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Transport Processes in a Plasma

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Abstract.

Plasma transport in toroidal magnetic systems is reviewed. The motion of single particles in nonuniform magnetic fields is treated and a qualitative picture of plasma transport is given. This discussion is followed by the kinetic theory of transport in the weakly collisional (banana), intermediate and collision dominated regime. The latter regime is also treated, using fluid equations.

Preface

This report is a corrected and slightly expanded version of the lecture notes distributed concurrently with the lectures at the MPI für Plasmaphysik. Most of the material has first been presented in an advanced graduate course at Cornell University.

The purpose of these lectures has been to give not only a survey of plasma transport theory, in particular the recent rapid developments in neoclassical transport theory, but mainly to provide the background and techniques necessary for a study of the current literature and for research in this area. The problems at the end of each chapter serve the same purpose. Following the intent of these lectures, it was not attempted to give a complete bibliography and the detailed discussion has been restricted in general to the simplest nontrivial case, e.g. a model axisymmetric toroidal magnetic field in the treatment of the kinetic theory of transport.

I should like to thank my colleagues at the MPI für Plasmaphysik and Cornell University for helpful comments.

Transport Processes in a Plasma

I. Introduction.

Classical, Neoclassical and Anomalous Transport

To achieve controlled thermonuclear fusion, [1] plasma physics must solve essentially two problems: heating the plasma to temperatures of about 10 keV and confinement of the plasma for times which satisfy the Lawson criterion $n\tau_E \gtrsim 10^{14}$. (n plasma density, τ_E energy confinement time). For both of these goals as well as for the fluid dynamical description of many plasma phenomena such as shock waves e.g., it is necessary to know the transport processes in the plasma. The transport processes we shall discuss in the following are diffusion and heat conduction across the magnetic field, resistivity, viscosity and energy transfer. These processes occur as a result of collisions between particles or collective effects (scattering of particles by waves).

In this and the following lectures we want to discuss plasma confinement. The energy confinement time τ_E is determined by the relation

$$\frac{1}{\tau_E} = \frac{1}{\tau_D} + \frac{1}{\tau_H} + \frac{1}{\tau_R} \qquad (1)$$

where τ_D is the particle confinement time,

τ_H determines the energy loss rate due to heat conduction and τ_R describes the loss due to radiation and charge exchange. For cylindrical geometry τ_D may be estimated as

$$\tau_D = \left(\frac{a}{2.4}\right)^2 1/D \tag{2}$$

where a is the plasma radius, D the coefficient for diffusion across the magnetic field and a density profile $n(r) \propto J_o(2.4\, r/a)$ was assumed. A similar relation holds for τ_H if D is replaced by the thermal conductivity. To these radial losses one must add end losses.

Heat conduction depends on the displacement of particles between collisions. If it is assumed that a particle may be displaced, a distance Δx with equal probability to the left or the right, in the time τ between collisions, then the particle flux in the direction of a density gradient is

$$\phi^+ = \frac{1}{2}\int_{x_o-\Delta x}^{x_o} 1/\tau\, n(x)\, dx = \frac{1}{2}\left[n(x_o) - \frac{\partial n}{\partial x}\frac{\Delta x}{2}\right]\frac{\Delta x}{\tau} \tag{3}$$

The net flux is $\phi = -D\frac{\partial n}{\partial x}$, where

$$D = \left\langle\frac{(\Delta x)^2}{2\tau}\right\rangle \tag{4}$$

is the diffusion coefficient. Similarly, one obtains for the heat flux $q^+ = \frac{\Delta x}{\tau} n \frac{\partial T}{\partial x} \Delta x$; i.e.

$$q = - \varkappa \frac{\partial T}{\partial x} \tag{5}$$

where

$$\varkappa = n \langle (\Delta x)^2 / \tau \rangle \tag{6}$$

is the coefficient of thermal conductivity.
In the absence of a magnetic field $\Delta x = v\tau$ which gives a diffusion coefficient $D_0 = v_e^2 \tau$ where $v_e = (T_e/m)^{1/2}$ is the electron thermal velocity. In a strong magnetic field, the particle is displaced by the gyro radius, as a result of a collision. Diffusion and heat conduction across a strong magnetic field are thus reduced by the factor $1/(\Omega_e\tau)^2$ where $\Omega_e = \frac{eB}{mc}$ is the gyro frequency. The corresponding confinement times are very large, typically several seconds. The observed diffusion loss across the magnetic field, generally is several orders of magnitude larger and shows a different dependence on the magnetic field as predicted by the classical transport theory.[2, 3, 4]

Bohm et.al.[5] related this <u>anomalous</u> loss to instabilities in the plasma and gave the following phenomenological formula

$$D_B = \frac{1}{16} \frac{cT_e}{eB} = \frac{\Omega_e \tau}{16} D_{cl} \quad . \tag{7}$$

The corresponding confinement time is typically a few micro seconds. It would pose a serious limitation for controlled fusion, had it not been shown very recently that (7) does not have the generally assumed universal character and that much longer confinement times can be achieved. Nevertheless, a large amount of work on plasma turbulence was stimulated by the ideas of Bohm et.al. In order to derive anomalous transport coefficients one must have a nonlinear theory for the wave spectrum which developes as a result of instabilities and one must then be able to relate this spectrum with plasma transport. This is generally a very difficult task [6, 7]. An elementary "derivation" of (7) may be made as follows: The guiding centers of a particle $\tilde{x} = x - v_y/\Omega$, $\tilde{y} = y + v_x/\Omega$ satisfy the equations

$$\frac{d\tilde{x}}{dt} = -\frac{c}{B} E_y \quad ; \quad \frac{d\tilde{y}}{dt} = \frac{c}{B} E_x \tag{8}$$

where $\underline{E}$ is the fluctuating electric field due to instabilities. The corresponding diffusion coefficient is

$$\begin{aligned} D_{yy} &= \frac{c^2}{B^2} \frac{1}{2\tau} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \langle E_x(\tau_1) E_x(\tau_2) \rangle \\ &\approx \frac{c^2}{B^2} \int_0^\infty ds \langle E_x(t) E_x(t-s) \rangle = \\ &= \frac{c^2}{B^2} \langle E_x^2 \rangle \tau_{corr} \end{aligned} \tag{9}$$

where τ_{corr} is the correlation time of the fluctuations, as seen by the moving particle. For strong turbulence τ_{corr} may be identified with the time it takes the particle to diffuse over one wavelength, since after this time it is out of phase with the accelerating electric field. Thus, $\frac{cE\tau_{corr}}{B} \sim 1/k$ and $D \sim \frac{c\phi_{rms}}{B}$ Equ.(7) is obtained if we take a typical value for the root mean square potential $\frac{e\phi_{rms}}{T_e} = \frac{n_1}{n} = 1/16$. A kinetic theory of diffusion, following these lines has been given by Dupree [8].

It may be remarked that the experiment of Bohm et.al found an interpretation within classical transport theory, i.e. making use only of binary collisions between particles. However, anomalous transport plays an important role in plasma physics. Anomalous electrical conductivity and viscosity are present in collisionless shocks. Efficient heating of high temperature plasma may be achieved if instabilities are induced in the plasma, which decrease the electrical conductivity [9]. These topics will be discussed in subsequent lectures.

Considerable progress has been made in controlling plasma instabilities. The unavoidable loss of plasma in open ended systems such as the linear pinch and even the mirror machine lead to the investigation of closed toroidal systems, in which losses can only result due to transport across the confining magnetic field. It was found that these losses considerably exceeded the predictions of classical transport theory, even in situations where instabilities did not play any role. Classical transport theory pertains to straight systems with an uniform magnetic field. The assumption that a small toroidicity and inhomogeneity of the magnetic field does not influence very much the plasma transport processes proved to be incorrect. Neoclassical transport theory which is based upon binary collisions but takes into account the actual magnetic field configuration gives results which differ substantially from classical theory, both in magnitude and magnetic field dependence. The remaining discrepancy with experiments may be due to residual plasma turbulence.

As early as 1951 Tamm [10] and Budker [11] have given a qualitative discussion of diffusion and heat conduction in a toroidal system. They found that, because of the toroidal drift, particles suffer a displacement in a collision which exceeds the Larmor radius.

Pfirsch and Schlüter [12] showed that taking into account the toroidicity, enhances the classical diffusion coefficient by the factor $1 + 8\pi^2/\iota^2$, where ι is the rotational transform of the magnetic field. Shafranov [13] found a similar factor for the thermal conductivity. These investigations were done in the hydrodynamic approximation, which is correct for a collision dominated plasma. In the opposite limit of small collision frequency the same problem was studied first by Galeev and Sagdeev [14]. They found that particles trapped in the magnetic field, so called bananas play the dominant role in transport of a weakly collisional plasma. Neoclassical theory of transport has now become a very active field of investigation.

Anomalous transport in toroidal systems, due to turbulence, has not yet been studied very extensively, but appears to be an important field of investigation [15].

Problems:

1. Derive (2)
2. Show that, more generally, the coefficient for diffusion across the magnetic field is given by $D_{cl} = \dfrac{D_0}{1+(\Omega\tau)^2}$
   Bohm diffusion is obtained as the maximum of this expression. Does this indicate an actual upper limit to diffusion?
3. Give estimates of the classical, neoclassical and anomalous confinement time for a typical Tokomak experiment.

II. Particle Motion in the Drift Approximation

In order to estimate diffusion or heat conductivity in the presence of nonuniform electric and magnetic fields, we have to find the particle orbits under the influence of these fields. Since these orbits are also the characteristics of the kinetic equation of transport, they will also be very useful in a more rigorous investigation of transport processes. The exact equations of motion of a nonrelativistic particle may be derived from the Lagrangian

$$L = mv^2/2 + e/c\, \underline{v} \cdot \underline{A} - e\,\underline{\Phi} \tag{10}$$

or the Hamiltonian

$$H = 1/2m \left(\underline{p} - e/c\, \underline{A} \right)^2 + e\,\underline{\Phi}\,. \tag{11}$$

In these equations $\underline{\Phi}$ is the electrostatic potential and $\underline{A}$ the vector potential, such that the electric and magnetic fields are given by

$$\underline{E} = -1/c \frac{\partial \underline{A}}{\partial t} - \nabla \underline{\Phi} \tag{12}$$

$$\underline{B} = \nabla \times \underline{A}\,. \tag{13}$$

A rigorous solution of the equations of motion for non-uniform and time dependent fields is usually far too difficult. However, symmetry properties of the system lead to rigorous conservation laws which are very useful. If e.g. the potentials Φ and $\mathbf{A}$ are independent of time then the total energy

$$\mathcal{E} = mv^2/2 + e\,\Phi \tag{14}$$

is a constant of motion.

If the Hamiltonian is independent of a space variable ξ then the corresponding canonical momentum $p_\xi = \frac{\partial L}{\partial \dot{\xi}}$ is a constant of motion. A number of toroidal systems, such as the Tokomak or the Levitron are axisymmetric, i.e. symmetric with respect to the azimuthal angle ξ. The corresponding canonical momentum

$$p_\xi = \rho\left[m v_\xi + \frac{e}{c} A_\xi\right] = \mathrm{const} \tag{15}$$

in this case. In (15), ρ is the distance from the axis of symmetry.(major axis of the torus).

We may also recall that particles in quasiperiodic motion possess various <u>adiabatic invariants</u> [16, 17]. The magnetic moment of the Larmor orbit

$$\mu = \frac{m v_\perp^2}{2B}$$

, where $v_\perp$ is the velocity component corresponding to rotation about to the magnetic field , is an important adiabatic invariant. μ remains constant if the fields vary slowly over distances of the order of the Larmor radius and times of the order of a gyroperiod. In such fields the motion of the particle resembles the superposition of a rapid rotation about the guiding center, a motion of the latter along the magnetic field line and a relatively slow drift transverse to the magnetic field.

$$\underline{x} = \underline{r} + \frac{v_\perp}{\Omega}\left[\underline{\xi}_1 \sin\alpha - \underline{\xi}_2 \cos\alpha\right] \tag{16}$$

$$\underline{v} = v_\parallel \underline{\xi}_0 + v_\perp\left[\underline{\xi}_1 \cos\alpha + \underline{\xi}_2 \sin\alpha\right] \tag{17}$$

$$\frac{d\alpha}{dt} = \Omega = -\frac{eB}{mc} \tag{18}$$

In these equations $\underline{r}$, is the position of the guiding center, α is the phase angle, $\underline{\xi}_0$ is a unit vector along the magnetic field line, $\underline{\xi}_1$ is the normal

$$(\underline{\xi}_0 \cdot \nabla)\underline{\xi}_0 = \underline{\xi}_1 / R \tag{19}$$

where R is the radius of curvature and

$$\underline{\xi}_2 = \underline{\xi}_0 \times \underline{\xi}_1 \tag{20}$$

is the binormal to the line of force.

Equations (16-18) hold rigorously only for a uniform magnetic field. The presence of two timescales for the motion in weakly inhomogeneous and time dependent fields allows one to derive approximate equations of motion for the drift variables $\underline{r}, \bar{v}_\perp, \bar{v}_\parallel, \bar{\alpha}$ [17, 18]. The bar indicates an average over the rapid motion of frequency Ω. The drift variables differ from the variables $\underline{x}, v_\perp, v_\parallel, \alpha$ by terms of order $1/\Omega$, cf. (16-18). The differences can be expressed in terms of the drift variables, e.g.

$v_\perp = \bar{v}_\perp + \delta v_\perp(r, \bar{v}_\perp, \bar{v}_\parallel, \bar{\alpha})$ where $\delta v_\perp = O(1/\Omega)$.

The equations of motion for the drift variables do not depend upon the phase angle and are of first order, instead of the original second order equations. The derivation of these drift equations is rather involved. For our purpose it is sufficient to replace the actual particle by a fictitious particle of mass m, charge e and magnetic moment μ. The force on this particle is $\underline{F} = e[\underline{E} + (\underline{v}/c) \times \underline{B}] - \mu \nabla B$ which allows us to write down the equations of motion for the guiding center

$$\frac{d\underline{r}}{dt} = v_\parallel \underline{\tau}_0 + \underline{v}_D \quad . \tag{21}$$

The drift velocity is

$$\underline{v}_D = \underline{\tau}_0/\Omega \times \left[\frac{e}{m}\underline{E} - v_\parallel^2 (\underline{\tau}_0 \cdot \nabla)\underline{\tau}_0 - \frac{v_\perp^2}{2B} \nabla B\right] \tag{22}$$

where $\Omega = -\frac{eB}{mc}$ and $v_\parallel, v_\perp$ are the average velocity components in the direction parallel and perpendicular to $\underline{B}$ at the location of the guiding center. The various terms in (22) represent the drift due to the electric field, the curvature of $\underline{B}$ (centrifugal force) and the gradient of B. The equations of motion are completed by the energy equation

$$\frac{d\, m v^2/2}{dt} = e\underline{E}\cdot\frac{d\underline{r}}{dt} + \frac{m v_\perp^2}{2B}\frac{\partial B}{\partial t} \tag{23}$$

where $v^2 = v_\parallel^2 + v_\perp^2$

and the conservation law

$$\mu = \frac{m v_\perp^2}{2B} = \text{const}. \tag{24}$$

For transport processes in a strong magnetic field, such that the characteristic scale length $L \gg \frac{v_\perp}{\Omega}$ and the characteristic time $T \gg 1/\Omega$ we may consider not the particle transport itself but the transport of guiding centers. Comparing with the case of a uniform magnetic field, in which there is no guiding center drift, we see that in this way we do not obtain the fluxes which are connected with the rapid gyration of the particles. These fluxes, however, are independent of the structure of the magnetic field, and are not of interest to us at present. They could be obtained by using the relations between the particle coordinates and the drift variables [19]. However, since the drift kinetic equations are correct only to first order in $1/\Omega$, one cannot take into account corrections of the transport processes due to viscosity etc, which are of order $1/\Omega^4$.

Equation (22) can be put into a somewhat more useful form

$$\underline{v}_D = \frac{c}{B^2}(\underline{E} \times \underline{B}) + \frac{mc}{eB^3}(v_\parallel^2 + v_\perp^2/2)\ (\underline{B} \times \nabla B) \quad (25)$$
$$+ \frac{mcv_\parallel^2}{eB^4}[\underline{B} \times \mathrm{curl}\, \underline{B}] \times \underline{B} .$$

For the magnetic field outside conductors $\mathrm{curl}\, \underline{B} = \frac{4\pi}{c}\underline{j} = 0$ and the last term in (25) does not contribute. If furthermore, the electric and magnetic fields are time independent then (23) becomes the conservation law for the total energy

$$\mathcal{E} = mv^2/2 + e\Phi = mv_\parallel^2/2 + \mu B + e\Phi = \mathrm{const.} \quad (26)$$

In terms of the constants of motion $\mathcal{E}, \mu$ and the potential Φ the longitudinal velocity becomes

$$v_\parallel = \pm\left[\frac{2}{m}(\mathcal{E} - e\Phi - \mu B)\right]^{1/2} . \quad (27)$$

Thus, with respect to the longitudinal motion $e\Phi + \mu B$ is the effective potential energy. We immediately see the possibility of magnetic trapping, in addition to electric trapping, for particles of small velocity $v_\parallel$. The transverse drift velocity can also be derived from a "potential":

$$\underline{v}_D = v_{\parallel} \underline{\Sigma}_0 \times \nabla \left({v_0}/{\Omega} \right)_{\varepsilon, \mu} . \tag{28}$$

The spatial gradient has to be taken at constant ε, μ. Equ.(28) includes the drift due to the potential Φ . The drift due to a nonpotential electric field, e.g. the applied electric field in a Tokomak, and a nonpotential magnetic field [last term in (25)] must be added to (28).

Let us now consider the drift motion in a toroidal, axi-symmetric magnetic field, e.g. in systems like the Tokomak or the Levitron. Toroidal coordinates are introduced as shown in Fig.1. R is the major radius, a the minor radius, A=R/a the aspect ratio and r/R the toroidicity. The toroidal magnetic field may be represented by that of a straight wire at the axis of symmetry.

$$B_{\xi} = \frac{B_0 R}{\rho} \tag{29}$$

where

$$\rho = R \left[1 + {r}/{R} \cos \delta \right] \tag{30}$$

is the distance from the axis of symmetry. For the magnetic field (29) we obtain from (25) the toroidal drift due to the curvature and gradient of B,

$$\underline{v}_g = - \frac{mc}{eB} \frac{(v_{\parallel}^2 + v_{\perp}^2/2)}{R} \underline{e}_z , \tag{31}$$

The drift velocity is directed along the axis of symmetry, in our case upwards for electrons and downwards for ions. The resulting charge separation would lead to an electric field in the z direction and thus an ExB drift which is directed radially outward. Plasma equilibrium can be achieved by adding a poloidal magnetic field B_δ , usually $\Theta = B_\delta / B_\zeta \ll 1$. B_δ provides the magnetic field with a rotational transform

$$\iota(r) = \frac{2\pi R}{r} \frac{B_\delta}{B_\zeta} = \frac{2\pi R}{r} \Theta(r) \tag{32}$$

$\iota(r)$ is the angle of rotation in one transit about the symmetry axis. The magnetic field lines form a system of nested surfaces, which in our case have circular cross section $r = \mathrm{const}$. Particles deviate from these magnetic surfaces as a result of the toroidal drift.
Taking components of (21-22) we obtain

$$\frac{dr}{dt} = v_g(r,\delta) \sin\delta \tag{33}$$

$$r\frac{d\delta}{dt} = v_g(r,\delta)\cos\delta + \Theta v_\parallel + v_E \tag{34}$$

where v_g is given by (31) and

$$v_E = \frac{c}{B}\frac{\partial\Phi}{\partial r} \tag{35}$$

assuming a radial electric field only and neglecting also the drift due to B_δ.

From the conservation law (26) we obtain the longitudinal velocity

$$v_{\parallel} = \pm \hat{v}_{\parallel} \sqrt{2\kappa^2 - 1 + \cos\delta} \tag{36}$$

where

$$\hat{v}_{\parallel} = \left(\frac{2\mu B_o r}{m R} \right)^{1/2} \tag{37}$$

and

$$2\kappa^2 = \frac{m v_{\parallel}^2 (r, \delta = 0)}{2\mu B_o r} \tag{38}$$

We want to solve these equations approximately by treating the toroidal drift as a small quantity. Thus, we must require

$$\frac{v_g}{\Theta v_{\parallel}} \sim \frac{v^2}{\Omega} \frac{2\pi R}{R L r v} \sim \frac{2\pi}{L} \frac{r_L}{r} \ll 1 \tag{39}$$

Usually $q = \frac{2\pi}{\iota} = 1\text{-}10$, but $r_L \ll r$. Neglecting the toroidal drift in (34) we obtain an approximate equation for δ :

$$r_o \frac{d\delta}{dt} = \Theta \hat{v}_{\parallel} \sqrt{2\kappa^2 - 1 + \cos\delta} \quad + v_E (r_o) \tag{40}$$

where $r = r_o$ defines the magnetic surface.

For the case $v_E = 0$ the trapping criterion is $\kappa^2 < 1$
The motion of trapped particles is periodic with period

$$T = \frac{4r_0}{v_\parallel \Theta}\int_0^{\delta_m} \frac{d\delta}{\sqrt{2\kappa^2-1+\cos\delta}} = \frac{4r_0\sqrt{2}}{\hat{v}_\parallel \Theta} K(\kappa) \qquad (41)$$

where δ_m is the maximum angle, given by $1-\cos\delta_m = 2\kappa^2$
The radial displacement may be found from

$$\frac{dr}{d\delta} = \frac{r\frac{dr}{dt}}{r\frac{d\delta}{dt}} = \frac{r_0 v_g \sin\delta}{\pm\Theta\hat{v}_\parallel\sqrt{2\kappa^2-1+\cos\delta}+v_E} \qquad (42)$$

For the case $v_E = 0$ and trapped particles $\kappa^2<1$, $v_\parallel < v_\perp$, (42) gives

$$\Delta r = r - r_0 = \pm\frac{2r_0 v_g}{\Theta v_\parallel}\sqrt{2\kappa^2-1+\cos\delta} + \text{const} \qquad (43)$$

The drift surface deviates from the magnetic surface by approximately

$$\Delta r = \frac{2r_0 v_g}{\Theta v_\parallel}\sqrt{2\kappa^2} \approx \frac{1}{\Theta\Omega}(r/R)^{1/2}\left(\frac{pB_0}{m}\right)^{1/2} \qquad (44)$$
$$\approx 2\pi/\iota\,(R/r)^{1/2} r_L$$

For transiting particles, $v_\parallel \approx \text{const}$,

$$\frac{dr}{d\delta} \approx \frac{r_0 v_g}{\Theta v_\parallel + v_E}\sin\delta \qquad (45)$$

The dispacement $\Delta r \approx \frac{2\pi}{\iota} r_L$ is reduced by a factor $(r/R)^{1/2}$ from that of trapped particles. It is not difficult to see the form of the orbit, if it is recalled that particles follow essentially the magnetic field lines, except for the toroidal drift which is always directed along the axis of symmetry. The projection of the orbit of trapped particles on the (r, δ) plane has the form of a banana, [20] The displacement Δr is largest for barely trapped particles, since they have the longest period of motion, during which the toroidal drift can act.

The effect of a <u>radial electric field</u> is to provide an additional rotational transform, as may be seen from (34). If $v_E/\Theta \gtrsim v_{th}$, then the number of trapped particles will be negligible, since the corresponding velocities $v_\parallel$ are now in the tail of the distribution function, instead of the bulk $v_\parallel \approx 0$. From (45), e.g., we also see that the electric drift may reduce the excursion of the guiding centers. For small rotational transform, the ion bananas may intersect the plasma boundary, creating a loss cone, preferentially for barely trapped ions. Because of the large mass ratio, the loss of electrons in this way is usually not significant. Thus a potential well developes, which by the effects just discussed reduces the ion loss [21]. The <u>selfconsistent potential</u> $\Phi(r, \delta)$ which arises from the guiding center drifts has been studied numerically by Smith and Bishop [22]. They have shown that the charge separation due to the drifts is only partially cancelled by electron streaming along the magnetic field lines, even in a collisionless plasma. Thus, the selfconsistent potential is not azimuthally symmetric [23].

The driving toroidal electric field in systems like the Tokomak produces an inward drift of trapped particles of velocity

$$v_D = \frac{-cE_\zeta}{B_\delta} \tag{46}$$

which is much larger than the drift $\frac{-cE_\zeta B_\delta}{B^2}$ in a straight geometry [24]. This toroidal pinch effect may be conveniently analyzed by making use of the conservation of angular momentum p_ζ , equ. (15). Applying Stokes theorem to the cross section $z = z_0, \rho = \rho_0$ we find

$$\begin{aligned} \oint \underline{A} \cdot d\underline{s} &= 2\pi A_\zeta \rho_0 = -\oint \operatorname{curl} \underline{A} \cdot d\underline{S} = -\Phi_z \\ &= \int_0^{\rho_0} 2\pi\rho\, B_z\, d\rho \equiv 2\pi R \Psi . \end{aligned} \tag{47}$$

$\Psi(\rho)$ is the flux per unit length of magnetic axis and is a constant on a given magnetic surface $\Psi = \Psi(r)$. Considering now the displacement Δr of the banana turning points ($v_\parallel \approx 0$) in time Δt, we have from (15)

$$[\rho A_\zeta](t) = [\rho A_\zeta](t + \Delta t)$$

or with (47), for $z = 0$ ($-B_z \to B_\delta$)

$$-\Delta r \rho B_\delta + \Delta t \rho \frac{\partial A_\zeta}{\partial t} = 0$$

thus $\quad \frac{\Delta r}{\Delta t} = -cE_\zeta / B_\delta .$

The electric field E_δ which arises from the potential $\phi(r, \delta)$ does not lead to a net drift of bananas. The $E \times B$ drift is exactly compensated by the imbalance in magnetic drift.

The conservation laws for $\mathcal{E}, \mu, p_\zeta$ determine the guiding center orbits in an axisymmetric system. For nonsymmetric systems the conservation of p_ζ must be replaced by the adiabatic invariance of

$$J = \oint m v_\parallel \, ds \tag{48}$$

where the integration is between the two mirror points on a given magnetic field line [25]. The <u>longitudinal adiabatic invariant</u> is conserved as the particle drifts slowly to a neighbouring field line [26]. Bananas must lie on surfaces of constant J. The average drift velocity of a banana on this surface may be expressed in terms of the gradient of J. Introducing spatial coordinates α, β, s such that $\underline{B} = \nabla\alpha \times \nabla\beta$ and s is the length along the field line (α, β) it has been shown [26] that the banana drift is given by

$$\langle \dot{\alpha} \rangle = \frac{c}{eT} \frac{\partial J(\alpha, \beta, \mathcal{E}, \mu)}{\partial \beta} \tag{49}$$

$$\langle \dot{\beta} \rangle = -\frac{c}{eT} \frac{\partial J}{\partial \alpha} \tag{50}$$

where the average is over the period of trapped particle motion

$$T = \frac{\partial J}{\partial \mathcal{E}} \tag{51}$$

In axisymmetric systems bananas drift along the magnetic surfaces. In nonsymmetric systems the banana itself may be trapped, thus forming a superbanana. The formation of bananas and the trapping of bananas may be the result of various inhomogeneities. In systems like the stellarator or the bumpy torus one must distinguish two types of magnetic field inhomogeneities; one due to the helical winding and another due to toroidicity. The magnetic field may be represented by

$$\underline{B} = \frac{B_0}{1 + r/R \cos\theta}\, \underline{e}_\zeta + \nabla \Phi_m \tag{52}$$

where the magnetic potential Φ_m has the form

$$\Phi_m = \gamma \alpha^{-1} B_0 I_n(n\alpha r) \sin n(\delta - \alpha \zeta) \tag{53}$$

for the n-winding stellarator and

$$\Phi_m = -\gamma \alpha^{-1} B_0 I_0(\alpha r) \sin \alpha \zeta \tag{54}$$

for the bumpy torus. In these equ. I_n is the modified Bessel function of n^{th} order, and γ, α characterize the field strength and spatial periodicity of the stabilizing magnetic field respectively. In addition to the usual group of particles which are trapped in the toroidal field one has now so-called localized particles which are trapped within a spatial period of the stabilizing magnetic field. Trapping of particles and the formation of superbananas can also take place in electric field inhomogeneities. One can have E trapped B bananas etc. [25]. The appearence of superbananas leads to intensified particle displacement.

Problems

1. Derive equ.(25).

2. Derive equ.(28) and consider the effect of an additional nonpotential electric field.

3. Find the particle motion in a corrugated magnetic field $B = B_0(1 - \epsilon \cos k s)$, where s is the distance along the line of force.

4. Show that the axial flux Ψ = const on a magnetic surface.

5. For the model toroidal field [12]

$\underline{B} = B_0/h\,[0,\ \Theta,\ 1]$; $h = 1 + (r/R)\cos\delta$

a) show that: div B = 0 requires $\Theta = \Theta(r)$

b) determine curl B (plasma current)

c) find $\Psi(r)$ and the vector potential $\underline{A}$

d) determine the additional drift velocities due to the poloidal field, assuming an uniform current density.

6. a) Show that for the representation $\underline{B} = \nabla \times \underline{A} = \nabla\alpha \times \nabla\beta$ of the magnetic field:

$\operatorname{div} \underline{B} = 0$; $A = \alpha \nabla \beta$; $\Phi \equiv \oint \underline{B} \cdot d\underline{S} = \oint \underline{A} \cdot d\underline{\ell}$

$= \oint \alpha\, d\beta = \int d\alpha\, d\beta$

b) The poloidal field in axisymmetric systems may be represented by $B_\chi = \nabla\Psi \times \nabla\zeta = \nabla\chi$.

Find the expressions for the magnetic surfaces, the surface and the volume elements in the orthogonal coordinate system formed by $\Psi,\ \chi,\ \zeta$.

c) Show that for our model toroidal field (problem 6)

$d\Psi = -B_0\,\Theta(r)\,dr$, $d\chi = B_\delta\, r\, d\delta$.

7. Show that equ.(49-51) can be written in canonical form with the Hamiltonian $H = c\,\mathcal{E}/e$, $\mathcal{E} = \mathcal{E}(\alpha, \beta, J, \mu)$

8. The magnetic field of a bumpy torus does not possess rotational transform. Give a qualitative discussion of single particle confinement on surfaces of constant flux.

III. Neoclassical Transport in Axisymmetric Toroidal Systems

3.1. Elementary kinetic theory

The orbits derived above may be used to estimate the transport coefficients in a toroidal magnetic field. The displacement of particles between collisions is essentially given by the maximum displacement of the guiding centers from the magnetic surfaces.

The diffusion coefficient due to trapped electrons thus becomes $D = \frac{(\Delta r)^2}{\tau}\frac{\Delta n}{n}$ where Δr is given by (44), $\frac{\Delta n}{n} \approx \frac{v_\perp}{v_e} \approx \left(r/R\right)^{1/2}$ is the fraction of trapped electrons and the effective collision time is $\tau = \tau_e\left(\frac{v_\parallel}{v_e}\right)^2 = \tau_e\, r/R$ noting that Coulomb collisions consist of a large number of small deflections. τ_e is the 90° deflection time.

The diffusion coefficient due to bananas becomes

$$D_{ban} \approx \frac{r_e^2}{\tau_e}\left(\frac{2\pi}{\iota}\right)^2\left(R/r\right)^{3/2} = \frac{r_\delta^2}{\tau_e}\,\epsilon^{1/2} \qquad (55)$$

where r_δ is the electron Larmor radius in the field B_δ.

For the transiting particles $\frac{\Delta n}{n} \sim 1$, $\Delta r \approx \frac{2\pi}{\iota} r_e$ and $\tau \approx \tau_e$

thus

$$D_{transit} \approx \frac{r_e^2}{\tau_e}\left(\frac{2\pi}{\iota}\right)^2 = \frac{r_\delta^2}{\tau_e}\,\epsilon^2 \qquad (56)$$

The banana diffusion coefficient exceeds (56) by the large factor $(R/r)^{3/2}$

However, for banana diffusion to take place, we must require that the particles suffer only few collisions during the period T of trapped particle motion, thus

$$T/\tau \approx \frac{L}{v_{\parallel}\tau_e}\left(\frac{v_e}{v_{\parallel}}\right)^2 \approx \frac{L}{\lambda}\left(\frac{R}{r}\right)^{3/2} < 1 \tag{57}$$

where $L = 2\pi R\,(2\pi/\iota)$ (58)

is the connection length.

and $\lambda = v_e\tau_e$. Galeev and Sagdeev[14] have shown that the diffusion coefficient in the banana regime $L/\lambda \lesssim (r/R)^{3/2}$ agrees with our simple estimate (55), except for a factor 1.6.

In the classical regime $\frac{L}{\lambda} \gtrsim 1$ all particles will be dominated by collisions. Fluid equations are valid in this regime. Pfirsch and Schlüter found a diffusion coefficient which is essentially given by (56). In the range $1 \gtrsim \frac{L}{\lambda} \gtrsim (r/R)^{3/2}$ bananas are strongly affected by collisions, while the bulk of the plasma may be considered collisionless. The diffusion coefficient in this plateau regime is independent of the collision frequency and may be estimated, if τ_e in (56) is replaced by the transit time L/v_e

The situation is very similar to damping of waves in a plasma. The plateau regime, e.g. corresponds to the regime of Landau damping in a collisionsless plasma.

Heat conduction and the toroidal pinch effect can be discussed in a similar manner, but it is clear that a rigorous analysis requires the use of a kinetic description in terms of distribution functions. From kinetic theory we will also see that diffusion in axisymmetric systems is ambipolar, whereas at first glance it would appear that ion diffusion is $(M/m)^{1/2}$ times greater than electron diffusion.

3.2. The drift kinetic equation

As discussed in Section II, in the case of a strong magnetic field we may consider the transport of guiding centers, rather than the transport of particles. Guiding centers may be characterized by the position $\underline{r}$ and the velocities $v_\perp$ and $v_\parallel$. For the velocity coordinates one may equally use the energy $\mathcal{E} = mv^2/2 + e\Phi$ magnetic moment $\mu = \frac{mv_\perp^2}{2B}$ and the sign of $v_\parallel = \pm\left[\frac{2}{m}(\mathcal{E} - \mu B - e\Phi)\right]^{1/2}$.

We may introduce a guiding center distribution, such that $dN = f(\underline{r}, \mathcal{E}, \mu)\,d\underline{r}\,d\underline{v}$ is the number of guiding centers in $d\underline{r}d\underline{v}$, where

$$d\underline{v} = 2\pi v_\perp dv_\perp dv_\parallel = 2\pi B\, d\mathcal{E}\, d\mu / (m^2 |v_\parallel|) \qquad (59)$$

In (59) a summation over the sign of $v_\parallel$ must be included.

If the drift equations (21-24) are valid, then f satisfies the <u>drift kinetic equation</u>

$$\frac{\partial f}{\partial t} + [v_\parallel \tau_0 + \underline{v}_D] \cdot \frac{\partial f}{\partial r} + \frac{d\mathcal{E}}{dt}\frac{\partial f}{\partial \mathcal{E}} = \frac{\delta f}{\delta t} \qquad (60)$$

where $\frac{\delta f}{\delta t}$ is the collision term. In the case of potential electric and magnetic fields we may substitute (28) for $\underline{v}_D$ and $\frac{d\mathcal{E}}{dt} = 0$.

We may allow for an additional toroidal electric field E. The drift approximation requires $E \ll (v/c)B$. Equ.(60) becomes

$$\frac{\partial f}{\partial t} + v_\parallel \Theta \frac{1}{r}\frac{\partial f}{\partial \delta} + v_\parallel \frac{\partial}{\partial r}\left(v_\parallel/\Omega\right)\frac{1}{r}\frac{\partial f}{\partial \delta} - v_\parallel \frac{1}{r}\frac{\partial}{\partial \delta}\left(v_\parallel/\Omega\right)\frac{\partial f}{\partial r} + eEv_\parallel \frac{\partial f}{\partial \mathcal{E}} = \frac{\delta f}{\delta t}. \tag{61}$$

From the solution of (61) we may find the flux of guiding centers through a magnetic surface $r = $ const.

$$\bar{\Gamma} = 4\pi^2 Rr \int \frac{d\delta}{2\pi} \left\langle \left(1+\frac{r}{R}\cos\delta\right) n(r,\delta) v_{Dr}(r,\delta)\right\rangle \equiv 4\pi^2 Rr\Gamma \tag{62}$$

Using (28) for the radial drift velocity, we have

$$\Gamma = -\int \frac{d\delta}{2\pi}\left(1 + r/R\cos\delta\right)\int d\underline{v}\, f(\underline{r},\mathcal{E},\mu)\, v_\parallel \frac{1}{r}\frac{\partial}{\partial\delta}\left(v_\parallel/\Omega\right). \tag{62a}$$

An integration by parts gives $[\Omega = \Omega_0/(1+r/R\cos\delta)]$

$$\Gamma = \int \frac{d\delta}{2\pi}\left(1+r/R\cos\delta\right)^2 \int d\underline{v}\, \frac{v_\parallel^2}{\Omega_0}\frac{1}{r}\frac{\partial f}{\partial \delta}. \tag{63}$$

For $\frac{\partial f}{\partial \delta}$ we may substitute from (61). Noting that v_{Dr} is even in $v_\parallel$

$v_{D\delta} \ll \Theta v_\parallel$, $\frac{\partial f}{\partial t} \ll \frac{\delta f}{\delta t}$

and f approximately even in $v_\parallel$, we obtain approximately [c.f. problem III.3) and (67)]

$$\Gamma = \int \frac{d\delta}{2\pi}\left(1+r/R\cos\delta\right)^2 \int d\underline{v}\left(v_\parallel/\Theta\Omega_0\right)\left[\frac{\delta f}{\delta t} - eEv_\parallel\frac{\partial f}{\partial \mathcal{E}}\right]. \tag{64}$$

The first term in the square bracket gives a contribution to the flux which is proportional to the momentum transfer in collisions.

Since collisions between particles of a given species do not change their total momentum, self diffusion does not exist in our order of approximation and the fluxes of electron and ion charge are equal [19, 27, 28] (ambipolar diffusion). Performing the velocity integration for the second term in (64) gives $-\frac{nEc}{B_\delta}$. An additional contribution to the toroidal pinch effect arises however from the $\frac{\delta f}{\delta t}$ term [29].

3.3. Solution of the drift kinetic equation.

The problem of neoclassical diffusion has now been reduced to the solution of (61) and the computation of the flux from (63). The solution of (61) may be written as an expansion in terms of the small parameters of the problem. The drift approximation already requires that the Larmor radius is small compared to the scale length of the magnetic field, $\frac{r_L}{r} \ll 1$. We also assume that the toroidicity $\epsilon = r/R \ll 1$ and that $q = \frac{2\pi}{\iota} = O(1)$, i.e. $\Theta = \frac{B_\delta}{B} = \frac{\iota r}{2\pi R} = O(\epsilon)$. The displacement of a particle from the magnetic surface is also assumed small $\frac{\Delta r}{r} = \frac{2\pi/\iota \; r_L \; v/v_\parallel}{r} \ll 1$. The effect of collisions is characterized by the parameter λ/L, cf. Sec. 3.1.

The zero order distribution may be taken to be locally Maxwellian.

$$f_o(\varepsilon\mu,r) = \frac{N(r)}{(2\pi v_{th}^2)^{3/2}} \exp\{-\varepsilon/T\} \tag{65}$$

where $v_{th}^2 = T/m$. The corresponding density is

$$n = N(r) \exp\{e\phi/T\} \quad . \tag{66}$$

The drift terms in equ.(61) are smaller then the free streaming terms by the factor $\frac{\Delta r}{r}$. Equ.(61) to zero order in the Larmor radius prescribes T = T(r), N = N(r) in (65) but allows for arbitrary $\phi = \phi(r, \delta)$. Quasineutrality to zero order requires $\phi_o = \phi_o(r)$. Equ.(o1) to first order in $\frac{\Delta r}{r}$ becomes

$$\oplus v_\parallel \frac{1}{r}\frac{\partial f_1}{\partial \delta} - v_\parallel \frac{1}{r}\frac{\partial}{\partial \delta}(v_\parallel/\Omega)\frac{\partial f_o}{\partial r} + eEv_\parallel\frac{\partial f_o}{\partial \varepsilon} = \frac{\delta f_1}{\delta t} \tag{67}$$

where $\frac{\delta f_1}{\delta t}$ is the linearized collision term.

Because diffusion is ambipolar it is sufficient to calculate the electron flux. However, equs. (67) for electrons and ions are generally coupled by the collision term

$$\frac{\delta f_{1e}}{\delta t} = C_{ee}(f_{oe}, f_{1e}) + C_{ei}(f_{1e}, f_{oi}) + C_{ei}(f_{oe}, f_{1i}) \tag{68}$$

. f_{1i} is essentially determined by the driving term $\frac{\partial f_{oi}}{\partial r}$.

Requiring that $\frac{\partial f_{oi}}{\partial r} = 0$ to lowest order in $(m/M)^{1/2}$ decouples the electron and ion equations. It must be recalled that $\frac{\partial f_o}{\partial r}$ is to be taken at constant ε, μ, thus from (65-66)

$$\left.\frac{\partial f_o}{\partial r}\right|_{\varepsilon,\mu} = \left[\frac{1}{n}\frac{\partial n}{\partial r} + \frac{e}{T}\frac{\partial \Phi}{\partial r}\right] f_o .$$

Requiring $\frac{\partial f_{oi}}{\partial r} = 0$, means that a frame is chosen in which the mean ion velocity in the δ direction vanishes. For the electrons we obtain then

$$\left.\frac{\partial f_{oe}}{\partial r}\right|_{\varepsilon,\mu} = \left(1 + T_i/T_e\right) \frac{1}{n}\frac{\partial n}{\partial r} f_{oe} . \tag{69}$$

For simplicity, it was assumed that T_e, T_i are independent of r.

The parameter λ/L determines how the collision term compares to the terms on the left hand side of (67). In the banana regime of weak collisions the left hand side is dominant. Thus (67) becomes to two orders in the collision frequency

$$\Theta v_{\parallel} \frac{\partial f_i^o}{\partial \delta} - v_{\parallel} \frac{1}{r} \frac{\partial}{\partial \delta}\left(v_{\parallel}/\Omega\right) \frac{\partial f_o}{\partial r} = 0 \tag{70}$$

and

$$\Theta v_{\parallel} \frac{1}{r} \frac{\partial f_i^1}{\partial \delta} + e E v_{\parallel} \frac{\partial f_o}{\partial \varepsilon} = \frac{\delta f_i^o}{\delta t} . \tag{71}$$

The solution of (70) is

$$f_1^0 = \frac{v_{\parallel}}{\Theta\Omega}\frac{\partial f_0}{\partial r} + g(\varepsilon, \mu, r) \tag{72}$$

where $\frac{\partial g}{\partial s} = 0$. g must satisfy the solubility condition for (71) which is obtained by integrating this equation over s . In the case of trapped particles the integration is between the turning points. g may be determined from the resulting equations if the appropriate boundary conditions are known. These can be obtained by considering (67) in the transition region between trapped and transiting particles. A very similar problem appears if one considers the effect of collisions on the damping of a finite amplitude wave. The resulting flux in the banana regime is [29]

$$\Gamma = -1.6\,(r/R)^{1/2}\left[\rho_\theta^2/\tau_e \frac{dn}{dr} + \frac{cnE}{B_\theta}\right] \tag{73}$$

where

$$\rho_\theta^2 = mc^2(T_e + T_i)/(e^2 B_\theta^2)$$

and the second term describes the toroidal pinch effect.

In the plateau regime $(r/R)^{3/2} < \frac{L}{\lambda} < 1$ slow transiting particles $v_{\parallel}^2 < v_{\perp}^2$ play the dominant role in transport. We may replace the collision term in (67) by $-\nu f_1$ where ν is the effective collision frequency. It is also convenient to change from the μ coordinate to the coordinate $v_\parallel$. Considering only the effect of toroidicity we obtain

$$\Theta v_\parallel \frac{1}{r}\frac{\partial f_1}{\partial \delta} + v_g \sin\delta \frac{\partial f_0}{\partial r} = -\nu f_1 \tag{74}$$

where $v_g = \dfrac{v_\perp^2/2 + v_\parallel^2}{R\,\Omega}$.

Equ.(74) has the solution $f_1 = f_1^+ + f_1^-$

where

$$f_1^\pm = \pm \frac{-v_g \frac{\partial f_0}{\partial r} e^{\pm i\delta}}{2i[\nu \pm i/r\, v_\parallel \Theta]} \tag{75}$$.

For small ν we have the usual resonant denominator, thus

$$f_1 = -v_g \frac{\partial f_0}{\partial r}\left[\pi\,\delta(v_\parallel \Theta/r)\sin\delta - P\left(\frac{r}{v_\parallel \Theta}\right)\cos\delta\right] \tag{76}$$.

f_1 may be substituted into (62) to compute the flux. Only the term $\propto \sin\delta$ contributes, since $v_{Dr} = v_g \sin\delta$

The result is

$$\Gamma = -(\pi/2)^{1/2}\left(1 + T_i/T_e\right)\frac{r_e^2 \nu_e}{\Theta}\left(r/R\right)^2 \frac{1}{r}\frac{\partial n}{\partial r} \tag{77}$$.

The contribution from the toroidal electric field (pinch effect) must be added to (77) [29].

In the classical regime of large collision frequency $L/\lambda \gtrsim 1$ (67) may be expanded as follows [28, 31]

$$\frac{\delta f_1^0}{\delta t} \equiv C(f_0, f_1^0) = 0 \tag{78}$$

$$\Theta v_{\parallel} \frac{1}{r} \frac{\partial f_1^0}{\partial \delta} = \frac{\delta f_1^1}{\delta t} \tag{79}$$

and

$$\Theta v_{\parallel} \frac{1}{r} \frac{\partial f_1^1}{\partial \delta} - v_{\parallel} \frac{1}{r} \frac{\partial}{\partial \delta} (v_{\parallel}/\Omega) \frac{\partial f_0}{\partial r} = \frac{\delta f_1^2}{\delta t} . \tag{80}$$

The solution of (78) is

$$f_1^0 = \frac{N_1^0(r, \delta)}{N_0(r)} f_0 \tag{81}$$

where f_0 is a Maxwellian distribution (65).
Equ.(79) for f_1^1 is essentially the problem solved by Spitzer and Harm[2] for the electrical conductivity. Integrating (80) over velocity space eliminates the collision term. One obtains a differential equation for $N_1^0(r, \delta)$ which depends only on the longitudinal electrical conductivity and the magnetic fields. We shall derive these equations for general magnetic field structures, starting from the fluid equations.

Similar calculations may be performed for the thermal conductivity [14, 27]. Since the excursions from the magnetic surface are much larger for ions than for electrons, the modification of the ion thermal conductivity is most significant.

Extending the calculations to nonaxisymmetric systems such as the stellarator or the bumpy torus introduces some new features [23, 27, 31, 32]. Diffusion is now ambipolar only for a certain radial electric field. There is self diffusion, i.e. the diffusion coefficient depends on the total collision frequency, $\nu_e = \nu_{ee} + \nu_{ei} + \nu_{en}$. The dependence on magnetic field strength and collision frequency is also different from the axisymmetric case, due to the occurence of transiting bananas and trapped bananas (superbananas). The displacement of the bananas from the surfaces J = const , is independent of field strength and may be very large, c.f. Sec. II . The maximum enhancement of diffusion occurs then the collision frequency ν equals the frequency of banana drift motion ω_D . In the axisymmetric case, we had maximum enhancement for $\nu \sim \omega_b$, the bounce frequency of guiding center motion.

Problems

1. Derive equ.(61). Show that all terms which arise from the toroidal electric field $E \ll v/c\, B$ are negligible, except for the term retained in (61). The time dependence of B and ϕ was assumed to be of higher order, in general one must add $\left[e \frac{\partial \phi}{\partial t} + \mu \frac{\partial B}{\partial t} \right] \frac{\partial f}{\partial \varepsilon}$ on the l.h.s. of (61)

2. Prove (63)

3. The expression (64) for follows from the linearized equation (67).

   a) Show that one obtains from (61) the exact (within the drift approximation) relation (equation of motion)[19]

$$\Gamma = \frac{1}{\Theta\Omega_0}\Big[\Big\langle h^2 v_\parallel\Big(\frac{\delta f}{\delta t} - eE v_\parallel \frac{\partial f}{\partial \varepsilon}\Big)\Big\rangle_v^\delta - \frac{\partial}{\partial t}\langle h^2 v_\parallel f\rangle_v^\delta - \frac{1}{r}\frac{\partial}{\partial r} r \langle h^2 v_\parallel v_{Dr} f\rangle_v^\delta\Big]$$

   where $\langle\ \ \rangle_v^\delta$ indicates an integration over $\underline{v}$ and an average over δ and $\Omega = \Omega_0/h$, $h = 1 + r/R\cos\delta$

   b) Modify the distribution function f_1, equ. (76), such as to include a radial electric field $E_r \sim \frac{T_e}{er}$ and show that the last term in the above expression for Γ is of order[19]

$$\frac{cE_r}{r\Omega B\Theta^2}\Gamma \sim \frac{r_L^2}{r^2\Theta^2}\Gamma$$

4. a) From the drift kinetic equation (61) prove the continuity equation for guiding centers: $\frac{\partial n}{\partial t} + \nabla\cdot\hat{\underline{\Gamma}} = 0$ where $\hat{\underline{\Gamma}} = \int d\underline{v}\,[v_\parallel \underline{\tau}_0 + \underline{v}_D] f$ is the guiding center flux.

   b) Show that the flux Γ, equ. (62), through a magnetic surface r = const, satisfies $\frac{\partial\langle n\rangle}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\Gamma = 0$ where $\langle n\rangle = \int \frac{d\delta}{2\pi}(1 + r/R\cos\delta)\, n(r,\delta)$.

   c) The above result may be generalized for arbitrary poloidal fields. Using the coordinates and results of problem II.7b, c show that[28]

$$\frac{\partial\langle n\rangle}{\partial t} + \Big(\oint \frac{d\chi}{B\chi}\Big)^{-1}\frac{\partial \Gamma}{\partial \psi} = 0$$

where $\langle n \rangle = \left(\oint \frac{dx}{B\chi^2} \right)^{-1} \int \frac{dx}{B\chi^2} \, n(\psi, x)$

and Γ is the inward flux per unit length of magnetic axis.

$$\Gamma = \int \frac{dx}{B\chi^2} \nabla\psi \cdot \int d\underline{v} \, \underline{v}_D f = B_0 \oint dx \int d\underline{v} \frac{v_\parallel}{B} \frac{\partial}{\partial x} (v_\parallel / \Omega)$$

$$= - B_0 \oint dx \int d\underline{v} \frac{v_\parallel^2}{B\Omega} \frac{\partial f}{\partial x} , \quad \text{(cf. 62a, 63)}.$$

5. Derive the drift kinetic equation in the coordinates $(\varepsilon, v_\parallel, r, s)$. Show that the left hand side corrections of (74) are negligible.

6. From equ.(78-81) derive the differential equation for $N_1(r, s)$.

IV. Transport Processes in a Collision Dominated Plasma (Classical Theory)

IV.1 Transport equations [2, 3, 4]

The macroscopic properties of a plasma, relating to transport of mass, momentum, energy, etc. may be derived by taking moments of the distribution functions $f_a(\underline{x}, \underline{v}, t)$ of each particle species a.

$$n\langle\phi(\underline{x}, t)\rangle = \int d\underline{v}\, \phi(\underline{x}, \underline{v}, t) f(\underline{x}, \underline{v}, t) \quad (82)$$

The first few moments are

$$n(\underline{x}, t) = \int d\underline{v}\, f(\underline{x}, \underline{v}, t) \qquad \text{density} \quad (83)$$

$$\underline{u}(\underline{x}, t) = \frac{1}{n}\int d\underline{v}\, \underline{v} f = \langle\underline{v}\rangle \qquad \text{mean velocity} \quad (84)$$

$$P_{\alpha\beta} = nm\langle w_\alpha w_\beta\rangle \qquad \text{pressure tensor} \quad (85)$$

$$Q_{\alpha\beta\gamma} = nm\langle w_\alpha w_\beta w_\gamma\rangle \qquad \text{heat flux tensor} \quad (86)$$

where $\underline{w} = \underline{v} - \underline{u}$.

The pressure tensor may be written as

$$P_{\alpha\beta} = p\,\delta_{\alpha\beta} + \Pi_{\alpha\beta} \quad (87)$$

where

$$p = nm\langle w^2\rangle/3 \tag{88}$$

is the scalar pressure and $\underline{\underline{\Pi}}$ the viscous stress tensor. The thermal energy density W and temperature T may be defined by

$$W = nm\langle w^2/2\rangle = 3/2\, nT. \tag{89}$$

The heat flux vector is defined by

$$\underline{q} = nm\langle w^2/2\,\underline{w}\rangle \quad . \tag{90}$$

For the total mechanical energy flux we have

$$\underline{T} = nm\langle v^2/2\,\underline{v}\rangle = E.\underline{u} + \underline{\underline{P}}\cdot\underline{u} + \underline{q} \tag{91}$$

where

$$E = nm\langle v^2/2\rangle = nm\, u^2/2 + W \tag{92}$$

is the total mechanical energy density.

Finally, the entropy density may be defined by

$$S = -\int d\underline{v}\, f(\underline{v})\left[\ln f(\underline{v}) - 1\right] \quad . \tag{93}$$

The distribution functions satisfy kinetic equations of the form

$$\frac{\partial f_a}{\partial t} + \underline{v} \cdot \frac{\partial f_a}{\partial \underline{x}} + \frac{e_a}{m_a}\left[\underline{E} + (\underline{v}/c) \times \underline{B}\right] \cdot \frac{\partial f_a}{\partial \underline{v}} = \frac{\delta f_a}{\delta t} . \tag{94}$$

The collision term of a stable plasma is a functional of all distribution functions.

$$\frac{\delta f_a}{\delta t} = \sum_b C_{ab}(f_a, f_b) \quad . \tag{95}$$

The collision term should satisfy the following conservation laws for density, momentum and energy

$$\int d\underline{v}\; C_{ab} = 0 \tag{96}$$

$$\underline{R}_{aa} \equiv \int d\underline{v}\, m_a \underline{v}\, C_{aa} = 0 \tag{97}$$

$$\int dv\, m_a v^2/2\; C_{aa} = 0 \quad . \tag{98}$$

If the collisions are elastic we have furthermore

$$R_{ab} + R_{ba} = \int d\underline{v}\, m_a \underline{v}\, C_{ab} + \int d\underline{v}\, m_b \underline{v}\, C_{ba} = 0 \tag{99}$$

$$\int d\underline{v}\, m_a v^2/2\; C_{ab} + \int d\underline{v}\, m_b v^2/2\; C_{ba} = 0 \quad . \tag{100}$$

$\underline{R}_{ab}$ is the rate of change in momentum of species a in collisions with particles of species b. We may introduce a heat transfer rate by

$$Q_{ab} = \int dv \, m_a w^2/2 \; C_{ab} \tag{101}$$

Equ.(100) may then be written as

$$Q_{ab} + Q_{ba} + \underline{R}_{ab}(\underline{u}_a - \underline{u}_b) = 0 \tag{102}$$

The collision term should vanish for Maxwellian velocity distributions, which correspond to (local) plasma equilibrium.

$$f_{0a} = \frac{n(\underline{x},t)}{(2\pi T/m)^{3/2}} \exp[-m(\underline{v}-\underline{u})^2/2T] \tag{103}$$

(Note: $T_a(\underline{x},t) = T$, $\underline{u}_a(\underline{x},t) = \underline{u}$)

Any initial distribution should relax to a Maxwellian in the course of time. (H-Theorem). Finally, $f_a(\underline{x},\underline{v},t)$ must be nonnegativ at all times.

Elastic collisions between charged particles can be described by the Landau collision integral, which satisfies all the conditions listed above. It takes the form

$$C_{ab} = \Gamma_{a} \sum_{\alpha,\beta} \frac{\partial}{\partial v_\alpha} \int d\underline{v}' \, \omega_{\alpha\beta}(\underline{v}, \underline{v}') \left[\frac{\partial f_a}{\partial v_\beta} f_b(v') - \frac{m_a}{m_b} f_a \frac{\partial f_b}{\partial v'_\beta} \right] \tag{104}$$

where $\omega_{\alpha\beta} = \dfrac{\partial^2 |v-v'|}{\partial v_\alpha \partial v_\beta} = [u^2 \delta_{\alpha\beta} - u_\alpha u_\beta]/u^3$

$\underline{u} = \underline{v} - \underline{v}'$

and $\Gamma_{ab} = \dfrac{2\pi e_a^2 e_b^2}{m_a^2} \ln \Lambda$,

$\ln \Lambda$ is the usual Coulomb logarithm, $\Lambda = 12\pi n \lambda_D^3$.

Equ.(104) can be put into the form of a Fokker Planck equation [33]

$$C_{ab} = \sum_{\alpha,\beta} \frac{\partial}{\partial v_\alpha} \left[\left\langle \frac{\Delta v_\alpha}{\Delta t} \right\rangle_{ab} f_a \right] + \frac{\partial^2}{\partial v_\alpha \partial v_\beta} \left[\left\langle \frac{\Delta v_\alpha \Delta v_\beta}{2 \Delta t} \right\rangle f_a \right] \tag{105}$$

where

$$\left\langle \frac{\Delta v_\alpha}{\Delta t} \right\rangle = \Gamma_a \frac{\partial}{\partial v_\alpha} H_{ab}(\underline{v}) \tag{106}$$

$$\left\langle \frac{\Delta v_\alpha \Delta v_\beta}{2 \Delta t} \right\rangle = \frac{1}{2} \Gamma_a \frac{\partial^2}{\partial v_\alpha \partial v_\beta} G_{ab}(\underline{v}) \tag{107}$$

$$H_{ab} = \left(\frac{e_b}{e_a} \right)^2 \frac{m_a + m_b}{m_b} \int dv' f_b(\underline{v}') / |v - v'| \tag{108}$$

$$G_{ab} = \left(\frac{e_b}{e_a} \right)^2 \int d\underline{v}' f_b(\underline{v}') \, |v - v'| \tag{109}$$

and

$$\Gamma_a = \frac{4\pi e_a^4}{m_a^2} \ln \Lambda .$$

Quite generally, it can be shown that the collision term is of the Fokker-Planck type if particles suffer a large number of statistically independent, small deflections, during a deflection time $\tau_D = 2v^2 / \langle \frac{\Delta v \Delta v}{\Delta t} \rangle$. In the following we shall make use of the characteristic times[3]

$$\tau_e = \frac{3 m^{1/2}}{4 e^4} \frac{1}{(2\pi)^{1/2}} \frac{T_e^{3/2}}{Z n \ln \Lambda} \tag{110}$$

$$\tau_i = \frac{3 M^{1/2}}{4 e^4} \frac{1}{\pi^{1/2}} \frac{T_i^{3/2}}{Z^3 n \ln \Lambda} \tag{111}$$

for electrons and ions respectively. (Z = ion charge)

A characteristic feature of a plasma is the very small ratio of electron and ion mass, $m/M \ll 1$. The relative velocity in (104) is essentially equal to the electron velocity. Expanding $\omega_{\alpha\beta}$ in terms of the ion velocity we obtain

$$C_{ei} = \frac{3\sqrt{2\pi}}{4} \left(\frac{T_e}{m}\right)^{3/2} 1/\tau_e \sum_{\alpha,\beta} \frac{\partial}{\partial v_\alpha} \Big[V^{\alpha\beta} \frac{\partial f_e}{\partial v_\beta} + \tag{112}$$

$$m/M \left(\frac{2 v_\alpha}{v^3} f_e + T_i/m \, \frac{3 v_\alpha v_\beta - v^2 \delta_{\alpha\beta}}{v^5} \frac{\partial f_e}{\partial v_\beta} \right) \Big]$$

where

$$V^{\alpha\beta} = \frac{1}{v^3} \left(v^2 \delta^{\alpha\beta} - v_\alpha v_\beta \right)$$

The electron velocity in (112) is computed from the mean ion velocity. The ion pressure was approximated by the scalar pressure $p_i = n_i T_i$. The first term in (112) is independent of the ion distribution. It is the collision term for a Lorentz gas, $m/M \to 0$ which vanishes for any isotropic distribution $f(|v|)$. The ion-electron collision integral can also be simplified by expanding $\omega_{\alpha\beta}$ in terms of the ion velocity. If the electron distribution is approximately Maxwellian one obtains

$$C_{ie} = \frac{Zm}{M\tau_e} \sum_\alpha \frac{\partial}{\partial v_\alpha} \left(v_\alpha f_i + \frac{T_e}{m} \frac{\partial f_i}{\partial v_\alpha} \right) - \frac{\underline{R}_i}{M n_i} \cdot \frac{\partial f_i}{\partial \underline{v}} . \qquad (113)$$

Here the ion velocity is computed from the mean ion velocity and $\underline{R}_i = \int d\underline{v}\, M \underline{v}\, C_{ie} = -\underline{R}_e$.
Equ.(113) has the form of a Fokker Planck equation that describes Brownian motion in a moving medium with temperature T_e .

From (112) and (113) one may compute the collisional momentum and heat transfer. For Maxwellian distributions one finds

$$\underline{R}_e^0 = - \frac{mn}{\tau_e} (\underline{u}_e - \underline{u}_i) \qquad (114)$$

$$Q_i = \frac{3m}{M} n/\tau_e \, (T_e - T_i) . \qquad (115)$$

Q_e is given by the relation (102).

From the above we find for temperatures of the same order

$$\tau_{ee} : \tau_{ii} : \tau_{ei}^{E} \approx 1 : (M/m)^{1/2} : M/m \qquad (116)$$

where τ_{ei}^{E} is the energy exchange time. Thus, electrons and ions reach separate equilibria much before it is established between the components. This circumstance makes it possible to obtain transport equations with different electron and ion temperatures and mean velocity. Since $\tau_{ee} \sim \tau_{ei} \sim \tau_e$, ions have an important effect on the relaxation of electrons. On the other hand $\tau_{ie} \sim \left(\frac{M}{m}\right)\tau_e$, $\tau_{ie} \ll \tau_{ii}$, thus the ions behave like a simple gas.

Transport equation which describe the macroscopic behavior of the plasma may be found by taking moments of the kinetic equation (94). One finds for each species

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot (n\underline{u}) = 0 \qquad (117)$$

$$\frac{\partial}{\partial t} nm\underline{u} + \frac{\partial}{\partial \underline{x}} \cdot nm\underline{u}\underline{u} = nm\frac{d\underline{u}}{dt} = \qquad (118)$$

$$en[\underline{E} + (\underline{u}/c) \times \underline{B}] - \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}} + \underline{R}$$

$$\tfrac{3}{2} n \frac{dT}{dt} + p \, \mathrm{div}\, \underline{u} + \underline{\underline{\Pi}} : \frac{\partial \underline{u}}{\partial \underline{x}} + \mathrm{div}\, \underline{q} = Q \tag{119}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \frac{\partial}{\partial \underline{x}}$ (120)

$$\underline{R}_a = \sum_b \underline{R}_{ab} \; ; \quad Q_a = \sum_b Q_{ab} \tag{121}$$

and the relations (83 - 90, 96 - 102) have been used.

In order to close the hierarchy of moment equations it is necessary to obtain relations between the parameters $n(\underline{x},t)$ $\underline{u}(\underline{x},t)$ and $T(\underline{x},t)$ and $\underline{R}$, Q, $\underline{\underline{\Pi}}$, $\underline{q}$. To this end one has to find a solution of the kinetic equation (94) which can be expressed in terms of the local variables n, $\underline{u}$, T. The possibiliby of such a description derives from the properties of the relaxation process due to collisions. If the plasma parameters change over time intervals much greater than the collision times and distances much larger than a mean free path, then the distribution functions will be of the form

$$f_a = f_{oa} + f_{1a} \tag{122}$$

where f_{oa} is a local Maxwellian with density $n_a(\underline{x},t)$ temperature

$T_u(\underline{x},t)$ and mean velocity $\underline{u}_u(\underline{x},t)$ and f_{1a} is a small correction which depends on the gradients of $n, \underline{u}, T$ and the electric and magnetic fields. From f_{1a} one obtains relations between the fluxes and the forces which produce the deviation from thermal equilibrium. The coefficients in these relations are called <u>transport coefficients.</u>

4.2. Transport coefficients for a collision dominated plasma in a magnetic field.

The method of obtaining transport equations from the kinetic equation differs in the plasma case from that of a neutral gas essentially as a result of the small electron to ion mass ratio and the presence of a magnetic field. Because of the small mass ratio it is possible to uncouple electron and ion kinetic equations (c.f. Sec. 4.1.) and to obtain separate transport equations for electrons and ions with different mean velocities and temperatures. The magnetic field introduces an anisotropy in the plasma. Transport processes along the magnetic field are generally the same as in the absence of a magnetic field. Across the magnetic field they depend upon the parameter $\Omega\tau$ (Ω gyro frequency, τ collision time).

The kinetic equations must be solved by approximation methods. Basically, one has two expansion parameters, $\delta = \lambda/L$ which characterizes the effect of collisions and $\beta = r_L/L$ which characterizes the effect of the magnetic field, (L scale length of inhomogeneity, $\lambda = v_{th}\tau$ mean free path, $r_L = v_{th}/\Omega$ Larmor radius). Other expansion parameters, such as the toroidicity, which characterizes the magnetic field inhomogeneity, or the mass ratio may also be used.

It is convenient to write (94) in the frame of the mean velocity $\underline{u}$

$$\frac{df}{dt} + \underline{w}\cdot\frac{\partial f}{\partial \underline{x}} + \left[\frac{e}{m}(\underline{w}/c)\times\underline{B}\right]\cdot\frac{\partial f}{\partial \underline{w}} + \underline{a}\cdot\frac{\partial f}{\partial \underline{w}} = \frac{\delta f}{\delta t} \qquad (123)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\frac{\partial}{\partial \underline{x}}$

$$a = \frac{e}{m}\left[\underline{E} + (\underline{u}/c)\times\underline{B}\right] - \frac{d\underline{u}}{dt} - \underline{w}\cdot\frac{\partial \underline{u}}{\partial \underline{x}} .$$

In the case of a <u>collision dominated plasma</u>, f is expanded in the small parameter $\delta = \lambda/L$, assuming that the collision term keeps corrections to f_{0a} small. One obtains the expansion

$$O(1/\delta) \qquad \Omega\frac{\partial f_0}{\partial \alpha} = C(f_0) \qquad (124)$$

$$O(1) \quad \frac{df_0}{dt} + \underline{w}\cdot\frac{\partial f_0}{\partial \underline{x}} + \underline{a}\cdot\frac{\partial f_0}{\partial \underline{w}} + \Omega\frac{\partial f_1}{\partial \alpha} = C(f_0, f_1) \qquad (125)$$

etc., where α is the phase angle of $\underline{w}_\perp$, and $\Omega = -\frac{eB}{mc}$.

Retaining the Ω term allows one to keep the parameter $B/S = \frac{1}{\Omega\tau}$ arbitrary, but $\Omega\tau \ll 1$ or $\Omega\tau \gg 1$ means unnecessary work. For $\Omega\tau \ll 1$ (weak magnetic field) the expansion could be modified to

$$C(f_0) = 0 \tag{126}$$

$$\frac{df_0}{dt} + \underline{w} \cdot \frac{\partial f_0}{\partial \underline{x}} + \underline{a} \cdot \frac{\partial f_0}{\partial \underline{w}} + \Omega \frac{\partial f_0}{\partial \alpha} = C(f_1) \tag{127}$$

etc. For strong magnetic fields $\Omega\tau \gg 1$ one may expand in the parameter $B = r_L/L$, the so-called finite Larmor radius expansion [34]

$$O(1/B) \qquad \frac{\partial f_0}{\partial \alpha} = 0 \tag{128}$$

$$O(1) \qquad \frac{df_0}{dt} + \underline{w} \cdot \frac{\partial f_0}{\partial \underline{x}} + \underline{a} \cdot \frac{\partial f_0}{\partial \underline{w}} + \Omega \frac{\partial f_1}{\partial \alpha} = C(f_0) \tag{129}$$

$$O(B) \quad \frac{df_1}{dt} + \underline{w} \cdot \frac{\partial f_1}{\partial \underline{x}} + \underline{a} \cdot \frac{\partial f_1}{\partial \underline{w}} + \Omega \frac{\partial f_2}{\partial \alpha} = C(f_1), \tag{130}$$

The drift kinetic equations are obtained from (128-130), after introducing the drift variables and separating average and α dependent quantities.

4.2. Transport coefficients for a fully ionized electron-ion plasma.

The solution of the kinetic equations in the collision dominated case have been discussed by many authors. Thus we shall merely summarize the results for the transport coefficients of a fully ionized electron-ion plasma ($Z=1$) in a magnetic field and give a qualitative interpretation. Expressions will be given to dominant order in $1/\Omega\tau$. More general results may be found elsewhere[3].

a. Momentum transfer

We have $\underline{R}_e = \int d\underline{v}\, m\underline{v}\, C_{ei}$ and $\underline{R}_i = -\underline{R}_e$, $\underline{R}_e = \underline{R}_u + \underline{R}_T$ consists of the friction force

$$\underline{R}_u = n|e|(\eta_{\parallel} \underline{j}_{\parallel} + \eta_{\perp} \underline{j}_{\perp}) \tag{131}$$

and the thermal force

$$\underline{R}_T = -0.71\, n \nabla_{\parallel} T_e - 3/2 \frac{n}{\Omega_e \tau_e} [\underline{\tau}_0 \times \nabla T_e] \tag{132}$$

where $\eta_{\parallel}, \eta_{\perp}$ are the longitudinal and perpendicular electrical resistivity respectively

$$\eta_{\parallel} = \alpha \frac{m}{ne^2 \tau_e} \qquad \alpha = 0.51 \text{ for } Z = 1 \tag{133}$$

$$\eta_{\perp} = \frac{m}{ne^2 \tau_e} . \tag{134}$$

$\underline{j} = -n|e|(\underline{u}_e - \underline{u}_i)$ is the current and $\underline{\tau}_0$ is an unit vector in the direction of the magnetic field. The factor 0.51 in (133) accounts for the distortion of the electron

distribution from a shifted Maxwellian, due to the velocity dependence of the collision time, $\tau \propto v^3$
The thermal force is also the result of the velocity dependence of the collision time, although τ_e does not appear explicitely in (118). This may be understood as follows: Through the surface $x = x_0$ there will be two compensating electron fluxes $n v_e$, with corresponding friction forces of order $n m v_e / \tau_e$ However, if there is a temperature gradient dT/dx, these forces will not cancel but result in a net force $R_T = \Delta T \frac{d}{dT}\left[\frac{n m v_e}{\tau_e}\right]$
With $\Delta T \sim v_e \tau_e \frac{dT}{dx}$ and $v_e/\tau_e \propto 1/T_e$ we obtain the longitudinal component of (132). Across the magnetic field, the temperature difference is carried by the gyro motion, $v_e \tau_e \Rightarrow r_e = v_e/\Omega_e$, resulting in the transverse thermal force.

b. Heat conduction

The electron heat flux has the form $\underline{q}^e = \underline{q}_u + \underline{q}_T^e$ with

$$\underline{q}_u = 0.71\, n T_e \hat{u}_\parallel + 3/2 \frac{n T_e}{\Omega_e \tau_e}\left[\underline{\tau}_0 \times \underline{\hat{u}}\right] \tag{135}$$

$$\underline{q}_T^e = -\kappa_\parallel^e \nabla_\parallel T_e - \kappa_\perp^e \nabla_\perp T_e - \frac{5 c n T_e}{2|e|B}\left[\underline{\tau}_0 \times \nabla T_e\right] \tag{136}$$

where $\underline{\hat{u}} = \underline{u}_e - \underline{u}_i$ and

$$\kappa_\parallel^e = 3.16 \frac{n T_e \tau_e}{m} \tag{137}$$

$$\varkappa_{\perp}^{e} = 4.66 \frac{nT_e\tau_e}{m(\Omega_e\tau_e)^2}, \tag{138}$$

The relation between (132) and (138) is an expression of the Onsager-Casimir symmetry relations between transport coefficients [35]. The ion heat flux is

$$q_i = -\varkappa_{\parallel}^{i} \nabla_{\parallel} T_i - \varkappa_{\perp}^{i} \nabla_{\perp} T_i + \frac{5cn_iT_i}{2Z|e|B}\left[\zeta_0 \times \nabla T_i\right]$$

where (139)

$$\varkappa_{\parallel}^{i} = 3.9 \frac{n_iT_i\tau_i}{M} \approx \left(\frac{m}{M}\right)^{1/2} \varkappa_{\parallel}^{e} \tag{140}$$

$$\varkappa_{\perp}^{i} = 2 \frac{n_iT_i\tau_i}{M(\Omega_i\tau_i)^2} \approx \left(\frac{M}{m}\right)^{1/2} \varkappa_{\perp}^{e}, \tag{141}$$

As in the case of the momentum transfer, it is possible to give a similar elementary interpretation for the heat conduction [3].

c. Heat transfer

The heat transfer to the ions as a result of collisions is given by (115)

$$Q_i = \frac{3m}{M}\frac{n}{\tau_e}(T_e - T_i). \tag{115}$$

The electron heating rate is related to Q_i by (102)

$$Q_e = -\underline{R}_e \cdot \underline{u} - Q_i = \eta_{\parallel} j_{\parallel}^2 + \eta_{\perp} j_{\perp}^2 + \frac{1}{ne} \underline{R}_T \cdot \underline{j} - Q_i \tag{142}$$

The first two terms in (142) represent Joule heating.

d. Viscous stress tensor

The stress tensor in the absence of a magnetic field is

$$\Pi_a^{\alpha\beta} = -\mu_{oa} W_a^{\alpha\beta} \qquad a = e, i \tag{143}$$

where

$$W_a^{\alpha\beta} = \frac{\partial u_a^{\alpha}}{\partial x_{\beta}} + \frac{\partial u_a^{\beta}}{\partial x_{\alpha}} - 2/3\, \delta^{\alpha\beta} \operatorname{div} u_a \tag{144}$$

is the rate of strain tensor and

$$\mu_{oe} = 0.73\, n T_e \tau_e \tag{145}$$

$$\mu_{oi} = 0.96\, n_i T_i \tau_i \,. \tag{146}$$

In the case of a magnetic field, the relation between $W^{\alpha\beta}$ and $\Pi^{\alpha\beta}$ becomes much more complicated [3]. A qualitative discussion of viscosity in this case has been given by Kaufman [36].

4.3 Hydrodynamic description of a plasma

Adding the equations (117-119) for all species we obtain

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot (\varrho \underline{u}) = 0 \tag{147}$$

$$\frac{\partial \varrho \underline{u}}{\partial t} + \frac{\partial}{\partial \underline{x}} \varrho \underline{u}\,\underline{u} = \varrho \frac{d \underline{u}}{d t} = -\frac{\partial}{\partial \underline{x}} \cdot \overline{\underline{\underline{P}}} + \sigma \underline{E} + (\underline{j}/c) \times \underline{B} + \underline{R} \tag{148}$$

$$\frac{\partial \overline{W}}{\partial t} + \frac{\partial}{\partial \underline{x}} (\underline{u} \overline{W}) + \bar{p} \operatorname{div} \underline{u} = 3/2 \frac{d \bar{p}}{d t} + 5/2\, \bar{p} \operatorname{div} \underline{u} \tag{149}$$

$$= -\frac{\partial}{\partial \underline{x}} \cdot \overline{\underline{q}} - \overline{\underline{\underline{\Pi}}} : \frac{\partial \underline{u}}{\partial \underline{x}} + [\underline{E} + (\underline{u}/c) \times \underline{B}] \cdot [\underline{j} - \sigma \underline{u}] + Q$$

where

$$\varrho = \sum_a n_a m_a \tag{150}$$

is the total mass density

$$\underline{j} = \sum_a n_a e_a \underline{u}_a = \underline{J} + \sigma \underline{u}\,; \quad \sigma = \sum_a n_a e_a \tag{151}$$

the total current and change density respectively

$$\underline{u} = \frac{1}{\varrho} \sum_a n_a m_a \underline{u}_a \tag{152}$$

the center of mass velocity

$$\overline{\underline{\underline{P}}} = \sum_a \overline{\underline{\underline{P}}}_a = \sum_a [\, \underline{\underline{P}}_a + n_a m_a (\underline{u}_a - \underline{u})(\underline{u}_a - \underline{u})] \tag{153}$$

the total barycentric pressure and $\overline{\underline{q}}$ the total barycentric heat flow, etc. The bar indicates that for barycentric quantities the random velocities are to be counted from the center of mass velocity $\underline{u}$ rather than the mean velocity of each species,

$$\overline{\underline{\underline{P}}}_a = n_a m_a \langle \underline{v}-\underline{u}\ \underline{v}-\underline{u} \rangle_a \tag{154}$$

etc.

For binary collisions $\underline{R} = \sum_a \underline{R}_a = 0$, $Q = \sum_a [Q_a + \underline{R}_a(\underline{v}_a - \underline{u})] = 0$

Multiplying (118) by e_a/m_a and summing over all species an equation for the current is obtained.

The resulting equ. for an electron-ion plasma is

$$\frac{m}{ne^2(1+\alpha)} \frac{D\underline{j}}{Dt} = [\underline{E} + (\underline{u}/c) \times \underline{B}](1+\alpha\bar{\sigma}) - \tag{155}$$

$$\frac{1}{nec}(\underline{j} \times \underline{B}) \frac{1-\alpha}{1+\alpha} + \frac{1}{n|e|(1+\alpha)} \frac{\partial}{\partial \underline{x}} \cdot (\overline{\underline{\underline{P}}}_e - \alpha \overline{\underline{\underline{P}}}_i) - \bar{\sigma} \frac{m}{|e|} \frac{du}{dt} - \underline{A}$$

where

$$\frac{D\underline{j}}{Dt} = \frac{\partial \underline{j}}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot (\underline{u}\,\underline{j}) + \underline{j} \cdot \frac{\partial}{\partial \underline{x}} \underline{u} \tag{156}$$

$$\bar{\sigma} = \frac{\sigma}{n|e|(1+\alpha)} \tag{157}$$

$$\alpha = Zm/M \tag{158}$$

$$\underline{A} = \frac{1}{n|e|} [\underline{R}_e - \frac{\alpha}{1+\alpha} \underline{R}] \tag{159}$$

The Hall term $\underline{j} \times \underline{B}$ can be eliminated by making use of the equation of motion (148).

From (155) we can obtain an Ohms Law for a plasma, if we restrict ourselves to phenomena of low frequency and weak spatial variation. We may then neglect the left hand side of (155), i.e. electron inertia, assume quasineutrality $\tilde{\sigma} \ll 1$, and also use $\alpha \ll 1$
Ohms Law takes the form

$$\underline{E}' \equiv \underline{E} + (\underline{u}/c) \times \underline{B} + \frac{1}{ne}\left[\frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{p}}_e - \underline{R}_T\right] = \underline{\underline{\eta}} \cdot \underline{j} + \frac{1}{n|e|c}\, \underline{j} \times \underline{B} \tag{160}$$

If the Hall term is eliminated, using (148), (160) becomes

$$\underline{E} + (\underline{u}/c) \times \underline{B} - \frac{1}{ne}\left[\frac{\partial}{\partial \underline{x}} \cdot (\underline{\underline{p}}_i + \rho \underline{u}\,\underline{u}) + \underline{R}_T + \frac{\partial \rho \underline{u}}{\partial t}\right] = \underline{\underline{\eta}} \cdot \underline{j} \tag{161}$$

Equ.(160) may be solved for the current

$$\underline{j} = \frac{\underline{E}'_{\parallel}}{\eta_{\parallel}} + \frac{[\underline{E}'_{\perp} + \Omega_e \tau_e (\underline{\xi}_0 \times \underline{E}'_{\perp})]}{\eta_{\perp}[1 + (\Omega_e \tau_e)^2]} \tag{162}$$

which defines the longitudinal, Pedersen and Hall conductivities respectively

$$\underline{\underline{\sigma}} = \sigma_{33}\, \underline{\xi}_0 \underline{\xi}_0 + \sigma_{11} (\underline{\underline{I}} - \underline{\xi}_0 \underline{\xi}_0) + \sigma_{22} (\underline{\xi}_0 \times \underline{\underline{I}}) \tag{163}$$

For $\Omega_e \tau_e \gg 1$ the current across $\underline{B}$ is nearly perpendicular to $\underline{E}_{\perp}$. If the current in the y direction is prevented from flowing then $j_x = \sigma_{\perp} E_x$ and $E_y = -\Omega_e \tau_e E_x$. $(\underline{B} \parallel \underline{z})$.

$\sigma_\perp = 1/\eta_\perp$ is called the Cowling conductivity [37].

Corrections

Equ.(105) add Ref. 33

Equ.(113) A term due to friction $\frac{R_i}{n_i M} \cdot \frac{\partial f_1}{\partial \mathbf{w}}$ must be added on the rh s. of (115)

Equ.(11); Replace 2 by 3

Problems

1. Obtain the collision term Γ_{ee}, equ.(105) for $v \ll v_e$ and $v \gg v_e$.

2. Show that the Lorentz collision term in spherical coordinates v, Θ, ϕ takes the form

$$\frac{\delta f^L}{\delta t} = \Gamma/v^3 \left[\frac{\partial}{\partial \mu}(1-x^2)\frac{\partial}{\partial \mu} + \frac{1}{1-x^2}\frac{\partial^2}{\partial \phi^2} \right] f(v, \phi, \mu)$$

where $x = \cos\Theta = v_\parallel / v$, $\Gamma = \frac{2\pi n e^4}{m^2} \ln \Lambda$

Show that $\frac{\delta f^L}{\delta t}$ has the eigenfunctions $P_l^m(x)\, e^{im\phi}$

3. Show that in the drift coordinates $\varepsilon, \mu = \frac{m v_\perp^2}{2B}, r, s$ the Lorentz collision term becomes

$$\frac{\delta f^L}{\delta t} = \frac{\nu}{B} m v_\parallel \frac{\partial}{\partial \mu} \left[\mu \; v_\parallel \frac{\partial f}{\partial \mu_m} \right]$$

where $\nu(v) = 3/2 \, (v_e/v)^3 (2\pi)^{1/2} \, 1/\tau_e$

V. Diffusion and Heat Conduction in a Collision Dominated Toroidal Plasma

We wish to investigate stationary plasma states (equilibria). Currents, electric fields and diffusion may be found from the following set of equations, describing plasma equilibrium

$$\nabla\cdot(\rho\underline{u}) = 0 \tag{164}$$

$$\nabla\cdot\underline{j} = 0 \tag{165}$$

$$\nabla\cdot(\underline{\underline{p}} + \rho\underline{u}\,\underline{u}) = (\underline{j}/c)\times\underline{B} \tag{166}$$

$$\underline{E} + (\underline{u}/c)\times\underline{B} = \frac{1}{n|e|}\nabla\cdot(\underline{\underline{p}}_i + \rho\underline{u}\,\underline{u}) + \frac{\underline{R}_T}{n|e|} + \underline{\underline{\eta}}\cdot\underline{j} \tag{167}$$

(the bars indicating barycentric quantities, have been dropped). From Maxwell's equations we need

$$\nabla\cdot\underline{B} = 0 \tag{168}$$

$$\nabla\times\underline{B} = \frac{4\pi}{c}\,\underline{j} \;. \tag{169}$$

(168-169) may be combined by introducing the vector potential B = curl $\underline{A}$, where $\underline{A}$ satisfies

$$\nabla^2\underline{A} = -\frac{4\pi}{c}\,\underline{j} \;. \tag{170}$$

From curl $\underline{E} = -1/c \frac{\partial B}{\partial t}$ follows in the stationary case $\underline{E}_{int} = -\nabla \underline{\Phi}$. We may however, allow for an external driving field. From (160) we obtain

$$\underline{j} \cdot (\nabla \cdot \hat{\underline{\underline{P}}}) = 0 \tag{171}$$

$$\underline{B} \cdot (\nabla \cdot \hat{\underline{\underline{P}}}) = 0 \tag{172}$$

where $\hat{\underline{\underline{P}}} = \underline{\underline{P}} + \rho \underline{u}\,\underline{u}$ is the total stress tensor. A function $\Psi(\underline{x})$ such that $\underline{B} \cdot \nabla\Psi = 0$ is called a magnetic surface function; Ψ = const describes a magnetic surface. In the axisymmetric case Ψ may be identified with the flux per unit length of magnetic axis, and $\underline{B} = \nabla\Psi \times \nabla \zeta$ is the poloidal magnetic field. If we approximate $\hat{\underline{\underline{P}}}$ by the scalar pressure p then $p = p(\Psi)$ is a surface function and the current flows in the magnetic surface.

The perpendicular components of the current and plasma velocity may be obtained from (166-167).

$$\underline{j} = \frac{c}{B^2}\left[\underline{B} \times \nabla \cdot \hat{\underline{\underline{P}}} \right] + j_{\parallel} \underline{\tau}_0 \tag{173}$$

$$\underline{u} = \frac{c}{B}\left[\left(\underline{E} - \frac{1}{n|e|}\nabla \cdot \hat{\underline{\underline{P}}}_i\right) \times \underline{\tau}_0\right] - \frac{c^2 \eta_\perp}{B^2}\left[(\nabla \cdot \hat{\underline{\underline{P}}})_\perp - \frac{3}{2} n \nabla_\perp T_e\right] + u_{\parallel} \underline{\tau}_0 \tag{174}$$

where $\hat{\underline{\underline{P}}}_i = \underline{\underline{P}}_i + \rho \underline{u}\,\underline{u}$. $j_\perp$ represents the confining diamagnetic current in a plasma with pressure gradients. The terms proportional to $\eta_\perp$ in (174) represent classical diffusion due to electron-ion collisions, consisting of resistive and thermal diffusion. The corresponding velocity across the magnetic field may be written as (neglecting viscosity and inertia)

$$\underline{v}_D = -D_{cl}\left[\frac{\nabla_\perp n}{n} + \frac{1}{T_e+T_i}\nabla_\perp\left(T_i - \tfrac{1}{2}T_e\right)\right] \qquad (175)$$

where

$$D_{cl} = \frac{c^2 n \eta_\perp}{B^2}(T_e+T_i) = \frac{1}{(\Omega_e\tau_e)^2}\frac{T_e+T_i}{m}\tau_e \qquad (176)$$

is the classical diffusion coefficient.

Assuming that $T_e = T_e(p)$, because of the large longitudinal thermal conductivity (137), $\underline{v}_D$ may also be written as

$$\underline{v}_D = -\frac{c^2\eta_\perp}{B^2}\left[1 - \tfrac{3}{2}\, n\frac{dT_e}{dp}\right]\nabla_\perp p \qquad (177)$$

indicating that for certains profiles of pressure and temperature the net flux may vanish.

The electron and ion mean velocities are given by

$$\underline{u}_e = \underline{u} - \underline{j}/n|e| \qquad (178)$$

$$\underline{u}_i \approx \underline{u} \qquad (179)$$

If we neglect viscosity, we find that classical diffusion is ambipolar ($\underline{j} \cdot \nabla \rho = 0$) and independent of the magnetic field structure. For an uniform magnetic field and $\underline{E} - \frac{1}{ne} \nabla \cdot \rho_i = \nabla \Phi'$, the divergence of the first two terms in (174) vanishes, thus they do not contribute to diffusión. The second term in (174) also shows that ion viscosity leads to self diffusion, i.e. diffusion due to collisions between like particles [38].

If we consider plasma in a nonuniform magnetic field it is found that the ExB drift makes an important, usually the dominant contribution to the particle flux. Because of the rotational transform of magnetic fields, such as the Tokomak field, the longitudinal current $j_{\parallel}$ has an azimuthal component, giving rise to an electric field

$$E_{\parallel} = E_{\zeta} \frac{B_{\zeta}}{B} + E_{\delta} \frac{B_{\delta}}{B} = \eta_{\parallel} j_{\parallel} \tag{180}$$

and a radial drift

$$v_r = \frac{c}{B^2} \left[E_{\delta} B_{\zeta} - E_{\zeta} B_{\delta} \right] . \tag{181}$$

In (180) use was made of the longitudinal component of Ohm's law (167), assuming for simplicity that $\hat{p}_i$ and T_e are constant on the magnetic surfaces. Because of the axisymmetry, E_{ζ} is the external field .

$$E_{\zeta} = \frac{E_0}{1 + r/R \cos \delta} . \tag{182}$$

The additional flux $\Gamma_{\parallel}$ is related to the longitudinal current $j_{\parallel}$, whereas the classical diffusion flux $\Gamma_{\perp}$ is related to the diamagnetic current $j_{\perp}$. The longitudinal current may be determined from the continuity equation (165) and Ohms law (167). We set $\underline{j}_{\parallel} = \alpha\,\underline{B}$ and find a differential equation for α. From (165), (168) $\underline{B}\cdot\nabla\alpha = -\nabla\cdot\underline{j}_{\perp}$ and from (173) $\nabla\cdot(\underline{j}_{\perp}B^2) = \underline{j}_{\perp}\cdot\nabla B^2 + B^2\nabla\cdot\underline{j}_{\perp} = \nabla\cdot c\,[\underline{B}\times\nabla p] = c\,[\nabla p\cdot(\nabla\times\underline{B}) - \underline{B}\cdot\mathrm{curl}\,\nabla p] = 0$
In the last relation use was made of (169). We obtain the magnetic differential equation

$$\underline{B}\cdot\nabla\alpha = \frac{\underline{j}_{\perp}\cdot\nabla B^2}{B^2} \tag{183}$$

with the solution

$$\frac{\underline{j}\cdot\underline{B}}{B^2} = \alpha(\ell) = \alpha(0) + \int_0^{\ell} d\ell\ \frac{\underline{j}_{\perp}\cdot\nabla B^2}{B^3} \tag{184}$$

where the integration is along a magnetic field line and the constant $\alpha(0)$ is to be determined from Ohm's law. If the field line ergodically covers the magnetic surface, we may conclude that $\alpha(0)$ is a constant on the surface $\alpha(0) = \alpha(\psi)$.

As an example consider a Tokomak with the magnetic fields

$$B_{\zeta} = \frac{B_0}{1 + r/R\cos\delta}\ ,\quad B_{\delta} = \frac{\iota(v)\,r}{2\pi R}\,B_{\zeta} \tag{185}$$

$$B_r = 0$$

Equ.(183) becomes $\frac{d\alpha}{d\delta} = \frac{2cr}{RB_0B_\delta} \sin\delta \frac{dp}{dr}$

with the solution

$$\alpha \approx \alpha_0 - \frac{2cr}{RB_0B_\delta} \frac{dp}{dr} \cos\delta \quad . \tag{186}$$

Comparing $\alpha = \frac{E_\parallel}{\eta_\parallel B} = \frac{1}{\eta_\parallel B}\left[E_0(1 - r/R\cos\delta) + E_\delta B_\delta / B\right]$
with (186) we find
$$\alpha_0 = \frac{E_0}{\eta_\parallel B_0}$$
and

$$E_\delta = - \frac{2cr\, B^2 \eta_\parallel}{RB_\delta^2 B_0} \frac{dp}{dr} \cos\delta \quad . \tag{187}$$

Using (180) and averaging over δ , according to (62) we find the average radial velocity due to the internal electric field
$$\langle v_r \rangle = \int \frac{d\delta}{2\pi}(1 + r/R\cos\delta) v_r = v_{D0} \frac{2\eta_\parallel}{\eta_\perp}\left(\frac{2\pi}{\iota}\right)^2$$
or the total velocity of resistive diffusion across magnetic surfaces

$$v_D = v_{D0}\left[1 + \frac{\eta_\parallel}{\eta_\perp} \frac{8\pi^2}{\iota^2}\right] \tag{188}$$

where from (174) $\quad v_{D0} = - \frac{c^2 \eta_\perp}{B_0^2} \frac{dp}{dr}$

The enhancement factor in (188) was first computed by Pfirsch and Schlüter[12].

The enhanced particle flux (188) can be interpreted in a simple manner: The toroidal drifts lead to charge separation which has to be compensated by the longitudinal current $j_\parallel = \frac{4\pi}{c} j_\perp \cos S$. Because of the longitudinal resistivity this compensation is not complete, resulting in a (vertical) electric field, cf. (187), and a corresponding radial drift velocity.

The enhanced heat conduction in a toroidal geometry is due to effects which are completely analogous to toroidal diffusion[13]. From (136-141) we find that for $\Omega\tau \gg 1$ the longitudinal, drift, and perpendicular heat flows differ greatly in magnitude

$$q_\parallel : q_\wedge : q_\perp \approx 1 : \frac{1}{\Omega\tau} : 1/(\Omega\tau)^2 \quad (189)$$

Because of symmetry, we have in a straight cylindrical geometry $q_\parallel = 0$ and $\operatorname{div} q_\wedge = 0$; heat transfer thus takes place only due to the magnetized flux $q_\perp$. In toroidal geometry we obtain to lowest order in $1/\Omega\tau$

$$\operatorname{div} \underline{q}_\parallel^0 = 0 \quad (190)$$

from which it follows that the temperature $T^{(0)}$ is constant on the magnetic surfaces. The divergence of the drift heat flow $\underline{q}_\wedge^0$ is now nonzero and has to be balanced by a longitudinal heat flow $q_\parallel^1$

$$\operatorname{div} (\underline{q}_\parallel^1 + \underline{q}_\wedge^0) = 0 \quad (191)$$

in the same way as the charge drift had to be balanced by the longitudinal current $j_{\parallel}$, div $\underline{j}$ = 0. From

$$\underline{q}^{0}_{\wedge} = \frac{5}{2}\frac{c\,n\,T^{0}}{e\,B}\left[\underline{\tau}_{0} \times \nabla T^{0}\right] = \frac{5}{2}\frac{n\,T^{0}}{e}\frac{dT^{0}}{dp}\,\underline{j}_{\perp} \qquad (192)$$

and (191) we obtain a magnetic differential equation for $\alpha = q^{1}_{\parallel}/B$, analog to (183)

$$\underline{B}\cdot\nabla\alpha = \frac{\underline{q}^{0}_{\wedge}\cdot\nabla B^{2}}{B^{2}} \qquad (193)$$

Because of the finite longitudinal conductivity, $\underline{q}^{1}_{\parallel}$ results in a temperature perturbation $T^{(1)}(r,\delta)$

$$\underline{q}^{1}_{\parallel} = \alpha B = -\varkappa_{\parallel}\nabla_{\parallel}T^{(1)}(r,\delta) \qquad (194)$$

The total heat flux across the magnetic surface consists of the usual perpendicular flux $q^{0}_{\perp}$ and the toroidal drift flux $q^{1}_{\wedge}$

$$q_{r} = q^{0}_{\perp r} + q^{1}_{\wedge r} \qquad (195)$$

Averaging (195) over a magnetic surface we find for ions

$$\langle q_{r}\rangle = q^{0}_{\perp r}\left[1 + 1.6\,(2\pi/\iota)^{2}\right] \qquad (196)$$

where

$$q^{0}_{\perp r} = -\varkappa_{\perp}\frac{dT^{(0)}}{dr}$$

Except for the numerical factor, (196) agrees with the simple estimate from elementary kinetic theory.

The temperature gradient also gives rise to a density perturbation $n_1(r, \varsigma)$ and a potential $\phi_1(r, \varsigma)$ Neglecting viscosity and inertia we obtain from the longitudinal component of the equation of motion (148)

$$\underline{B} \cdot \nabla p = \underline{B} \cdot \nabla p_e + \underline{B} \cdot \nabla p_i = 0 \quad . \tag{197}$$

Since $\varkappa_{\parallel}^{e} / \varkappa_{\parallel}^{i} \approx (M/m)^{1/2}$ we may neglect $T_e^{(1)} \ll T_i^{(1)}$ and obtain

$$\underline{B} \cdot \nabla n = - \frac{(\underline{B} \cdot \nabla T_i)\, n}{T_e + T_i} \quad . \tag{198}$$

The resulting pressure gradient has to be balanced by an electric field which may approximately be determined from $n = n_0 \exp[e\Phi / T_e]$,

$$\frac{e\Phi^{(1)}}{T_e} \approx - \frac{T_i^{(1)}}{T_e + T_i} \quad . \tag{199}$$

This field turns out to be $(M/m)^{1/2}$ times larger than the Pfirsch Schlüter field (187). However, this field does not significantly influence diffusion, since according to (174) the drift velocity depends on $E' = E - \frac{1}{n|e|} \nabla p_i$ which vanishes for (199). It has been pointed out by Stringer[39-41] that inertia connected with plasma rotation and (ion) viscosity lead also to a density perturbation $n_1(r, \varsigma)$. Equ.(197) has to be replaced by

$$\nabla_{\parallel} p = \frac{B_S}{B} \frac{1}{r} \frac{\partial p}{\partial S} = - \rho v_0 \frac{1}{r} \frac{\partial u_{\parallel}}{\partial S} + \rho_{\parallel} \nabla_{\parallel}^2 u_{\parallel} \quad (200)$$

where $v_0 = \frac{c}{B} \frac{\partial \phi_0}{\partial r}$ is the electric drift velocity and $\rho_{\parallel}$ the longitudinal viscosity. Stringer noted that in the rest frame of the rotating plasma the toroidal magnetic field acts like an external force of frequency $\omega = -v_0/r$ parallel wave number $k_{\parallel} = \Theta/r$ and mode number m = 1 (cos S). If for a certain v_0 this force is in resonance with a natural mode of the plasma, then a large perturbation of the plasma and enhanced diffusion will result. It has been shown by Rosenbluth and Taylor [42] that v_0 is not an arbitrary parameter, but determined by the equation of motion, in which they included viscosity (cf.Problem V.6).In subsequent papers it has been shown that including a number of effect disregarded previously. such as longitudinal ion viscosity, perturbation of electron temperature and thermal force [43] and perturbation of the ion temperature [44] results in a velocity of rotation far away from resonance and a radial potential $e\phi_0 \sim T_e$ For not too large pressure gradients $\rho_i/a \ll \Theta$ (ρ_i ion Larmor radius, $\frac{1}{a} = \frac{1}{p}\frac{dp}{dr}$) diffusion and heat flux differ only by factors of order unity from the expressions given by Pfirsch and Schlüter (188) and Shafranov (196) [44]. The above results have been obtained by linearizing the equations of motion, continuity, Ohms law and the heat balance equations for electrons and ions with respect to toroidicity $\epsilon = r/R \ll 1$ Zehrfeld and Green [45, 46] considered an expansion in terms of the resistivity, but treated inertia effects exactly.

It is argued that nonlinear effects due to toroidicity strongly limit enhanced diffusion due to inertia.

Diffusion and dissipation

In the above discussion we have made use of the model toroidal field (185) introduced by Pfirsch and Schlüter. We have found that enhanced toroidal diffusion is related to dissipation by the longitudinal current. It can be shown for general magnetic field configurations that diffusion is related to dissipation (energy-balance equation) [12, 47, 48, 49] and that the Pfirsch-Schlüter enhancement factor for classical toroidal diffusion depends only on the magnetic field geometry and the ratio of longitudinal to perpendicular resistivity [50].

The plasma flux through an isobaric surface p = const is given by

$$\Gamma = \oint_{p=\mathrm{const}} d\underline{S}\cdot n\underline{u} = -\oint \frac{dS}{|\nabla p|}\nabla p\cdot n\underline{u} = \frac{d}{dp}\int d\tau\, n\underline{u}\cdot\nabla p. \qquad (201)$$

Scalar multiplication of the equation of motion (166) by $n\underline{u}$ and of Ohm's law (167) by $n\underline{j}$ results in

$$-n\underline{u}\cdot\nabla p = n\underline{j}\cdot\left(\underline{\underline{\eta}}\cdot\underline{j} + \frac{R_T}{n|e|} - \underline{E}\right) + \underline{j}/|e|\,\nabla\left(\underline{\underline{p}}_i + \rho\underline{u}\,\underline{u}\right) + n\underline{u}\cdot\nabla\,\underline{\underline{\Pi}} \qquad (202)$$

where

$$\Pi^{\alpha\beta} = p^{\alpha\beta} - p\,\delta^{\alpha\beta} + \rho u^{\alpha}u^{\beta}$$

Using (202) in (201) we see that the plasma flux consists of contributions due to resistivity, thermal force, electric field pressure tensor and inertia.

$$\Gamma = \Gamma_\eta + \Gamma_{RT} + \Gamma_E + \Gamma_{p_i} + \Gamma_\Pi . \qquad (203)$$

Using the longitudinal component of Ohm's law (203) may also be written as

$$\Gamma = \Gamma_{\eta_\perp} + \Gamma_{RT_\perp} + \Gamma_{E_\perp} + \Gamma_{p_{i\perp}} + \Gamma_\Pi . \qquad (204)$$

The general form of the energy balance equation has been discussed by Wimmel [49]. The contribution due to an isotropic pressure p_i

$$\Gamma_{p_i} = \int \frac{dS}{|\nabla p|} \underline{j}/e \cdot \nabla p_i = -\frac{d}{dp} \int d\tau \, \underline{j}/e \cdot \nabla p_i =$$
$$- \frac{d}{dp} \oint d\underline{S} \cdot \underline{j} \, p_i/e = 0$$

if $\underline{j} \cdot \nabla p = 0$, cf. (171). Similarly if in addition $\underline{j} . \nabla n = 0$, e.g. $n = n(p)$ then the contribution from the internal electric field $\underline{E} = -\nabla \Phi$ vanishes. For $T_e = T_e(p)$ we obtain $\Gamma_{RT} = \Gamma_{RT_\perp} = -3/2 \; n \frac{dT_e}{dp} \Gamma_{\eta_\perp}$ cf. (177). The usual classical diffusion (175) is described by the terms

$$\Gamma_{\eta_\perp} = \oint \frac{dS}{|\nabla p|} n \eta_\perp j_\perp^2 = \oint dS \, \frac{n c^2 \eta_\perp}{B^2} \nabla p \qquad (205)$$

and $\Gamma_{RT_\perp}$

Assuming $\nabla(\underline{\underline{P}}_i + \varrho \underline{u}\,\underline{u}) = -\nabla p_e \approx 0$ and $T_e = T_e(\rho)$ and comparing (203) and (204) we find that the additional flux is given by

$$\Gamma_{PS} = \Gamma_{E_\perp} = \Gamma_{\eta_\parallel} + \Gamma_{E_{ext}} = \oint \frac{dS}{|\nabla\rho|}\left[\eta_\parallel j_\parallel^2 - \underline{j}\cdot\underline{E}_{ext}\right] \qquad (206)$$

For a straight cylindrical geometry (206) vanishes. The Pfirsch-Schlüter enhancement factor becomes for a general magnetic field

$$1 + \frac{\Gamma_{\eta_\parallel}}{\Gamma_{\eta_\perp}} = 1 + \frac{\oint \frac{dS}{|\nabla\rho|}\left[\eta_\parallel j_\parallel^2 - \underline{j}\cdot\underline{E}_{ext}\right]}{\oint \frac{dS}{|\nabla\rho|}\,\eta_\perp\left(\frac{c\nabla p}{B}\right)^2} \qquad (207)$$

The longitudinal current is given by $j_\parallel = \alpha B$ where α is given by (184). The constant $\alpha(0)$ is obtained from the longitudinal component of Ohm's law

$$\oint \frac{dS}{|\nabla\rho|}\,\underline{B}\cdot\left[\underline{E}_{ext} - \underline{\underline{\eta}}\cdot\underline{j}_\perp\right] = \oint \frac{dS}{|\nabla\rho|}\,\underline{B}\cdot\underline{\underline{\eta}}\cdot\underline{j}_\parallel \qquad (208)$$

$$= \oint \frac{dS}{|\nabla\rho|}\,B^2\eta_\parallel\,\alpha \, .$$

The left hand side is a known function, thus $\alpha(0)$ may be determined on comparison with (184). Since $\rho = \rho(\psi)$ the enhancement factor (207) depends only (for $E_{ext} = 0$) on the magnetic field geometry and the ratio of longitudinal and perpendicular resistivity [50].

Problems

1. From the drift velocity (21) obtain the continuity equation for guiding centers, linearized in $\epsilon = r/R$ and averaged over Maxwellian velocity distributions. Show that quasi neutrality requires a longitudinal current $j_\parallel = \frac{4\pi}{\iota} j_\perp \cos\vartheta$ [40]

2. Obtain the averaged particle and guiding center flux across a magnetic surface, assuming constant temperatures, but allowing for potential and density perturbations. Show that the two fluxes are equal [40].
More generally, one has the relation between particles and guiding center flux [3]

$$\langle n\underline{v}\rangle = \langle n\underline{v}_c\rangle - \mathrm{curl}\left(\frac{c\,n\,T}{e\,B^2}\underline{B}\right) \quad .$$

3. From the fluid equations obtain a general expression for the averaged electron flux across a magnetic surface in terms of $n_1, \phi_1, T_e^{(1)}$. Eliminate the perturbation of the potential $\phi_1(r,\vartheta)$, using the longitudinal component of Ohm's law and discuss the corrections to Pfirsch-Schlüter diffusion (188) [44].

4. Show that $\nabla\cdot(\underline{q}_\parallel^{\,0}) = 0$ implies T_o = const on the magnetic surface.

5. Obtain the temperature perturbation from (194), for a Tokomak field (185), making use of the analogy between diffusion and heat conduction. Compare the resulting electric field (199) with the Pfirsch-Schlüter field (187).

6. From the ς component of the equ. of motion (148) and the condition of ambipolar diffusion show that the velocity of rotation satisfies [42]

$$\varsigma \frac{\partial v_\theta}{\partial t} = - \frac{2\epsilon(T_e + T_i)}{r} \oint \frac{d\varsigma}{2\pi} n_1(r,\varsigma) \sin\varsigma$$

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Figure Captions

Fig. 1 Toroidal Coordinates.

a minor radius R major radius

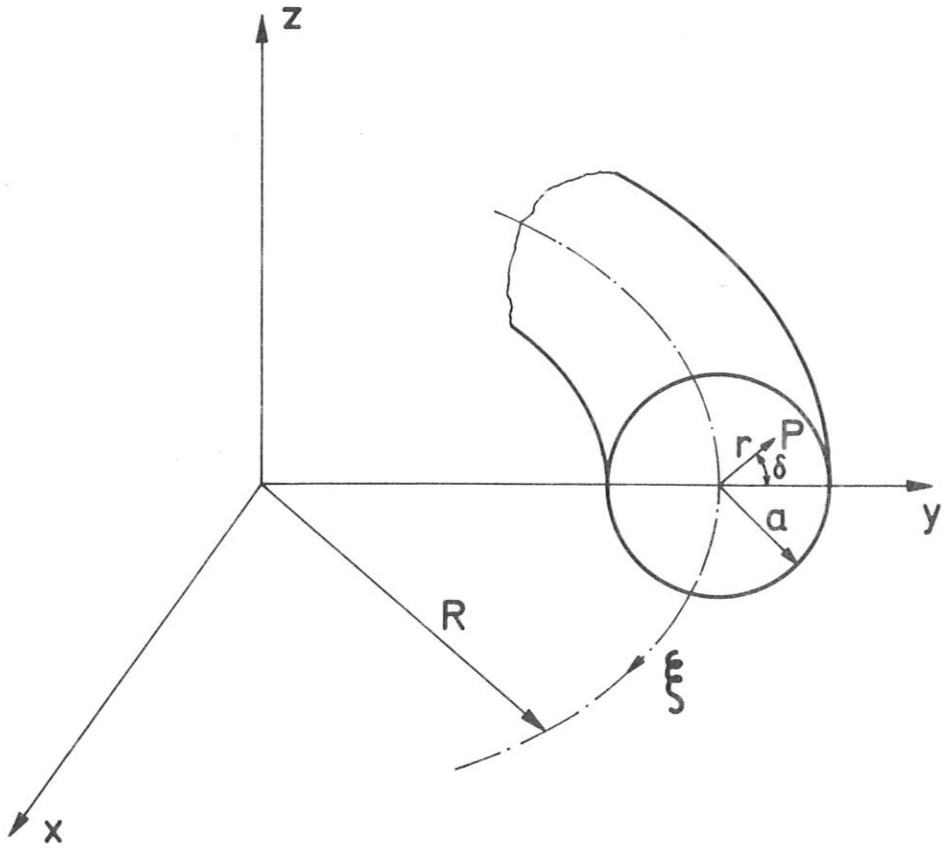

Fig. 1