Calculations of self-inductances of thick air-core coils, mutual inductances and axial forces between such coils in coaxial systems by means of a digital computer

Berechnung der Selbstinduktivitäten dickwandiger Luftspulen, Wechselinduktivitäten und axialen Kräfte zwischen solchen Spulen in koaxialen Systemen mit Hilfe eines digitalen Rechners

R. Pöhlchen

IPP 4/93 November 1971

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Calculations of self-inductances of thick air-core coils, mutual inductances and axial forces between such coils in coaxial systems by means of a digital computer

Berechnung der Selbstinduktivitäten dickwandiger Luftspulen, Wechselinduktivitäten und axialen Kräfte zwischen solchen Spulen in koaxialen Systemen mit Hilfe eines digitalen Rechners

R. Pöhlchen

IPP 4/93 November 1971

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Max-Planck-Institut für Plasmaphysik und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

ABSTRACT

The magnetic flux of a semi-infinite circular current sheet (ideal solenoid) can be described with the "general complete elliptic integral".From the expression for the magnetic flux of an ideal solenoid,formulas for the axial forces and mutual inductances between two ideal solenoids in coaxial geometry can be derived.Thick-walled solenoids are computed by dividing each solenoid into a number of current sheets and adding the contributions of each individual current sheet according to a "composite Gaussian quadrature" formula.
A program has been written in ALGOL and FORTRAN at the "Institut fuer Plasmaphysik" in Garching (near Munich) for computing all inductances and axial forces for an arbitrary number of thick coaxial coils.The only input data which are needed for the calculation are the geometrical dimensions of the coil system,the current densities,the winding densities and the number of abszissas.The accuracy of the results depends on the number of abszissas.

ACKNOWLEDGEMENT

It is a pleasure to acknowledge the support given by H.Wilhelm, who translated the mathematical formulation into ALGOL 60 and FORTRAN IV .

CONTENTS

INTRODUCTION

The computer program (FORTRAN, IBM 360/91) described here was developed to calculate the self-inductances of iron-free thick coils with circular-cylindrical geometry and the mutual inductances and the forces acting within linear coaxial coil systems. The present form of the program is restricted to coils with windings of rectangular cross-sections. The current density and winding density must be homogeneous for a given coil. However, the program can easily be extended to compute coils of arbitrary cross-sectional areas and coils with current density and winding density depending on the radius.
The program was developed to meet the needs of "fusion technology". High magnetic fields in large volumes are generally created with iron-free coils. The multiturn circular-cylindrical coil is the crucial component of all "fusion machines" and of many coil configurations for detailed investigations on plasmas. Our program was written for such coils. The axial forces must be known in order to design the supporting structure of the coil system. After the matrix of the partial inductances has been computed, the total inductance (energy) ,which depends on the electrical connection of the individual coils, can be determined.

Besides, the program may prove to be useful for many other applications. Single coils and coil systems without ferromagnetics have long been used in electrical engineering and physics. For example, to calculate the short-circuit forces of large transformers, the transformer windings can be regarded as iron-free coils. Therefore in the literature on the subject a great number of approximate formulas, tables and diagrams are available for computing inductances and forces. (see ref. 5 and 6 as well as handbooks of electrical engineering). However, even with these tools it is generally very hard to get any results. The formulas have an uncertain range of applicability. In addition, one often exceeds the range of the diagrams and the tables. All these disadvantages are avoided by using a digital computer.

MATHEMATICAL FORMULATION

The magnetic flux of a semi-infinite solenoid $\Phi(S)$.

To simplify the following , an expression is given for the magnetic flux of a semi-infinite solenoid.A circular-cylindrical current sheet with the left end (or right end) extending to infinity is called a semi-infinite solenoid.(denoted by S in the following)

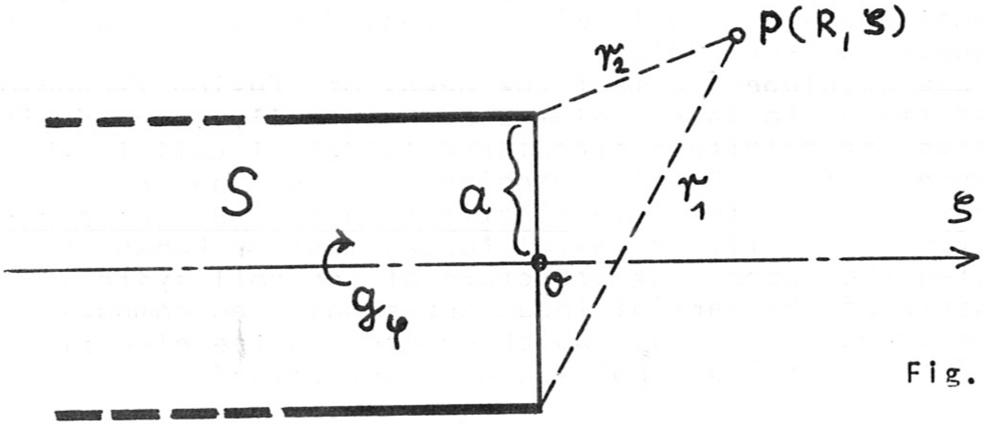


Fig. 1

g_{φ} : current density ; independent of ξ.

The magnetic flux of S as well as the axial force and the mutual inductance between two coaxial solenoids can be described by the "general complete elliptic integral":

$$\text{(1)} \qquad \mathrm{cel}(kc, p, a, b) = \int_0^{\pi/2} \frac{(a\cos^2\varphi + b\sin^2\varphi)\,d\varphi}{(\cos^2\varphi + p\sin^2\varphi)\sqrt{\cos^2\varphi + kc^2\sin^2\varphi}}$$

The parameter p is called c'^2.The following special forms of equation (1) are needed:

$$\mathrm{cel}(kc, 1, 1, 1) \equiv K$$

$$\mathrm{cel}(kc, 1, 0, 1) \equiv (K - E)/k^2$$

$$\mathrm{cel}(kc, c'^2, 0, 1) \equiv (\pi - K)/(1 - c'^2)$$

K, E, π are named the "complete elliptic integrals" of the 1st, 2nd and 3nd kinds.

The parameters kc, k, c' result from the geometrical quantities in Fig.1:

$k^2 = 4\,a\,R/r_1^{\,2}$ ......module of the elliptic integrals

$kc^2 = 1 - k^2$ ....complementary module

$c'^2 = (a-R)^2 \,/\, (a+R)^2$..only of interest for the computation of π, or of a linear combination, K, E and π; otherwise $p = c'^2 = 1$ in equation (1).

It is decisive that the Bartky transformation leads to recursion formulas which allow fast numerical computation of the quantities K, E and in particular, π and the linear combinations of π and K and of π and E without extinction (see R. Bulirsch in (4): Bartky transformation, recursion formulas, ALGOL procedure). It may be mentioned that it is now possible to compute exactly $\phi(s)$ as well as $B_z(s)$ and $B_R(s)$ by using K, E and π.

The magnetic flux of the semi-infinite solenoid is given by

(3)
$$\phi(s) = 8\pi 10^{-9} g_s(\pi a^2/4 + \varphi(s)) \quad \text{if} \quad R > a$$

$$\phi(s) = 8\pi 10^{-9} g_s(\pi R^2/4 + \varphi(s)) \quad \text{if} \quad R < a$$

with $\phi(s)$ in (Vs), when writing g_s in (A/cm).

The magnetic flux of a solenoid with finite length L is easily derived as the difference of the $\phi(s)$ of two semi-finite solenoids with the same radius and current density, but with the end of one of the solenoids located at $\xi = 0$ and that of the other at $\xi = -L$ (see Fig. 1).
The $\varphi(s)$ in the system of equations (3) is given by:

(4)
$$\varphi(s) = \frac{a R \xi}{r_1}\left(c'^2\, \mathrm{cel}(kc, c'^2, 0, 1) - \mathrm{cel}(kc, 1, 0, 1)\right)$$

Similar expressions for $\varphi(s)$ and $\phi(s)$ can be found in the publication of J. V. Jones. In this paper, Jones derives formulas for the mutual inductance between a circular-cylindrical helix and a coaxial circular filament. N. B. Alexander and A. C. Downing published a table for $\phi(s)$ as well as for $B_Z(s)$ and $B_R(s)$. However, they used a time consuming procedure (numerical integration) (see (7)).

The axial force between a couple of coaxial solenoids F (S_1, S_2)

The axial force results from the difference between two magnetic fluxes (mutual inductances): one which is originated by S_1 and which penetrates the left end-filament of S_2 and another which is originated by S_1 and which penetrates the right end-filament of S_2 (see Fig. 2).

The following is a generally valid theorem:

The axial forces of a coaxial system can be determined by computing the difference of mutual inductances of the next simpler system.

This can easily be shown (see (5), p. 94; also Jones in (1)).

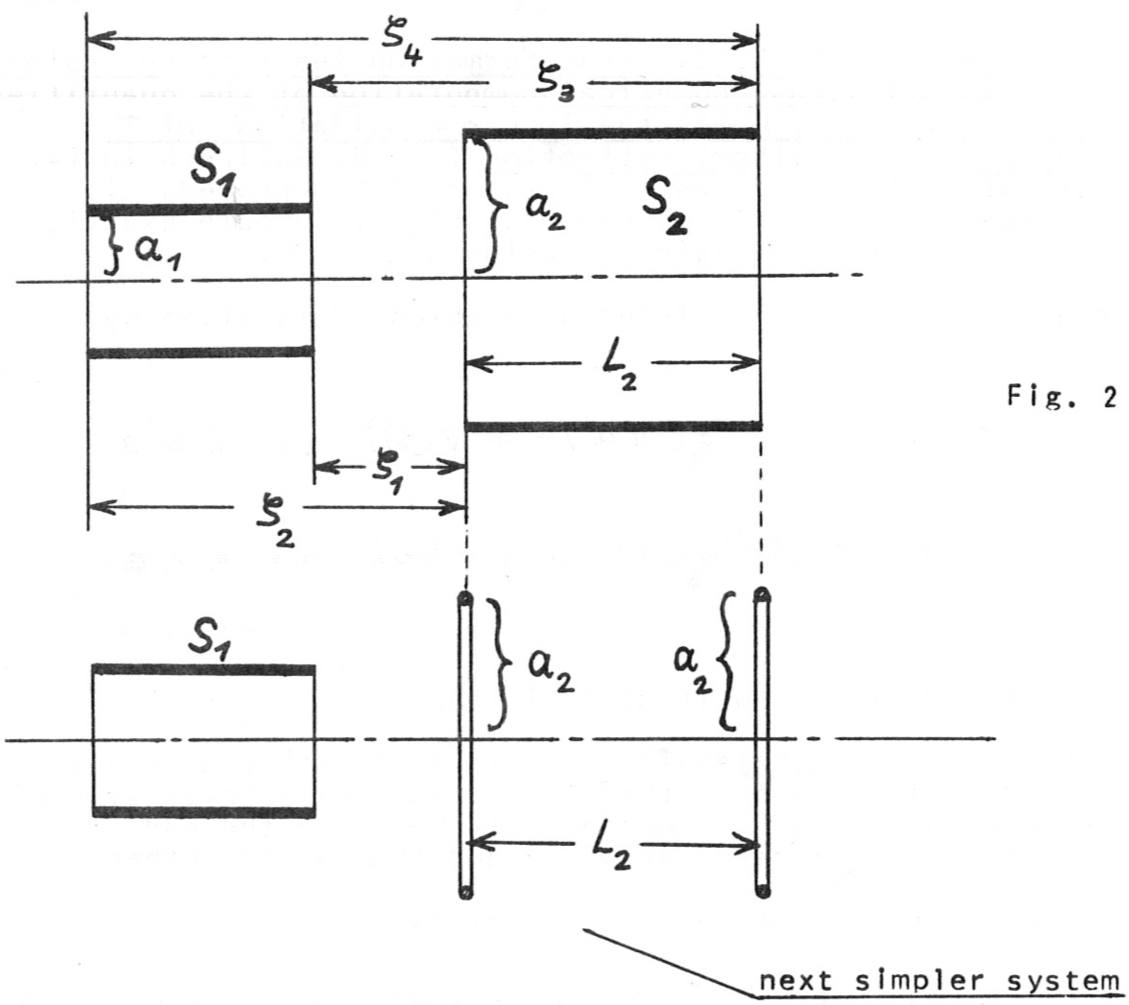


Fig. 2

Consequently one can write

$$(5) \quad F(S_1, S_2) = \frac{8\pi}{10^9} g_{S1} g_{S2} \sum_{m=1}^{4} (-1)^m \varphi_m(S_1)$$

in [VAs/cm.]

$$\text{with } \varphi_m(s) = \frac{a_1\, a_2\, \xi_m}{r_{1m}} \left(c'^2\, cel\left(kc_m, c'^2, 0, 1\right) - cel\left(kc_m, 1, 0, 1\right)\right)$$

and with g_{s1}, g_{s2}: current densities (A/cm).

Instead of the coordinates of the field point (R, ξ) one uses a_2 and $\xi_1, \xi_2, \xi_3, \xi_4$. In particular, $(\varphi_2(S_1) - \varphi_1(S_1))$ stands for the flux of S_1 through the left end-filament of S_2.

Axial force F (C_1, C_2) between two thick coils

The force between thick-walled coils is obtained by numerical integration over the radial dimension of the windings. Physically, this means a replacement of the thick coils by many circular-cylindrical current sheets. For numerical integration the intervals of integration (R_i, R_a) are divided into M/2 and N/2 subintervals respectively, and each of these subintervals is integrated according to Gauss with 2 abscissas.

Abscissas for coil 1:

$$a_p^{\pm} = R_{i1} + h_1 (1 \pm 1/\sqrt{3} + 2p)$$

(6a)

$$h_1 = (R_{a1} - R_{i1})/M$$

$$p = 0, 1, 2, 3 \ldots (M/2 - 1)$$

Abscissas for coil 2:

$$a_q = R_{i2} + h_2 (1 \pm 1/\sqrt{3} + 2q)$$

$$q = 0, 1, 2, 3 \ldots (N/2 - 1)$$

$$(6)\quad \varphi_{m,p,q} = \xi_m \frac{a_p^{\pm} \cdot a_q^{\pm}}{r_{1m,p,q}} \left(c_{p,q}'^2\, cel\left(kc_{m,p,q}, c_{p,q}'^2, 0, 1\right) - cel\left(kc_{m,p,q}, 1, 0, 1\right)\right)$$

For a fixed triplet of numbers (m, p, q) four combinations for $(a_p \cdot a_q)$ result in $4\,\varphi_{m,p,q}$. The same holds for the parameters of the system of equations (2).

$$(7)\quad F(C_1, C_2) = \frac{8\pi}{10^9}\, g_1\, g_2\, h_1\, h_2 \left[\sum_{m=1}^{4} (-1)^m \sum_{p=0}^{\frac{M}{2}-1} \sum_{q=0}^{\frac{N}{2}-1} \varphi_{m,p,q} \right]$$

with $F(C_1, C_2)$ in [VAs/cm], if g_1 and g_2 in [A/cm²]

Mutual inductances and self-inductances of thick coils.

Expressions very much like equation (7) can be derived for the mutual inductance of thick-walled coils. From $L(S_1,S_2)$, $L(C_1,C_2)$ is obtained by numerical integration over the radial thickness of the coils C_1 and C_2. The self-inductance of a thick-walled coil is computed as the mutual inductance of two coincident coils.

The mutual inductance $L(S_1,S_2)$ between two coaxial cylindrical current sheets is given by [2] (similar notation: see ref. 2) :

$$\frac{1}{w_{s1} \cdot w_{s2}} L(S_1, S_2) = \frac{2\pi}{10^4} \sum_{m=1}^{4} (-1)^m \xi_m \frac{F(S_1, S_2)}{g_{s1} \cdot g_{s2}} + \tag{8}$$

$$+ \frac{8\pi}{3 \cdot 10^7} \sum_{m=1}^{4} \left(a_1 a_2 r_{1m} \left(\mathrm{cel}(kc_m, 1, 1, 1) - (2-k^2)\, \mathrm{cel}(kc_m, 1, 0, 1) \right) \right)$$

w_{s1}, w_{s2} :winding densities (number of turns/cm)

Numerical integration analogous the method for $F(C_1,C_2)$ results in:

For the second term of equation (8) :

$$\Psi_{m,p,q} = a_p^{\pm} a_q^{\pm} r_{m,p,q} \left[\mathrm{cel}(kc_{m,p,q}, 1, 1, 1) + (k_{m,p,q}^2 - 2)\, \mathrm{cel}(kc_{m,p,q}, 1, 0, 1) \right]$$

$$l_{1,2} = \sum_{m=1}^{4} (-1)^m \sum_{p=0}^{\frac{M}{2}-1} \sum_{q=0}^{\frac{N}{2}-1} \Psi_{m,p,q}$$

For the first term of equation (8):

$$\Omega_{m,p,q} = \frac{a_p \cdot a_q}{r_{m,p,q}} \left[\mathrm{cel}(kc_{m,p,q}, 1, 0, 1) - c_{p,q}^2\, \mathrm{cel}(kc_{m,p,q}, c'^2_{p,q}, 0, 1) \right]$$

$$f_{1,2} = \sum_{m=1}^{4} (-1)^m \cdot \xi_m^2 \cdot \sum_{p=0}^{\frac{M}{2}-1} \sum_{q=0}^{\frac{N}{2}-1} \Omega_{m,p,q}$$

Mutual inductance between two thick-walled coils:

$$\underline{L(C_1, C_2) = \frac{8\pi}{3 \cdot 10^7} h_1 h_2 w_1 w_2 (l_{1,2} + 3 f_{1,2})} \tag{9}$$

$L(C_1,C_2)$ in Vs/A = Henry

w_1, w_2: winding densities (number of turns/cm^2)

h_1, h_2: see equations (6a).

DIRECTIONS FOR THE USE OF THE PROGRAM

All coils of a given configuration may have different dimensions, current densities and winding densities. Therefore, the dimensions of every coil:

outer radius RA,
inner radius RI,
length of coil L

have to be fed into the computer.

In addition, the dimension for D,

which represents the distance from the center of the coil on the axis of rotation to **the arbitrarely** chosen **origin** of the coordinates, has to be entered into the program (RA, RI, L, D in cm).

For computation of the forces the

current density (A/cm^2)

has to be known.
The computation of the inductances requires the

winding density (number of turns/cm^2).

For coils which are excited in opposite directions, the forces and inductances turn out with the correct sign provided current density and winding density are read in with their correct signs.
For every coil one has to provide an even number of

abscissas N

for the numerical integration.

For example: For N = 4 abscissas, the coil is replaced by four circular-cylindrical current sheets. However, the sheets are not equally spaced and the contribution of each sheet has to be multiplied by a weight factor according to a composite quadrature formula. (Gaussian two point formula pro subinterval). Increasing N (M) usually improves the accuracy of the results.

Finally, the

total number of coils

is read in.

Two steering parameters

provide that only the forces or the inductances or both are printed.

EXAMPLE OF COMPUTATION

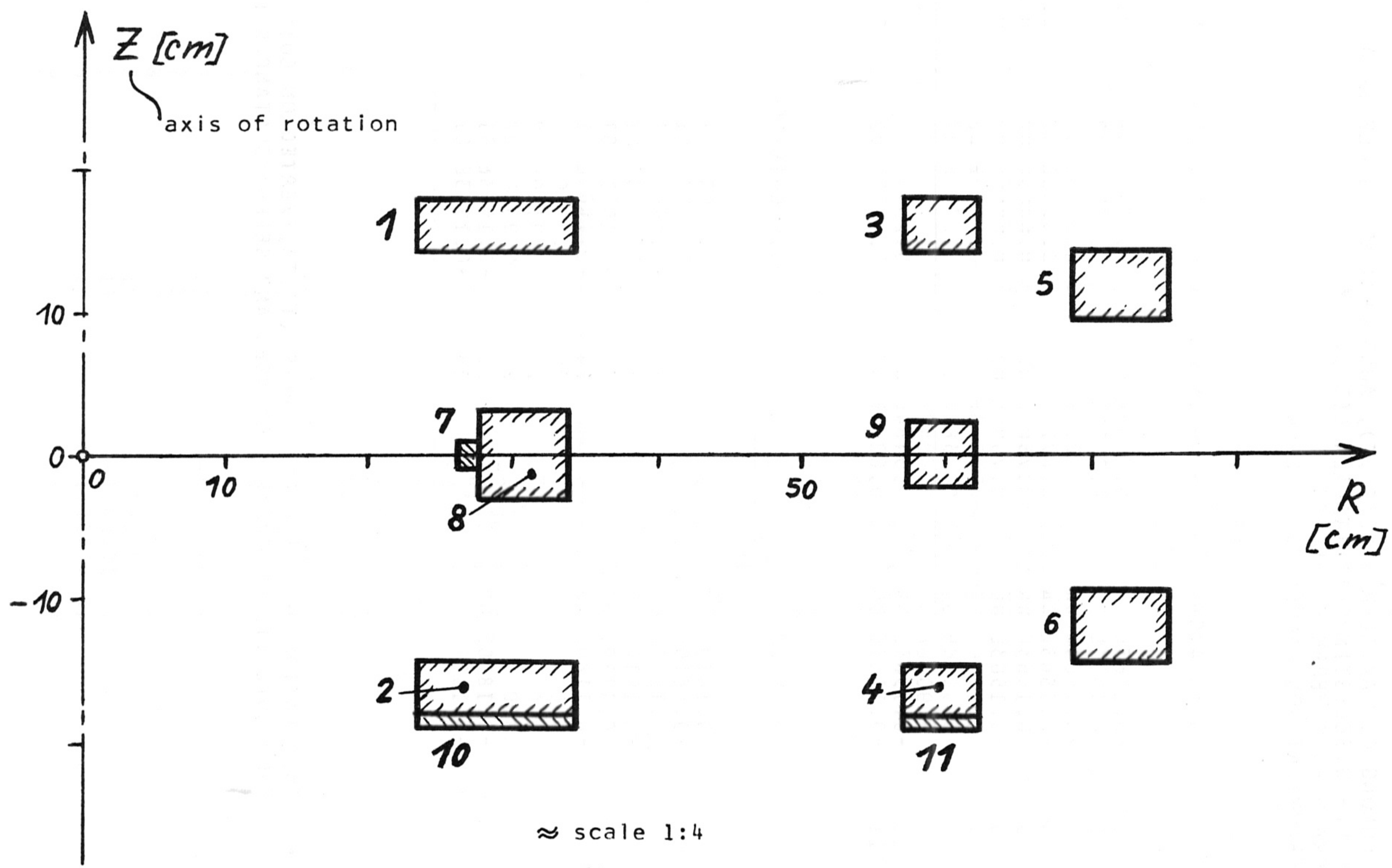
The procedure above described may be applied to determine some of the relevant data of a coil configuration planned for the ''Superconducting Quadrupol W VI'' in Division II of the ''Institut fuer Plasmaphysik'' in Garching (near Muenchen). (Status April 1970).

Figure 3 represents the coil system (scale ≈ 1:4). In order to compute the inductances and forces it is assumed that the total system is composed of 11 single coils. The field of the quadrupole is produced by coils 7, 8 and 9. The outside coils 1, 2, 3, 4, 5 and 6 are needed to compress the field into a finite volume. Coils 10 and 11 are for compensating the weights of coils 7, 8 and 9. The coils 7, 8 and 9 will be built of superconducting material, and the remaining coil system will be constructed of watercooled copper coils (see F. Rau, reference (8)).

The calculations were made on the IBM 360/91 of the IPP, Garching. The computation time was approximately 1 minute. Input data as well as results are printed by the IBM 1741 Communications Terminal as demonstrated in Table I.

Remarks on the results

If coils 10 and 11 are neglected, the coil system becomes symmetrical with regard to the plane z=0 (see Fig.(3)). Among the coils 1-6 there are 3 couples of coils (1+2, 3+4, 5+6) with equal dimensions, current densities and winding densities. Consequently, the absolute values of forces and inductances for these couples should be identical. Actually, the corresponding values for the forces are accurate within 3 digits, those for the inductivities within 2 digits (see underlined values, Table I) However, the degree of accuracy might easily be improved by choosing a greater number of abscissas.



≈ scale 1:4

figure 3

TABLE I

BERECHNUNG DER AXIALKRÄFTE F(TT,T), AUSGEUEBT VON SPULE TT AUF SPULE T,
WECHSEL- U.SELBSTINDUKTIVITAETEN L(TT,T) *
PROGRAMM FUER HERRN POEHLCHEN
PROGRAMMIERT VON HANNELORE MUELLER

T	G IN A/CM**2	RA IN CM	RI IN CM	L IN CM
1	-0.1563E 04	0.3464E 02	0.2344E 02	0.4000E 01
2	-0.1563E 04	0.3464E 02	0.2344E 02	0.4000E 01
3	-0.1563E 04	0.6224E 02	0.5696E 02	0.4000E 01
4	-0.1563E 04	0.6224E 02	0.5696E 02	0.4000E 01
5	-0.1563E 04	0.7505E 02	0.6865E 02	0.4800E 01
6	-0.1563E 04	0.7505E 02	0.6865E 02	0.4800E 01
7	0.9150E 04	0.2759E 02	0.2630E 02	0.2200E 01
8	0.9150E 04	0.3380E 02	0.2760E 02	0.6600E 01
9	0.9020E 04	0.6240E 02	0.5760E 02	0.4400E 01
10	-0.1563E 04	0.3464E 02	0.2344E 02	0.8000E 00
11	-0.1563E 04	0.6224E 02	0.5696E 02	0.8000E 00

D IN CM	N	WDG.DICHTE/QCM
0.1610E 02	24	-0.1563E 01
-0.1610E 02	20	-0.1563E 01
0.1610E 02	22	-0.1563E 01
-0.1610E 02	26	-0.1563E 01
0.1180E 02	22	-0.1563E 01
-0.1180E 02	24	-0.1563E 01
0.0	20	0.9150E 02
0.0	24	0.9150E 02
0.0	22	0.9020E 02
-0.1850E 02	22	-0.1563E 01
-0.1850E 02	24	-0.1563E 01

* CALCULATION OF THE AXIAL FORCES F(TT,T), EXERTED ON COIL T FROM COIL TT, MUTUAL INDUCTANCES AND SELF-INDUCTANCES L(TT,T).

TABLE I

TT	T	F(TT,T) IN KP
1	2	2.8424E 02
1	3	0.0
1	4	1.0890E 02
1	5	2.7934E 01
1	6	1.0954E 02
1	7	-2.9398E 02
1	8	-4.6960E 03
1	9	-6.2522E 02
1	10	4.9383E 01
1	11	2.0864E 01
2	3	-1.0894E 02
2	4	0.0
2	5	-1.0956E 02
2	6	-2.7932E 01
2	7	2.9417E 02
2	8	4.6984E 03
2	9	6.2528E 02
2	10	6.5869E 02
2	11	4.6166E 00
3	4	2.0325E 02
3	5	3.3029E 02
3	6	3.2899E 02
3	7	-3.2288E 01
3	8	-6.7055E 02
3	9	-2.7176E 03
3	10	2.0847E 01
3	11	3.6811E 01
4	5	-3.2898E 02
4	6	-3.3033E 02
4	7	3.2284E 01
4	8	6.7054E 02
4	9	2.7173E 03
4	10	4.6179E 00
4	11	4.8929E 02
5	6	7.9471E 02
5	7	-2.0653E 01
5	8	-4.2169E 02
5	9	-3.0021E 03
5	10	2.2012E 01
5	11	6.0829E 01
6	7	2.0653E 01
6	8	4.2169E 02
6	9	3.0022E 03
6	10	8.5541E 00
6	11	8.8881E 01
7	8	0.0
7	9	0.0
7	10	-4.8494E 01
7	11	-6.8393E 00
8	9	1.8268E-02
8	10	-7.7864E 02
8	11	-1.4047E 02
9	10	-1.3141E 02
9	11	-4.6312E 02
10	11	0.0

TABLE I

TT	T	L(TT,T) IN VS/A
1	1	4.0403E-03
1	2	5.8842E-04
1	3	7.3232E-04
1	4	4.3585E-04
1	5	8.4476E-04
1	6	6.5043E-04
1	7	-4.8097E-03
1	8	-7.9754E-02
1	9	-3.5870E-02
1	10	1.0491E-04
1	11	8.1890E-05
2	2	4.0476E-03
2	3	4.3579E-04
2	4	7.3317E-04
2	5	6.5045E-04
2	6	8.4571E-04
2	7	-4.8106E-03
2	8	-7.9786E-02
2	9	-3.5909E-02
2	10	7.3517E-04
2	11	1.4622E-04
3	3	2.8136E-03
3	4	6.7079E-04
3	5	2.2568E-03
3	6	1.2319E-03
3	7	-1.9480E-03
3	8	-3.7068E-02
3	9	-6.7736E-02
3	10	8.1987E-05
3	11	1.2476E-04
4	4	2.8125E-03
4	5	1.2321E-03
4	6	2.2563E-03
4	7	-1.9475E-03
4	8	-3.7057E-02
4	9	-6.7719E-02
4	10	1.4551E-04
4	11	5.0709E-04
5	5	7.1738E-03
5	6	2.6011E-03
5	7	-2.5119E-03
5	8	-4.7862E-02
5	9	-1.1237E-01
5	10	1.2452E-04
5	11	2.3114E-04
6	6	7.1718E-03
6	7	-2.5118E-03
6	8	-4.7852E-02
6	9	-1.1234E-01
6	10	1.6688E-04
6	11	4.3296E-04

TABLE I

TT	T	L(TT,T) IN VS/A
7	7	8.2830E-02
7	8	7.2983E-01
7	9	1.2851E-01
7	10	-8.3047E-04
7	11	-3.7286E-04
8	8	1.3372E 01
8	9	2.4900E 00
8	10	-1.3833E-02
8	11	-7.0823E-03
9	9	9.4727E 00
9	10	-6.8544E-03
9	11	-1.2313E-02
10	10	1.7873E-04
10	11	2.9174E-05
11	11	1.2625E-04

References

(1) Prof. J. V. Jones: "On the calculation of the coefficient of mutual induction of a circle and a coaxial helix, and of the electromagnetic force between a helical current and a uniform coaxial circular cylindrical current sheet". Proceed. of the Roy. Soc. of London, 1898, Vol. LXIII

(2) Chester Snow: "Mutual inductance and force between two coaxial helical wires". Journal of Res. of the Nat. Bureau of Standards, Vol. 22, Febr. 1939.

(3) Milan Wayne Garrett:"Calculation of fields, forces and mutual inductances of current systems by elliptic integrals." Journal of Appl. Phys./Vol. 34/No. 9/1963.

(4) N.B. Alexander, A.C. Downing: "Tables for a semi-infinite circular current sheet". Oak Ridge Nat. Lab. 1959/ORNL-2828.

(5) J. Hak : "Eisenlose Drosselspulen". K.F. Koehler Verlag, Leipzig, 1938.

(6) Frederick W. Grover: "Inductance calculations" (working formulas and tables) Dover Publ. Inc., New York, 1962

(7) Sauer/Szabo: "Math. Hilfsmittel des Ingenieurs", Teil III, Springer Verlag 1968.

(8) F. Rau: "Magnetic Field of Levitated Superconducting Quadrupole W VI", IPP 2/67, April 1968.