6}
0.2	0.7	0.1682	0.8412	-0.3326	0.0198	-1.62×10^{-3}	1.52×10^{-4}	-1.51×10^{-5}	1.24×10^{-6}
0.2	1.0	0.1429	0.7143	-0.2831	0.0136	-8.88×10^{-4}	6.57×10^{-5}	-5.08×10^{-6}	3.25×10^{-7}
0.2	1.2	0.1254	0.6272	-0.2490	0.0102	-5.57×10^{-4}	3.45×10^{-5}	-2.22×10^{-6}	1.18×10^{-7}
0.2	1.5	0.1009	0.5044	-0.2007	0.0062	-2.59×10^{-4}	1.21×10^{-5}	-5.82×10^{-7}	2.29×10^{-8}
0.2	2.0	0.0674	0.3371	-0.1345	0.0026	-6.58×10^{-5}	1.87×10^{-6}	-5.47×10^{-8}	1.31×10^{-9}
0.2	2.5	0.0438	0.2192	-0.0876	0.0010	-1.58×10^{-5}	2.71×10^{-7}	-4.79×10^{-9}	6.90×10^{-11}
0.2	3.0	0.0281	0.1405	-0.0562	0.0004	-3.70×10^{-6}	3.84×10^{-8}	-4.11×10^{-10}	3.56×10^{-12}
0.5	0.51	0.4572	0.9145	-0.8523	0.1437	-3.30×10^{-2}	8.01×10^{-3}	-1.61×10^{-3}	1.77×10^{-4}
0.5	0.7	0.4234	0.8467	-0.7944	0.1215	-2.51×10^{-2}	5.45×10^{-3}	-9.73×10^{-4}	9.45×10^{-5}
0.5	1.0	0.3602	0.7204	-0.6846	0.0856	-1.42×10^{-2}	2.45×10^{-3}	-3.44×10^{-4}	2.59×10^{-5}
0.5	1.5	0.2544	0.5088	-0.4939	0.0405	-4.32×10^{-3}	4.70×10^{-4}	-4.12×10^{-5}	1.92×10^{-6}
0.5	3.0	0.0709	0.1418	-0.1414	0.0027	-6.34×10^{-5}	1.53×10^{-6}	-2.96×10^{-8}	3.02×10^{-10}
0.8	1.0	0.5883	0.7354	-1.0510	0.2148	-5.12×10^{-2}	1.01×10^{-2}	-1.29×10^{-3}	7.44×10^{-5}
1.0	1.01	0.7474	0.7474	-1.2650	0.3052	-7.44×10^{-2}	1.32×10^{-2}	-1.39×10^{-3}	6.39×10^{-5}
1.0	1.2	0.6655	0.6655	-1.1501	0.2437	-5.17×10^{-2}	7.91×10^{-3}	-7.14×10^{-4}	2.78×10^{-5}
1.0	1.5	0.5371	0.5371	-0.9601	0.1590	-2.59×10^{-2}	3.01×10^{-3}	-2.05×10^{-4}	5.98×10^{-6}
1.0	2.0	0.3561	0.3561	-0.6700	0.0697	-6.97×10^{-3}	4.96×10^{-4}	-2.06×10^{-5}	3.65×10^{-7}
1.0	3.0	0.1472	0.1472	-0.2907	0.0111	-4.05×10^{-4}	1.05×10^{-5}	-1.58×10^{-7}	1.02×10^{-9}

Table 1 Some examples for several sets of parameters A and C . $N_{max} = 6$ and $J_{max} = 34$.

J_{max}	N_{max}	T	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
34	6	5.5×10^{-4}	-0.68461	0.08561	-0.01422	2.45×10^{-3}	-3.44×10^{-4}	2.59×10^{-5}		
19	8	5.6×10^{-5}	-0.68467	0.08576	-0.01450	2.75×10^{-3}	-5.30×10^{-4}	9.12×10^{-5}	-1.15×10^{-5}	7.36×10^{-7}
19	6	4.9×10^{-4}	-0.68464	0.08563	-0.01422	2.44×10^{-3}	-3.40×10^{-4}	2.55×10^{-5}		
19	4	6.3×10^{-3}	-0.68398	0.08336	-0.01152	1.00×10^{-3}				
19	2	9.4×10^{-2}	-0.65930	0.05261						
6	6	1.1×10^{-14}	-0.47860	-0.03443	0.02599	-5.95×10^{-3}	7.01×10^{-4}	-3.49×10^{-5}		

Table 2 The case A = 0.5, C = 1 for different values of N_{max} and J_{max}

		S			T		
A	C	Tab.1	Eq.(50)	Eq.(34)	Tab.1	Eq.(44)	Eq.(33)
0.2	0.5	0.183	0.183	0.183	1.2×10^{-7}	1.3×10^{-3}	1.1×10^{-2}
0.2	0.7	0.168	0.168	0.169	4.1×10^{-7}	9.7×10^{-4}	2.9×10^{-2}
0.2	1.2	0.125	0.125	0.141	1.7×10^{-6}	1.3×10^{-3}	2.2×10^{-1}
0.2	3.0	0.028	0.028	1.388	3.2×10^{-6}	1.0×10^{-3}	< -1
0.5	0.7	0.423	0.420	0.428	2.8×10^{-4}	2.3×10^{-1}	> 1
0.5	1.0	0.360	0.357	0.382	5.5×10^{-4}	1.8×10^{-1}	> 1
0.8	1.0	0.588	0.571	0.640	1.4×10^{-2}	> 1	< -1
1.0	2.0	0.356	0.337	1.708	6.6×10^{-2}	< -1	< -1
1.0	3.0	0.147	0.140	7.750	6.1×10^{-2}	> 1	< -1

Table 3 Comparison of the two approximations: eqs. (50) and (34) for S
eqs. (44) and (33) for a_n

A	C	S	S/A	a_1	a_2	a_3	a_4	a_5	a_6
0.2	0.21	0.1968	0.9840	-0.3880	0.0285	-2.95×10^{-3}	3.55×10^{-4}	-4.56×10^{-5}	5.14×10^{-6}
0.2	0.5	0.1827	0.9137	-0.3607	0.0240	-2.22×10^{-3}	2.36×10^{-4}	-2.67×10^{-5}	2.55×10^{-6}
0.2	0.7	0.1682	0.8412	-0.3326	0.0198	-1.62×10^{-3}	1.52×10^{-4}	-1.51×10^{-5}	1.24×10^{-6}
0.2	1.0	0.1429	0.7143	-0.2831	0.0136	-8.88×10^{-4}	6.57×10^{-5}	-5.08×10^{-6}	3.25×10^{-7}
0.2	1.2	0.1254	0.6272	-0.2490	0.0102	-5.57×10^{-4}	3.45×10^{-5}	-2.22×10^{-6}	1.18×10^{-7}
0.2	1.5	0.1009	0.5044	-0.2007	0.0062	-2.59×10^{-4}	1.21×10^{-5}	-5.82×10^{-7}	2.29×10^{-8}
0.2	2.0	0.0674	0.3371	-0.1345	0.0026	-6.58×10^{-5}	1.87×10^{-6}	-5.47×10^{-8}	1.31×10^{-9}
0.2	2.5	0.0438	0.2192	-0.0876	0.0010	-1.58×10^{-5}	2.71×10^{-7}	-4.79×10^{-9}	6.90×10^{-11}
0.2	3.0	0.0281	0.1405	-0.0562	0.0004	-3.70×10^{-6}	3.84×10^{-8}	-4.11×10^{-10}	3.56×10^{-12}
0.5	0.51	0.4572	0.9145	-0.8523	0.1437	-3.30×10^{-2}	8.01×10^{-3}	-1.61×10^{-3}	1.77×10^{-4}
0.5	0.7	0.4234	0.8467	-0.7944	0.1215	-2.51×10^{-2}	5.45×10^{-3}	-9.73×10^{-4}	9.45×10^{-5}
0.5	1.0	0.3602	0.7204	-0.6846	0.0856	-1.42×10^{-2}	2.45×10^{-3}	-3.44×10^{-4}	2.59×10^{-5}
0.5	1.5	0.2544	0.5088	-0.4939	0.0405	-4.32×10^{-3}	4.70×10^{-4}	-4.12×10^{-5}	1.92×10^{-6}
0.5	3.0	0.0709	0.1418	-0.1414	0.0027	-6.34×10^{-5}	1.53×10^{-6}	-2.96×10^{-8}	3.02×10^{-10}
0.8	1.0	0.5883	0.7354	-1.0510	0.2148	-5.12×10^{-2}	1.01×10^{-2}	-1.29×10^{-3}	7.44×10^{-5}
1.0	1.01	0.7474	0.7474	-1.2650	0.3052	-7.44×10^{-2}	1.32×10^{-2}	-1.39×10^{-3}	6.39×10^{-5}
1.0	1.2	0.6655	0.6655	-1.1501	0.2437	-5.17×10^{-2}	7.91×10^{-3}	-7.14×10^{-4}	2.78×10^{-5}
1.0	1.5	0.5371	0.5371	-0.9601	0.1590	-2.59×10^{-2}	3.01×10^{-3}	-2.05×10^{-4}	5.98×10^{-6}
1.0	2.0	0.3561	0.3561	-0.6700	0.0697	-6.97×10^{-3}	4.96×10^{-4}	-2.06×10^{-5}	3.65×10^{-7}
1.0	3.0	0.1472	0.1472	-0.2907	0.0111	-4.05×10^{-4}	1.05×10^{-5}	-1.58×10^{-7}	1.02×10^{-9}

Table 1 Some examples for several sets of parameters A and C . $N_{max} = 6$ and $J_{max} = 34$.

J_{max}	N_{max}	T	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
34	6	5.5×10^{-4}	-0.68461	0.08561	-0.01422	2.45×10^{-3}	-3.44×10^{-4}	2.59×10^{-5}		
19	8	5.6×10^{-5}	-0.68467	0.08576	-0.01450	2.75×10^{-3}	-5.30×10^{-4}	9.12×10^{-5}	-1.15×10^{-5}	7.36×10^{-7}
19	6	4.9×10^{-4}	-0.68464	0.08563	-0.01422	2.44×10^{-3}	-3.40×10^{-4}	2.55×10^{-5}		
19	4	6.3×10^{-3}	-0.68398	0.08336	-0.01152	1.00×10^{-3}				
19	2	9.4×10^{-2}	-0.65930	0.05261						
6	6	1.1×10^{-14}	-0.47860	-0.03443	0.02599	-5.95×10^{-3}	7.01×10^{-4}	-3.49×10^{-5}		

Table 2 The case A = 0.5, C = 1 for different values of N_{max} and J_{max}

		S			T		
A	C	Tab.1	Eq.(50)	Eq.(34)	Tab.1	Eq.(44)	Eq.(33)
0.2	0.5	0.183	0.183	0.183	1.2×10^{-7}	1.3×10^{-3}	1.1×10^{-2}
0.2	0.7	0.168	0.168	0.169	4.1×10^{-7}	9.7×10^{-4}	2.9×10^{-2}
0.2	1.2	0.125	0.125	0.141	1.7×10^{-6}	1.3×10^{-3}	2.2×10^{-1}
0.2	3.0	0.028	0.028	1.388	3.2×10^{-6}	1.0×10^{-3}	< -1
0.5	0.7	0.423	0.420	0.428	2.8×10^{-4}	2.3×10^{-1}	> 1
0.5	1.0	0.360	0.357	0.382	5.5×10^{-4}	1.8×10^{-1}	> 1
0.8	1.0	0.588	0.571	0.640	1.4×10^{-2}	> 1	< -1
1.0	2.0	0.356	0.337	1.708	6.6×10^{-2}	< -1	< -1
1.0	3.0	0.147	0.140	7.750	6.1×10^{-2}	> 1	< -1

Table 3 Comparison of the two approximations: eqs. (50) and (34) for S
eqs. (44) and (33) for a_n

References

/1/ E. Fünfer, J. Junker, M. Kaufmann, J. Neuhauser, Report IPP-1/111, Max-Planck-Institut für Plasmaphysik, Garching 1970.
Conf. on Plasma Phys. and Contr. Nucl. Fusion Res., Madison, Wisc., USA, June 1971, Paper CN-28/j-3.

/2/ B.B. Kadomtsev, Soviet Phys. JETP 37, 962 (1960).

/3/ Courant-Hilbert, Methods of Mathematical Physics, Vol. 2, Interscience Publ., N.Y. 1962.