On the Asymptotic Behaviour of the Two-Stream Instability

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ABSTRACT: Some theoretical ideas are investigated to better understand the results of numerical simulations of the 1-dim. - two-stream instability. The observed relation $v_{drift} = v_{the}$ implies a principle of maximum dissipation. On a microscopic level it is shown that the quasi-linear solution is inconsistent with $v_{drift} = v_{the}$, and the assumption of an adiabatically growing BGK-wave cannot explain the observed similarity behaviour. It is proposed that the anomalous resistivity be due to an instability of a BGK-wave. Speculations on the multi-dimensional case are made in connection with preliminary results of 2-dimensional computations.

I. Introduction

The instability excited by a sufficiently strong current in a collisionless plasma, is a fundamental phenomenon which has been studied extensively theoretically as well as experimentally. It is generally believed to cause anomalous resistivity and turbulent heating in many plasma devices. Though it is rather simple to make an analysis of the linear stability properties, assuming shifted Maxwellian distributions for electrons and ions for instance, only recently progress has been made, using numerical simulation techniques, in understanding the nonlinear behaviour, especially in the practically most important case, when an external electric field is applied to drive the current. For 1-dimensional plasmas sufficiently large systems ($\sim$1000 initial Debye lengths) can be followed over sufficiently long times ($\sim$ 1000 ω_{pe}^{-1}) to give a rather accurate picture of the longtime behaviour. The main features of the development of a plasma submitted to a constant electric field E_0 are (see Ref. 1, 2):

a) The energy supplied to the system is used predominantly to accelerate and heat the electrons. The kinetic energy of the ions is small; in particular the ions do not resonate with the waves, i.e. are not heated irreversibly.

b) The system seems to evolve in a self-similar way, $v_d \sim t$ $v_{the} \sim t$, $\langle \tilde{E}^2 \rangle / 8\pi n T_e \approx$ const, $\langle k \rangle^{-1} \sim t$. In particular, the electron distribution function remains similar.

c) Drift velocity $v_d = \langle v \rangle$ and electron thermal velocity v_{the}, $v_{the}^2 = \langle v^2 \rangle - \langle v \rangle^2$ are equal, $v_d \approx v_{the}$, with an unexpectedly high degree of accuracy, apart from oscillation both perform about each other.

In the present note we try to understand these features theoretically. We start from a macroscopic point of view giving a simple principle which "explains" the experimental fact $v_d = v_{the}$. We then discuss the dynamical process which continuously regenerates the instability on a microscopic level. Two convenient approaches which have been discussed previously to explain the behaviour of the two-stream instability are proved to be incorrect. In the last section we shall briefly report on some numerical results of 2-dimensional computations we obtained.

II. Principle of maximum dissipation.

In a 1-dimensional system, the ion-sound instability with $v_d \ll v_{the}$ (for $T_e \gg T_i$) is easily stabilized by plateau formation. Thus, only the (Buneman-) two-stream instability with $v_d \gtrsim v_{the}$ can effectively prevent the electrons from being freely accelerated by the external field E_o. The exact equality $v_d = v_{the}$, however, is rather surprising since, considering the actual distribution function f_e (see Fig. 1), v_{the} does not seem to have a simple meaning. We therefore first consider the problem from a macroscopic point of view.

The energy balance relation is the only macroscopic equation containing only v_d, v_{the}, and the driving field E_o, since the contributions from the ions and the fluctuation field $\tilde{E}$ are small and will not be considered here. We therefore obtain

$$(1) \qquad v_d^2(t)/2 + v_{the}^2(t)/2 = \frac{e}{m_e} E_o \int_0^t v_d(t')\,dt' .$$

We have neglected $v_d(0)$, $v_{the}(0)$ since we are interested in the long-time behaviour, where $v_{the}(t)$, $v_d(t) \gg v_{the}(0)$, $v_d(0)$. If the system evolves in a self-similar way, the ratio v_d/v_{the} is constant, and from Eq.(1) $v_d \propto t$, $v_{the} \propto t$.

We can now introduce a simple variational principle, asking for the maximum thermal energy produced by a given E_o. Writing $v_d = a \frac{e}{m_e} E_o t$ and varying $v_{the}^2/2$ with respect to a, one obtains

(2) $$\delta\left(\frac{v_{the}^2}{2}\right) = \left(\frac{e}{m_e}E_o\right)^2 \frac{t^2}{2}(1-2a)\,\delta a = \sigma$$

which gives

(3) $$a = \frac{1}{2}\ , \text{ hence } v_d = v_{the}$$

Thus the observed relation $v_d = v_{the}$ implies that the two-stream turbulence driven by a constant electric field, maximizes dissipation, i.e. the plasma is heated most efficiently. The variational principle (2) can be generalized, so that the self-similarity has not to be assumed but appears as a consequence, and time dependent driving fields $E_o(t)$ can be included. Introducing the natural time scale $\tau = \int_0^t E_o(t')dt'$, Eq.(1) can be written in the form:

(4) $$W_{the}(\tau) \equiv v_{the}^2(\tau)/2 = \int_0^\tau v_d(\tau')\,d\tau' - v_d^2(\tau)/2\ .$$

We require $v_d(\tau)$ to have the property that at any instant τ, $0 \leq \tau \leq \tau_o$ the ratio of thermal to input energy, $R(\tau) = W_{the} / \int_0^\tau v_d'\,d\tau'$ be a maximum. This can be written in the form

(5) $$\delta \int_0^{\tau_o} R(\tau)\,d\tau = 0$$

where the variation is taken with respect to v_d, with the boundary condition $\delta v_d(0) = \delta v_d(\tau_0) = 0$.
Using the expression (5), the condition (6) leads to the equation

$$(6) \qquad \frac{v_d}{\int_0^{\tau} v_d' \, d\tau'} + \frac{1}{2} \int_0^{\tau} \frac{v_d'^2}{\left(\int_0^{\tau'} v_d'' \, d\tau''\right)^2} d\tau' = 0 \quad , \quad v_d' \equiv v_d(\tau') \text{ etc.}$$

with the solution $v_d = \tau/2^0$. From Eq.(4) one obtains again $v_d = v_{the}$.

III. Discussion of different microscopic approaches

To understand how the plasma manages to stay at $v_d = v_{the}$, to achieve maximum heating, one has to investigate the nonlinear behaviour of the two-stream instability on a microscopic level. Considering our very limited knowledge of strong nonlinear plasma dynamics, it does not seem possible at present to give a complete answer to this problem. Here we only intend to discuss a few points. The ions are not really heated and hence do not play an important role in the stabilisation process. Since the turbulent field energy is small, it is tempting to assume that nonlinear corrections to the dispersion relation are negligible and that the quasi-steady state of the system will be described by the marginally stable mean distribution function $f_e(v)$. A typical form is shown in Fig.1. However, it is easy to see that $f_e(v)$ cannot be a solution of the quasi-linear equation. The quasi-linear problem has recently been investigated,[3] and a similarity solution has been given for the asymptotic state. Since there are no waves with positive phase velocities (in the frame indicated in Fig.1), all particles with $v > o$ in Fig.1 can be freely accelerated, so that for $t \rightarrow \infty$ the $v > o$ part of f_e has vanished. The asymptotic solution will have the form shown in Fig.2. Here the particles freely accelerated are represented by a δ-function at $v = v_o = \frac{e}{m_e} E_o t$ with density n_{free}, while the square-shaped part at $v_o \leq v \leq 0$ represents particles resonating with waves (ion waves and plasma waves) with density $n_{trap} = 1 - n_{free}$. The presence of a quasi-steady state implies that the distribution function is marginally stable. From the dispersion relation, with $v_{ph} = \omega/k$,

$$(7) \qquad k^2 - \frac{\omega_{pi}^2}{v_{ph}^2} - \omega_{pe}^2 \left[\frac{n_{free}}{(v_{ph} - v_o)^2} - \frac{n_{trap}}{v_{ph}(v_o - v_{ph})} \right] = 0$$

one easily obtains in this case $n_{trap} \simeq 2(m_e/m_i)^{1/2}$. This means that for $m_e/m_i \rightarrow 0$ all electrons are freely accelerated $v_d/v_{the} \rightarrow \infty$, which is in clear contradiction to the numerical results. Since the similarity solution is essentially unique, the quasi-linear description is inadequate.

Electrons trapped by the ion waves ($v_{ph} \simeq 0$) apparently play an important role. The width of the distribution function for $v > 0$ is just given by the trapping range Δv, $\frac{1}{2} m_e \Delta v^2 = e\bar{\phi}$, where $\bar{\phi}$ is the average amplitude of the ion waves. The actual distribution (Fig.1) suggests the model distribution shown in Fig.3 (instead of Fig.2). Again free particles are represented by a δ-function (which is a good approximation as is obvious from phase space plots of the electrons 2)), while the rest is assumed to be trapped by ion waves forming a symmetric, square-shaped distribution. As in the case of the distribution Fig.2, it is straightforward to determine n_{trap} by the condition of marginality. Here the dispersion relation

$$(8) \quad k^2 - \frac{\omega_{pi}^2}{v_{ph}^2} - \omega_{pe}^2 \left[\frac{n_{free}}{(v_{ph} - v_0)^2} - \frac{n_{trap}}{v_0^2 - v_{ph}^2} \right] = 0$$

yields $n_{trap} = n_{free}$, and $v_d = 0.5\ v_o$, $v_{the} \simeq 0.65\ v_o$, which is reasonably close to the experimental values.

However linear dispersion theory using unperturbed orbits, is strictly speaking, not applicable because of the importance of electron trapping. This leads to the following inconsistency. The distribution should be marginally stable in the sence that a small increase of the drift velocity should make the system slightly unstable. Growing ion waves then further increase v_{the} to maintain $v_d = v_{the}$. The model distribution, however, becomes more stable by increasing v_d and the stability sets in only if the whole distribution is shifted past the ions at $v = o$. This is in clear contrast to the results of numerical experiments. Consequently the behaviour of the instability cannot be explained by considering the linear stability properties of the average distribution function, and it is necessary to include non-linear effects explicitely.

The following mechanism of reexcitation of the instability is conceivable. Since the trapped electrons are bound to the ion waves with phase velocity $v_{ph} \simeq 0$, while the untrapped electrons may be freely accelerated, a well may appear in the electron distribution Fig.1 between $v = v_o$ and $v = 0$, which would be unstable against excitation of electron oscillations. These can then decay into ion waves. Since however this electron instability is rather fast, any deviation from the stable plateau distribution will immediately be flattened out so that only little energy can be transformed into plasma waves. Indeed when investigating the frequency spectrum $\tilde{E}^2(\omega)$ (a typical example is seen in Fig.5) only a small fraction of the intensity is found near ω_{pe}, and a closer analysis shows that even this amount comes from plasma waves with long wave length, and phase velocity outside the electron distribution, and not from waves located in the plateau region. Thus the wave decay process plasma wave→ion wave is by far to weak to account for the wild bursts of ion waves observed in the numerical experiments.

Hence it does not seem possible to explain the observed phenomena by some process based on the weak turbulence expansion. Going beyond weak turbulence theory, it has been proposed that the instability generates essentially a single Bernstein-Green-Kruskal ion wave, the amplitude of which adjusts itself adiabatically to the slowly changing parameters of the plasma [4]. However, not only the amplitude, but also the wavelength has to change, $\lambda \propto v_{the}$, which is not possible adiabatically. Thus the wave pattern must change in a more turbulent way.

Instead of adiabatic changes, the observed behaviour seems to be due to the instability of a BGK-wave. When the ratio v_d/v_{the} is growing, a certain (quasi-stationary) BGK-wave becomes unstable to coalescence. Two neighboring potential wells tend to fall together, which effectively increases the wave length to twice the original value. The instability gives rise to the sudden bursts observed in the fluctuation energy and also to the sudden increase of the average wave length, accompanied with the bursts (see Fig.4) This instability mechanism is also strongly suggested by considering electron phase space plots. The instability of BGK-waves is subject to further, separate investigations.

IV. Speculations on the multi-dimensional case

Finally, we consider briefly the case of a 2-dimensional plasma. In contrast to the 1-dimensional case the ion-sound instability effects the major part of the electron distribution function because of the existence of oblique modes with respect to the drift directions, and for sufficiently small driving field E_o will produce a quasi-stationary state with $v_d < v_{the}$ [5] (at least for a certain period). However, there is a critical value E_c, $E_c \sim \frac{m_e}{e} v_{the} \nu_{eff}$, ν_{eff} being the effective collision frequency of electrons by ion sound waves. For $E_o > E_c$ the majority of electrons tends to "run away", so that again only the Buneman instability, $v_d \gtrsim v_{the}$, will prevent them from being freely accelerated.

Applying the variational principle of section 2, we again find that $W_{the} = \frac{v_d^2}{2}$. Here the distribution of the thermal energy over both degrees of freedom is not specified. Previous numerical simulations [2] showed that because of mode alignment the temperature becomes highly anisotropic, so that the 2-dimensional system behaves very similar to the 1-dimensional. However, relevant computer experiments are more difficult in this case than for the ion sound instability treated in Ref.5, since the system must be rather large, to prevent finite size effects to influence mode growth before a quasi-asymptotic behaviour becomes visible (this condition is not satisfied in the 2-dimensional run reported in Ref.2). Preliminary results of some computer experiments, performed by us, indicate, that oblique modes are strongly excited and temperature remained fairly isotropic, $T_{\parallel}/T_{\perp} \sim 2$. Also in these cases we find that the relation $v_d^2/2 = W_{the}$ is satisfied approximately. If further computations confirm this relation, the principle of maximum dissipation, formulated in section 2) is not restricted to 1-dimensional systems but seems to have a fairly general significance.

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Figure Caption:

Figure 1 Typical experimental form of $f_e(v)$, $v_0 = \frac{e}{m_e} E_0 t$

Figure 2 Solution of the quasi-linear equation

Figure 3 Model distribution function including the effect of electron trapping

Figure 4 v_d, v_{the}, $\langle \tilde{E}^2 \rangle / 8\pi$ and $\langle \lambda \rangle$ (in arbitrary units) as functions of time, for a 1-dimensional run with system size L = 1000 initial Debye-length and $\frac{e}{m_e} E_0 / \omega_{pe} v_{the_0} = 0.02$

Figure 5 Typical form of the frequency spectrum $\tilde{E}^2(\omega)$.

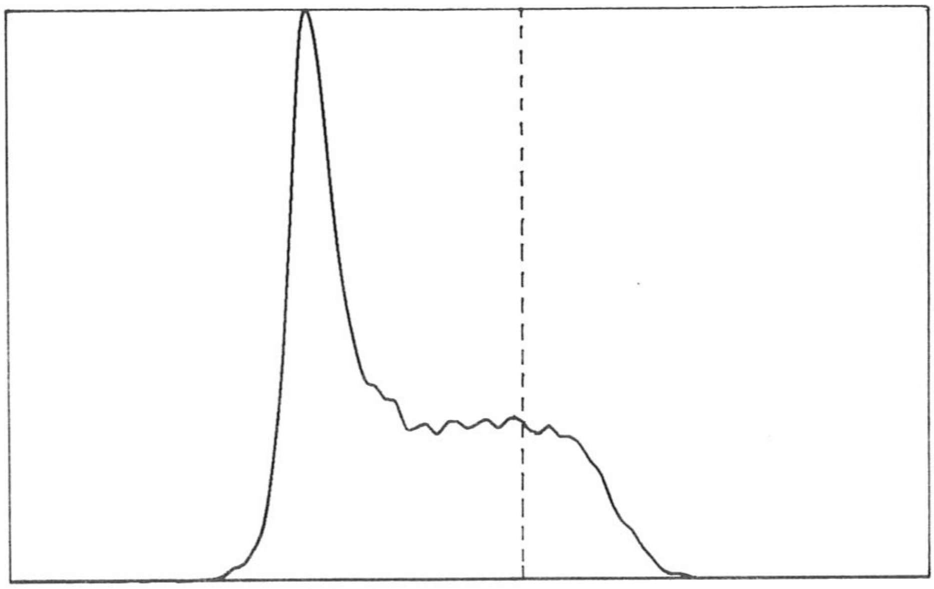

Fig. 1

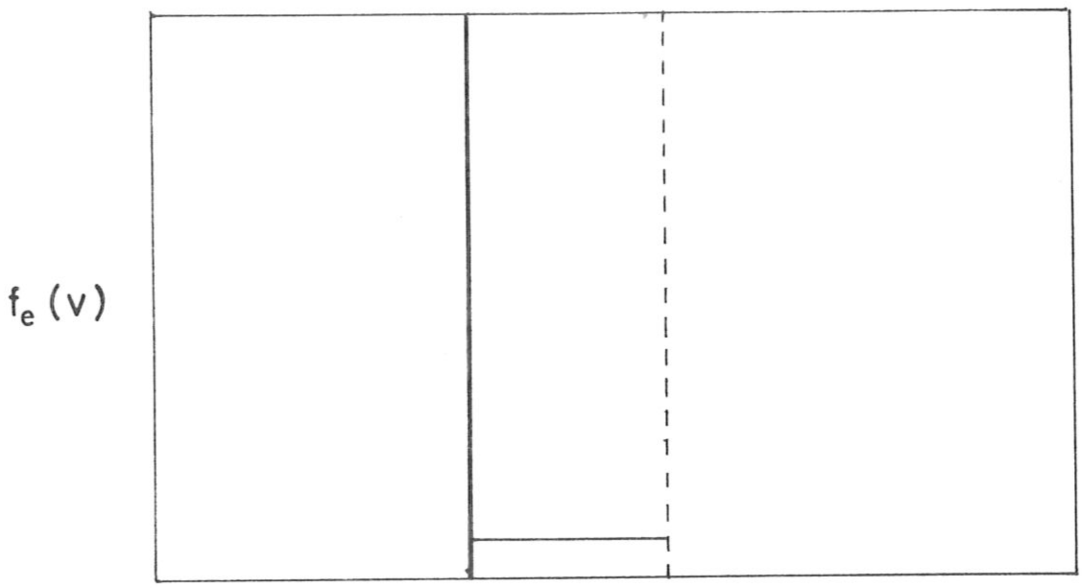

Fig. 2

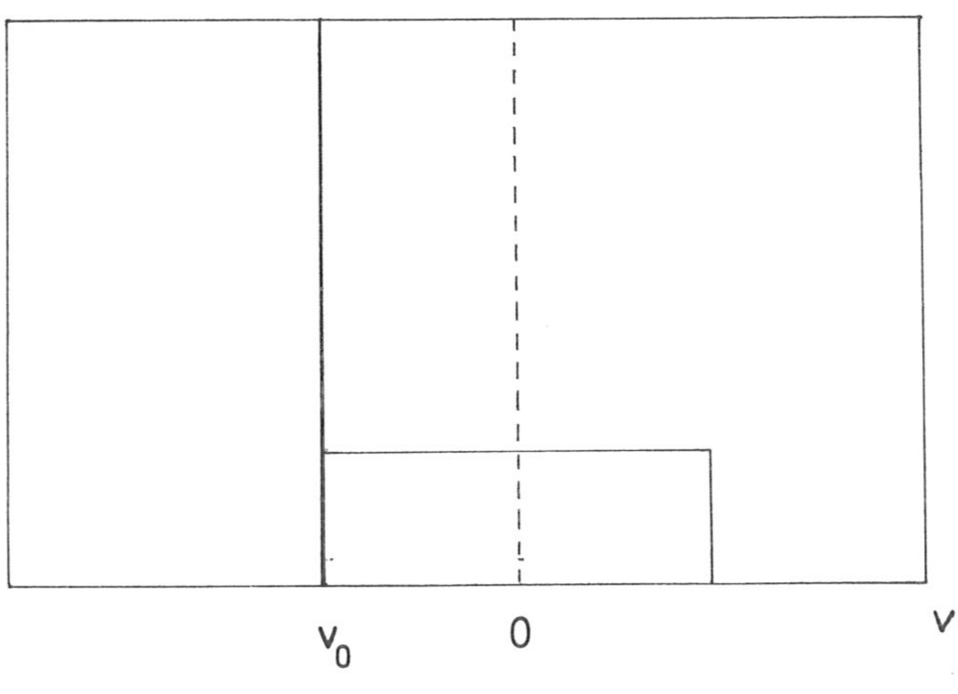

Fig. 3

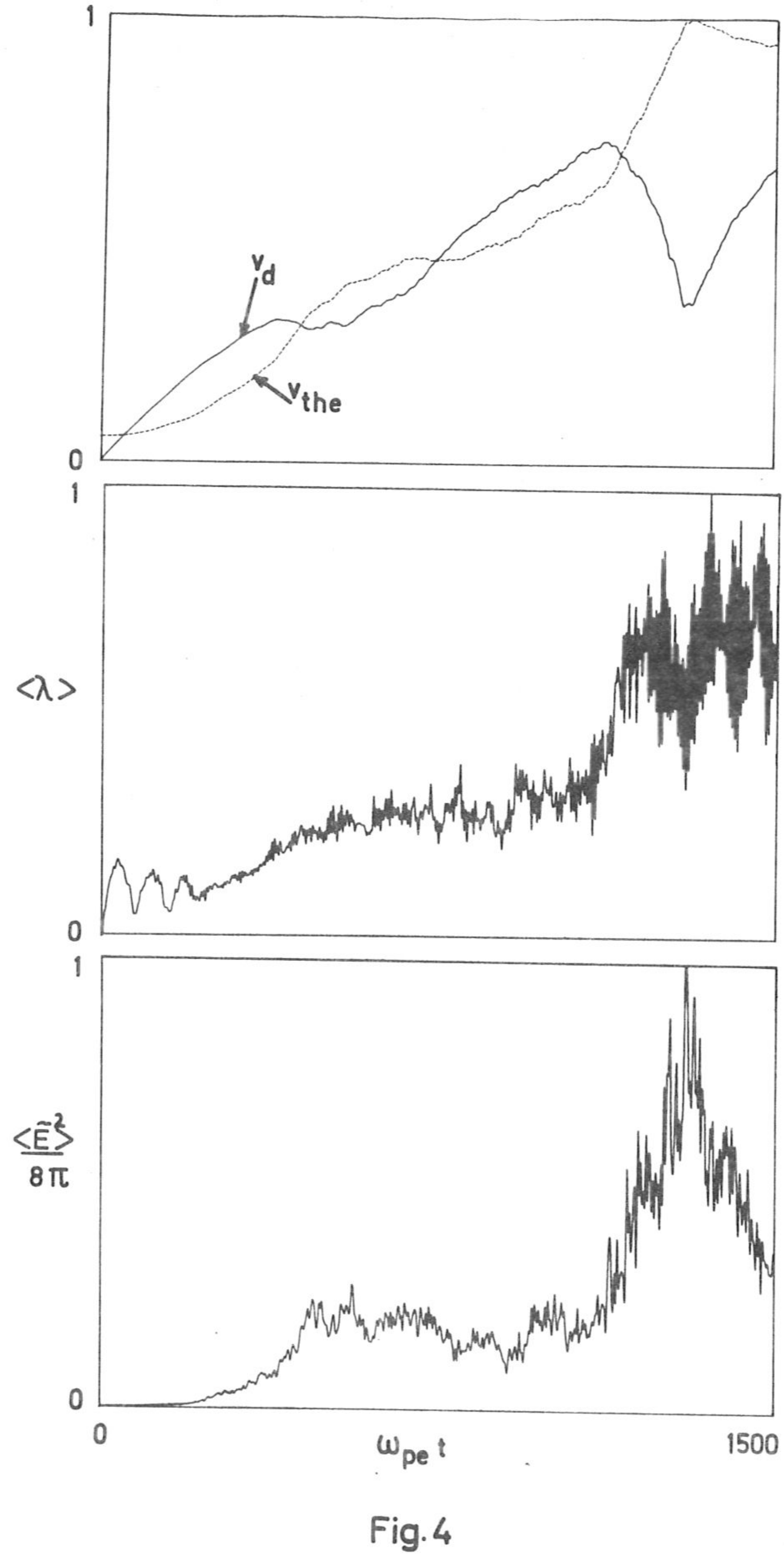
1
0
v_d
v_{the}
1
$\langle\lambda\rangle$
0
1
$\frac{\langle\tilde{E}^2\rangle}{8\pi}$
0
0
$\omega_{pe}\, t$
1500

Fig. 4

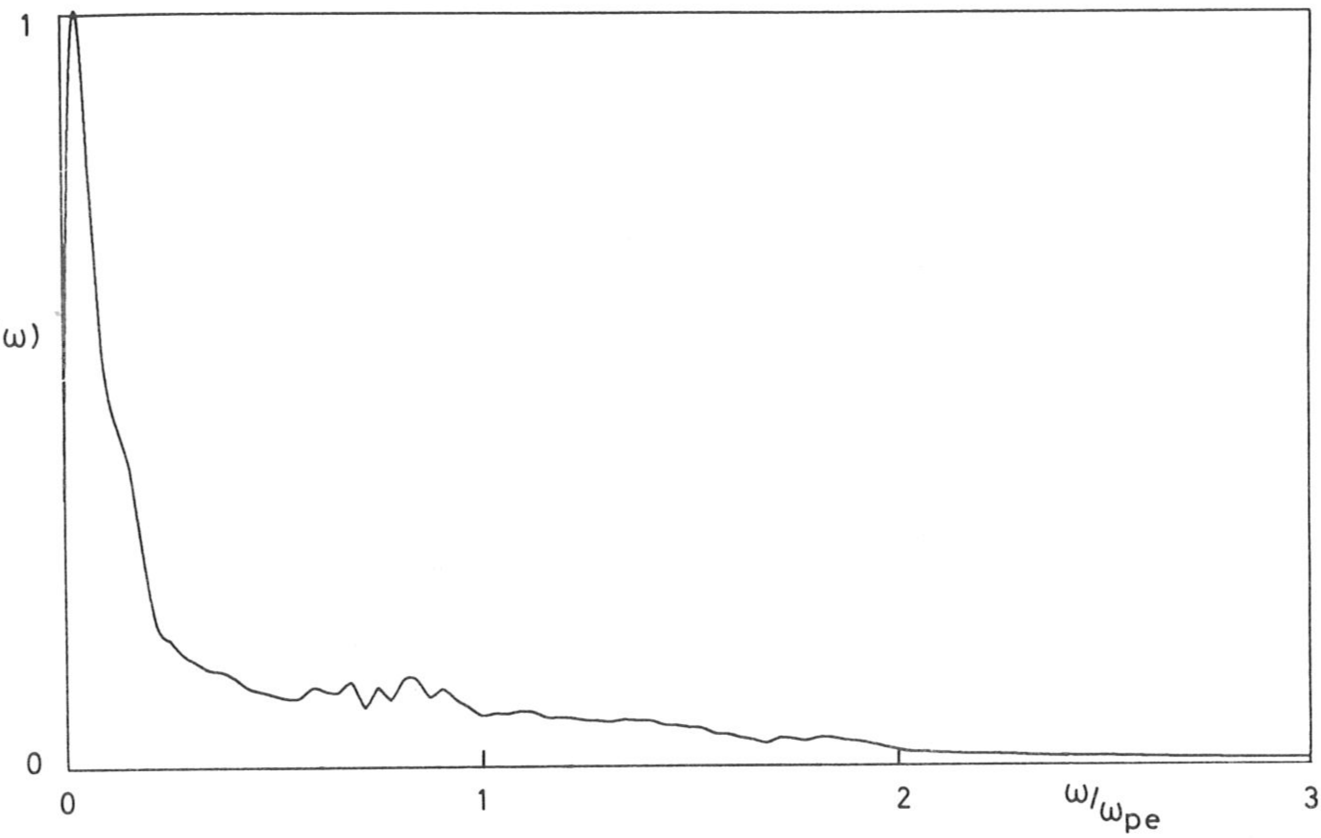

Fig. 5