MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

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Stability to Localized Modes for a Class of Axisymmetric MHD Equilibria

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Abstract

Mercier's criterion is computed numerically for an exact class of axisymmetric MHD equilibria containing possibly elliptical cross sections. The main result is that the toroidal current density and the plasma beta can be increased by at least one order of magnitude if a) the eccentricity of the magnetic surfaces, b) the poloidal current, and c) the aspect ratio are adequately chosen.

I. Introduction

Evaluations of Mercier's criterion have recently been made [1] , [2] . The shape of the magnetic surfaces is triangular or elliptical. All these calculations are done by an expansion in the aspect ratio.

Here Mercier's criterion is numerically evaluated without any expansion for a class of algebraic axisymmetric equilibria with nearly homogeneous current distribution.

In [3], [4] the same equilibrium is given and for special cases a stability calculation was done [4] . A previous paper [5] gives for a special case an exact expression explicit in the equilibrium parameters. This is used here as a test for the numerical calculations. The disagreement between the results of our stability calculation and those in [2] we cannot understand.

II. Equilibrium

The equilibrium is described by the equation for the stream function Ψ :

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} = -r^2 \frac{dp}{d\Psi} - T\frac{dT}{d\Psi} \qquad (1)$$

$p(\Psi)$ being the plasma pressure and $T(\Psi) = r B_\varphi$, where r, φ, z are the usual cylindrical coordinates. If

$$T^2 = T_0^2 + 4\gamma \frac{|p'|}{2(1+\alpha^2)} \Psi , \quad p' = \frac{dp}{d\Psi} \qquad (2)$$

and

$$p = -|p'|\Psi + p_{MAX} \tag{3}$$

with constant T_0 , γ , and p' , then a class of solutions can easily be found:

$$\Psi = \left[z^2(r^2-\gamma) + \frac{\alpha^2}{4}(r^2-R^2)^2\right]\frac{|p'|}{2(1+\alpha^2)} \tag{4}$$

It is convenient to introduce new coordinates

$$X^1 = \sqrt{\frac{\bar{\Psi}}{\alpha^2}} \ , \quad 2\cot X^3 = \alpha\frac{r^2-R^2}{z\sqrt{r^2-\gamma}} \ , \quad X^2 = \varphi \tag{5}$$

with

$$\bar{\Psi} = z^2(r^2-\gamma) + \frac{\alpha^2}{4}(r^2-R^2)^2 \tag{6}$$

from which it follows that

$$r = \sqrt{R^2+2X^1\cos X^3} \ , \quad z = \frac{\alpha X^1 \sin X^3}{\sqrt{R^2+2X^1\cos X^3-\gamma}} \tag{7}$$

The Jacobian can be calculated from

$$r\,dr\,d\theta\,dz = J_n\,dX^1\,dX^2\,dX^3$$

We obtain

$$J_n = \frac{\alpha X^1}{\sqrt{R^2-\gamma+2X^1\cos X^3}} \tag{8}$$

From eq. (4) we can calculate the poloidal magnetic field components.

$$B_r = \frac{2z}{r}\left(r^2-\gamma\right)\frac{|p'|}{2(1+\alpha^2)} \tag{9}$$

$$B_z = -\left[2z^2+\alpha^2\left(r^2-R^2\right)\right]\frac{|p'|}{2(1+\alpha^2)} \tag{10}$$

The toroidal current j_φ is given by

$$j_\varphi = -r p' - \frac{1}{r} T T' \tag{11}$$

The poloidal current density is proportional to γ. If γ is positive this current is confining the plasma. From eq. (9) it follows that B_r vanishes at a certain r in the plasma region , if γ is strong enough. In this case there are two magnetic axis and in some sense there exists a critical value of the plasma β for the equilibrium.
In the case

$$\alpha = \alpha_{circle} = \sqrt{1-\frac{\gamma}{R^2}} \tag{12}$$

one gets a circular near axis cross section of the magnetic surfaces. In general, one has an elliptic near axis cross section. For $\alpha > \alpha_{circle}$ "vertical" ellipses are obtained. The equilibrium is characterized by the ratio

$$\frac{B_\varphi j_p}{B_p j_\varphi} = \frac{\gamma}{R^2}\,\frac{1}{(1+\alpha^2)-\frac{\gamma}{R^2}}$$

where B_p and j_p are the poloidal magnetic field and poloidal current respectively.

For values less than 1 the equilibrium is tokamak-like and screw-pinch-like otherwise.

III. Stability to localized modes

Mercier's criterion is used in the form given in [6]. It is a necessary criterion for stability. If it is violated the plasma is MHD unstable.

The form is

$$Q = \frac{1}{4\oint\left(1+\frac{T^2}{r^2B^2}\right)\frac{J\,d\chi}{r^2}}\left[\frac{\partial}{\partial\psi}\oint\frac{TJ\,d\chi}{r^2} - 2p'T\oint\frac{J\,d\chi}{r^2B^2}\right]^2 \tag{13}$$

$$+\, p'\oint\frac{\partial}{\partial\psi}J\,d\chi - p'^2\oint\frac{J\,d\chi}{B^2} > 0$$

$$J\,d\chi = \frac{1+\alpha^2}{\alpha|p'|}\,\frac{d\chi^3}{\sqrt{R^2+2X^1\cos X^3-r}}$$

The stability calculation was done numerically. First it can be shown numerically that Q is a monotonically increasing function of X^1. This means that if Q becomes negative this happens near the axis.

From eqs. (2), (4), (5), (9), (10), and (13) it can easily be seen that near the axis Q depends on p' and T only in the following way

$$Q(p', T) = Q\left(\frac{p'}{T_0}\right)$$

Therefore, Q can be calculated in the neighbourhood of the magnetic axis as a function of $\frac{p'}{T_0}$ for different values of α and γ , and one can determine that value of $\frac{p'}{T_0}$ for which $Q\left(\frac{p'}{T_0}\right) = 0$

Equation (11) yields

$$\dot{j}_\varphi = R|p'|\left(1 - \frac{\gamma}{R^2}\frac{1}{(1+\alpha^2)}\right) \tag{14}$$

from which one can calculate that value of $\dot{j}_\varphi$ for which Q is equal to zero. This yields the stability diagram in Fig. 1.

The results are:

1. For $\gamma = 0$, i.e. the poloidal current is equal to zero, a circular cross section is more stable than an elliptic one in the sense that a circular cross section allows higher values of $\dot{j}_\varphi$

2. In the case of non-vanishing poloidal current the stability, i.e. the limit of $\dot{j}_\varphi$, increases with increasing γ . But the maximum value of γ is determined by the equilibrium in such a way that B_r goes to zero identically for a given value of r on the plasma boundary. The second magnetic axis appears at the boundary. This yields

$$\gamma \leq R^2 - 2X^1_{boundary} \approx R^2(1-2\varepsilon) \tag{15}$$

where ε is the inverse aspect ratio. Equations (5) and (6) are used. X^1 boundary is defined by

$$p = 0 = p'\psi + p_{MAX}$$

To provide high values of γ , it follows from eq. (15) that the torus has to be slim. In other words, a certain inverse

aspect ratio ε allows a certain maximum value $\gamma_{MAX}(\varepsilon)$ of γ
In the following $\gamma = \gamma_{MAX} \approx R^2(1-2\varepsilon)$
From eq. (4) one gets for the ratio A of the minor to the major axis in the case of elliptic cross sections

$$A = \sqrt{\frac{R^2-\gamma}{R^2\alpha^2}} \approx \frac{1}{\alpha}\sqrt{2\varepsilon} \tag{16}$$

From the stability diagram it can be seen that for each choice of γ, $\dot{j}_\varphi$ reaches practically its maximal value for $\alpha = 1$. This corresponds for a given aspect ratio to an elliptic cross section of the plasma with an eccentricity e

$$e = \sqrt{1-2\varepsilon} \tag{17}$$

Furthermore, it can be seen that for increasing values of γ the ratio $\dot{j}_\varphi(\alpha=1)$ to $\dot{j}_\varphi(\alpha = \alpha_{circle})$ increases from 1 to 5.
From eqs. (2) and (3) the toroidal plasma β can be calculated:

$$\beta_T = 2\,\frac{P_{MAX}}{B_\varphi^2} = \frac{S^2\varepsilon^2\alpha^2}{(1+\alpha^2)\left(1-\frac{\alpha^2}{1+\alpha^2}(1-2\varepsilon)S^2\varepsilon^2\right)} \tag{18}$$

where

$$S = \frac{|p'|}{T_0}R^3$$

In Fig. 2 β_T is calculated as a function of α for different values of γ and for values $\frac{p'}{T_0}$ for which $Q = 0$. The result is that for elliptic cross sections β_T becomes greater than for a circular cross section. For high values of γ the relative growth is greater, though absolutely β_T

is higher for smaller γ .

We can say in conclusion that in a certain range of α both much higher values of j_φ and much higher values of β_T compared with the corresponding values for circular cross section can be reached.

References:

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Fig.1 The limit of the toroidal current j_φ versus α for different values of $|\gamma|$. $|\gamma|$ is proportional to the poloidal current density; α denotes for a given $|\gamma|$ a certain eccentricity of the elliptic near axis cross sections of the magnetic surfaces. The values of α corresponding to a circular cross section are marked with a point.

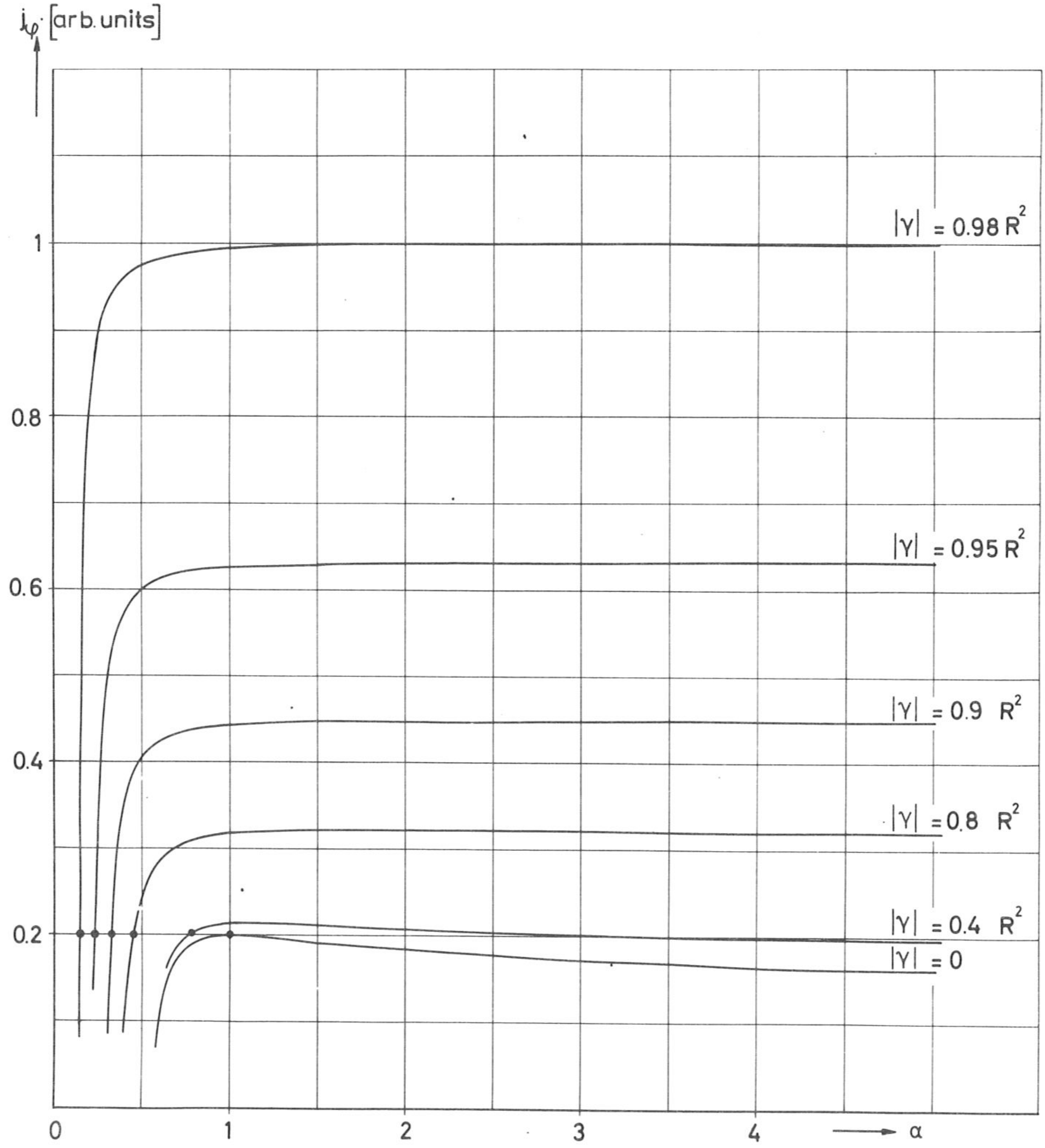
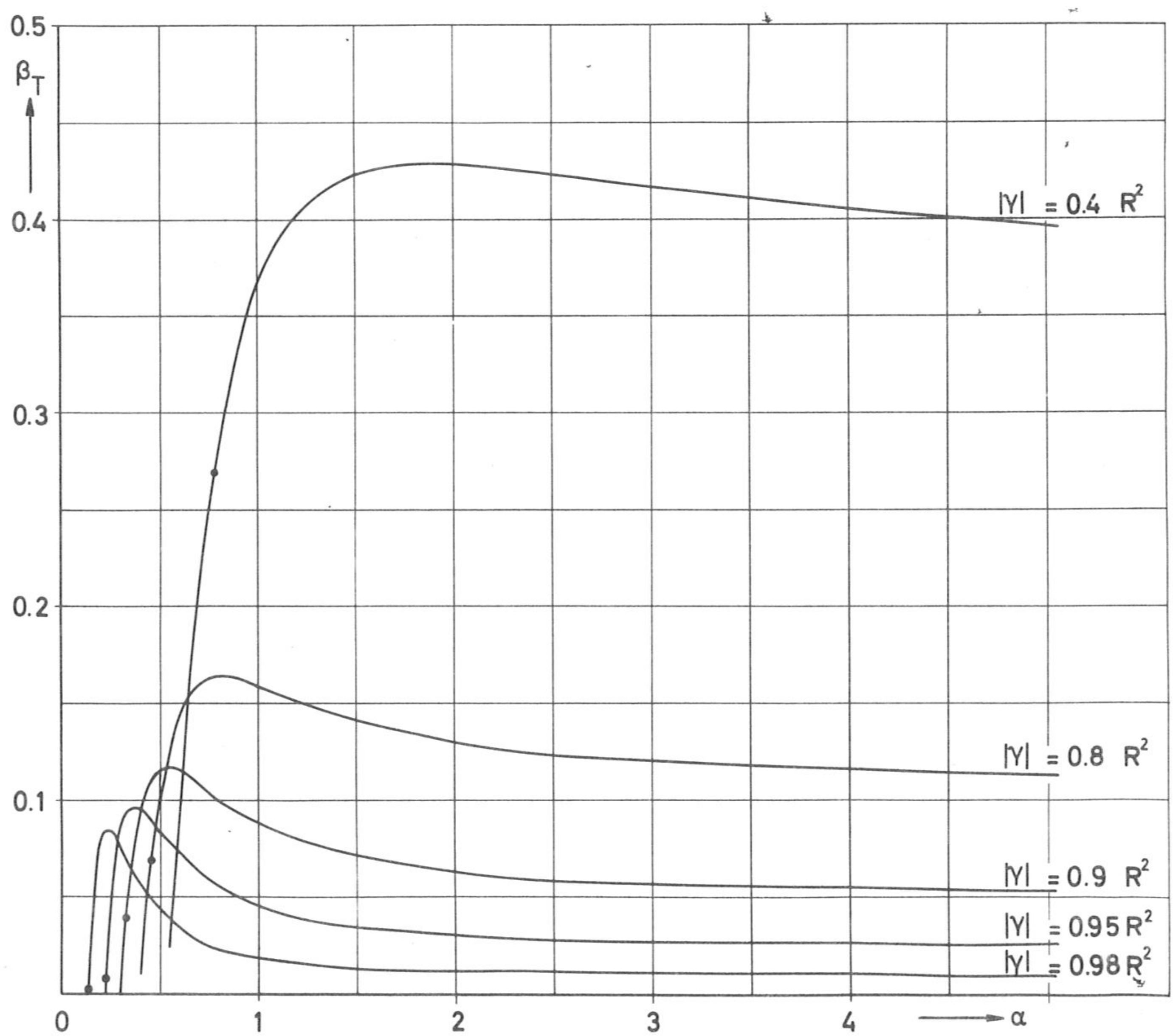


Fig. 2 The limit of the toroidal plasma β versus α for different values of $|\gamma|$. $|\gamma|$ is proportional to the poloidal current density; α denotes for a given $|\gamma|$ a certain eccentricity of the elliptic near axis cross sections of the magnetic surfaces. The values of α corresponding to a circular cross section are marked with a point.



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