Plasma Equilibria of Tokamak Type

Part I

by

H.P. Zehrfeld and B.J. Green

IPP III/1 June 1970

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Abstract

We present a general formalism for the description of an axisymmetric plasma equilibrium. This is a model for the steady operation of a Tokamak device. We use the hydromagnetic equations taking into account effects such as tensorial resistivity and finite thermal conductivity. The reformulation of this set leads to an equivalent set which includes the generalisation to toroidal geometry of the Bennett-Pinch relation, and an expression for the resistive plasma loss which shows explicitly the effect of the discharge current.

This mathematically concise presentation of the full resistive equilibrium problem is appropriate to practical calculations. As an example we consider a steady state with no mass sources for the case of small inverse aspect-ratio.

I. Introduction

The exciting experimental plasma parameters recently achieved in the Tokamak device has led to a re-awakening of theoretical interest in the problem of toroidal axisymmetric plasma equilibrium. While there exist many discussions of magnetohydrostatic equilibrium, the effect of finite plasma resistivity on the equilibrium has rarely been treated (but see for example [1]). One reason is of course, that the static problem is difficult enough and the consideration of finite resistivity further complicates the already nonlinear situation.

We consider here a simple one-fluid plasma model and derive several important relationships; for example we derive the generalisation to toroidal geometry of the well known Bennett-Pinch relation. The effect of the discharge current on plasma loss is explicitly shown.

We discuss possible differential geometric methods for treating the system of equations. With a parameter ordering typical of experiment, we calculate a solution for a steady state with no mass sources.

II. Basic Equations

To describe the stationary state of a Tokamak plasma, we use the following MHD equations (in M.K.S. units)

$$\underline{j} \times \underline{B} = \nabla p \tag{1}$$

$$\mathrm{rot}\,\underline{B} = \mu_0 \underline{j} \tag{2}$$

$$\mathrm{div}\,\underline{B} = 0 \tag{3}$$

$$\underline{E} + \underline{v} \times \underline{B} = \eta \cdot \underline{j} \tag{4}$$

$$\mathrm{div}\,\rho \underline{v} = Q \tag{5}$$

where $\underline{j}$ is the current density, $\underline{E}$ and $\underline{B}$ are the electric and magnetic fields respectively, p is the pressure , ρ is the mass density and $\underline{v}$ the plasma velocity. η is the tensorial resistivity and is assumed to be of the form (see [2])

$$\eta = \eta_{\perp} \mathbf{1} - (\eta_{\perp} - \eta_{\parallel}) \underline{B}\,\underline{B}/B^2 \tag{6}$$

The following observations should be made:

(A) We consider a stationary state, by which is meant, that the time variation of quantities is slow enough to neglect all partial derivatives with respect to time.

(B) The plasma losses across the magnetic field due to ion-electron collisions, as described by the resistivity η , are balanced by a plasma source Q (rate of mass injection).

(C) In the stationary state under investigation, plasma flows are retained, but are such that inertia effects are negligible.

For the plasma equation of state, we have adopted that of an ideal gas, so that the plasma temperature T is given by

$$p = \rho k T/m \tag{7}$$

where m is the mass of a plasma "particle" in the sense of the fluid picture. We assume that T is constant along magnetic field lines. As energy equation we take the following form

$$\operatorname{div}(e\rho\underline{v}) = \eta : \underline{j}\,\underline{j} - p \operatorname{div}\underline{v} + \operatorname{div}(\kappa\nabla T) + Q_E \tag{8}$$

where e is the specific internal energy of the plasma. This equation expresses the energy balance between ohmic heating, expansion energy, energy conduction due to temperature gradients and internal energy transport. Q_E describes possible

energy sources.
We imagine that the plasma described by equations (1) to (8) is enclosed in a toroidal container made of ideally conducting material, and that the meridional cross-section is of arbitrary form unless otherwise specified (fig.1).

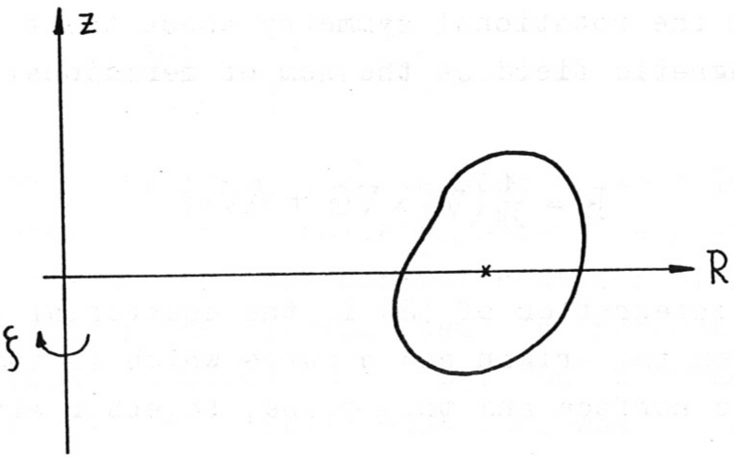

Fig.1

III. Equilibrium Equation

The usual description of an axisymmetric situation is carried out with the use of the cylindrical co-ordinate system (R,z,ς), where axisymmetry implies the absence of ς dependence in scalars. The introduction of flux functions leads to an elegant and convenient description which we now briefly review (for further elaboration see, for example, Kruskal and Kulsrud [3]). We employ for $\underline{B}$ and $\underline{j}$ the fluxes the "long" and "short" way, where each flux is evaluated in the appropriate direction, between the magnetic axis and a magnetic surface enclosing this axis. We call the magnetic fluxes the long and the short way F and G respectively. The corresponding fluxes of $\underline{j}$, the currents, we call I and J. By their definition, the equilibrium condition (1), the equation of state (7) and the assumption on T, the functions F, G, I, J, T, p and ϱ are surface quantities. The labelling of magnetic surfaces can be done in terms of any one of these quantities. We usually employ the poloidal magnetic flux G.

From (3) and the rotational symmetry about the z axis, we can write the magnetic field as the sum of meridional and toroidal components

$$\underline{B} = \frac{1}{2\pi}(\nabla\xi \times \nabla G + \Lambda\nabla\xi) \tag{9}$$

The surface integration of (2) in the equatorial plane of the torus, between the origin and a curve which is the intersection of a magnetic surface and this plane, together with Stokes theorem gives

$$\Lambda = \mu_0(J_A - J) \tag{10}$$

where Λ is identified as the line integral of the toroidal magnetic field along the above-mentioned curve, and J_A is the constant current flux through any surface bounded by the magnetic axis. Clearly Λ is a surface quantity too.

The current density we calculate from equations (2) and (9). The evaluation of the toroidal component appears difficult, but this is not really so, for

$$\frac{1}{2\pi}\,\mathrm{rot}(\nabla\xi \times \nabla G) = \alpha\nabla\xi \tag{11}$$

from vector considerations in this axisymmetric case. α is a scalar factor and is calculated by taking the scalar product of (11) with $\nabla\xi$, which, after simple vector manipulation, gives

$$\alpha = \frac{R^2}{2\pi}\,\mathrm{div}\,\frac{\nabla G}{R^2} \tag{12}$$

where $|\nabla\xi| = 1/R$. Finally, $\underline{j}$ is given as

$$j = \frac{1}{2\pi\mu_0}\left(\dot{\Lambda}\,\nabla G \times \nabla\xi + R^2 \operatorname{div}\frac{\nabla G}{R^2}\,\nabla\xi\right) \tag{13}$$

The dot identifies a derivative with respect to G. Using expressions (13) and (9) in equation (1) we have

$$\operatorname{div}\frac{\nabla G}{R^2} + \frac{\Lambda\dot{\Lambda}}{R^2} + 4\pi^2\mu_0\dot{p} = 0 \tag{14}$$

The first term is the invariant version of what in cylindrical co-ordinates is the so called $\Delta_5 G$ term (e.g. [4]). The solution of this equation, the general equation of magnetohydrostatic axisymmetric equilibrium, usually proceeds by selecting relatively simple functions $\Lambda(G)$, $p(G)$ and solving for G. The full solution of this elliptic equation, where Λ and p satisfy the other equations, has rarely been attempted because of analytical difficulty. In principle, numerical procedures can be used, although there still exist both theoretical and practical difficulties, for which the nonlinearity of this equation system is directly responsible. In the next section we discuss how the functions Λ and p are related to the structure of the as yet unconsidered equations (4) to (8).

IV. Equilibrium Relations for a Slowly Diffusing Plasma

In practice an ideally conducting torus must be "cut" so that the main toroidal field can enter the plasma region. This cut takes the form of a slit running around the torus in the ξ direction. In addition the torus must also be cut in a meridional plane so that a toroidal electric field can enter and induce a toroidal plasma current. We assume that this latter slit does not disturb the axisymmetry, a plausible assumption when one considers the function of the "liner " in experimental devices. The longitudinally induced current produces the meridional magnetic field, and is itself produced by a change of external magnetic flux in the z direction. The magnetic flux variation is accomplished by a transformer centred about the z axis, and ideally constructed so that the $\partial \underline{B}/\partial t$ in the region of the plasma, vanishes. That is, time-varying stray fields from the transformer in the plasma region are zero, so that

$$-\frac{\partial \underline{B}}{\partial t} = \mathrm{rot}\,\underline{E} = 0 \tag{15}$$

It follows that in the plasma

$$\underline{E} = -\nabla\Phi \tag{16}$$

and the total induced voltage around the torus ("ring voltage") is

$$U = -\oint \underline{E}\cdot d\underline{x} \tag{17}$$

This leads to

$$\Phi = \frac{U}{2\pi}\xi + \varphi \tag{18}$$

where φ is a single-valued point function in the plasma region. Ohm's law (4) then reads

$$\underline{v}\times\underline{B} = \frac{U}{2\pi}\nabla\xi + \nabla\varphi + \eta\cdot\underline{j} \tag{19}$$

We wish to derive certain relationships, which lead to a rather simplified representation of the equation system (1) to (8).
First let us consider (19), Ohm's law, and the continuity equation (5). The parallel component of (19),

$$\underline{B}\cdot\nabla\varphi = -\eta_{\parallel}\underline{j}\cdot\underline{B} - \frac{U}{2\pi}\underline{B}\cdot\nabla\xi \tag{20}$$

after integration gives the electric potential in terms of the magnetic field and the ring voltage up to an arbitrary surface potential φ_s . From the perpendicular component of the same equation it follows then that there is an arbitrary part in $\underline{v}_\perp$, which is given by $\underline{B}\times\nabla\varphi_s/B^2$. Introducing $\underline{v}_\perp$ in the continuity equation and denoting the part determined by the magnetic field and U by $\hat{\underline{v}}_\perp$, we get

$$\operatorname{div}\rho\left(\underline{v}_\parallel + \frac{\underline{B}\times\nabla\varphi_s}{B^2}\right) = Q - \operatorname{div}\rho\hat{\underline{v}}_\perp \tag{21}$$

This equation can be seen to be of the form

$$\underline{B}\cdot\nabla\left\{\rho\left(\frac{v_\parallel}{B} + \frac{\dot{\varphi}_s\Lambda}{B^2}\right)\right\} = Q - \operatorname{div}\rho\hat{\underline{v}}_\perp \tag{22}$$

Formal integration leads to a second arbitrary surface quantity ψ_s . Together with the perpendicular component of Ohm's law we can conclude then that $\underline{v}$ consists of two parts; a part $\hat{\underline{v}}$ which is expressible by the magnetic field and the ring voltage, and a divergence-free part everywhere tangential to the magnetic surfaces:

$$\underline{v} = \hat{\underline{v}} + \left(\psi_s - \frac{\Lambda}{B^2}\dot{\varphi}_s\right)\underline{B} + \frac{\dot{\varphi}_s}{B^2}\underline{B}\times\nabla G \tag{23}$$

The latter part is undetermined because of the arbitrariness in φ_s and ψ_s , quantities for which we have no further equations. This freedom in the tangential velocity is limited only by the assumption that inertia effects are negligible.

In order to get single-valued solutions of (20) and (22) we must satisfy corresponding consistency relations. Integrating equation (20) through the volume enclosed by a magnetic surface corresponding to the value G of the poloidal magnetic flux G, the right hand side must satisfy

$$\int_{G(x)\leq G}\eta_\parallel\,\underline{j}\cdot\underline{B}\,d^3\tau + \frac{U}{2\pi}\int_{G(x)\leq G}\underline{B}\cdot\nabla\xi\,d^3\tau = 0 \tag{24}$$

Here the U term can be re-expressed, so that

$$\int_{G(x)\leq G} \eta_{\parallel}\, j \cdot B \, d^3\tau + UF = 0 \tag{25}$$

is equivalent to (24). From (9) and (13) we find

$$-j \cdot \underline{B} = \frac{1}{\mu_o}\left(\dot{\Lambda} B_M^2 - \frac{\Lambda}{4\pi^2} \, div \frac{\nabla G}{R^2}\right) \tag{26}$$

where B_M is the meridional magnetic field component. From the definition of I, and use of (13) we have

$$I \equiv \frac{1}{2\pi} \int j \cdot \nabla \xi \, d^3\tau = \frac{1}{4\pi^2 \mu_o} \int div \frac{\nabla G}{R^2} \, d^3\tau \tag{27}$$

which by use of the well known identity for any single-valued function H

$$\frac{d}{dG}\left(\int_{G(x)\leq G} H \, d^3\tau\right) = \int_{G(x)=G} H \frac{dS}{|\nabla G|} \tag{28}$$

becomes

$$I = \frac{1}{4\pi^2 \mu_o} \frac{d}{dG} \int \frac{|\nabla G|^2}{R^2} \, d^3\tau = \frac{1}{\mu_o} \frac{d}{dG} \int B_M^2 \, d^3\tau \tag{29}$$

The rotational transform divided by 2π , ι , can be expressed, with the aid of (28), as

$$\frac{1}{\iota} \equiv \dot{F} = \Lambda \dot{\Omega} \;, \qquad \Omega := \int \frac{d^3\tau}{(2\pi R)^2} \tag{30}$$

With the introduction of ι we can differentiate (25) with respect to G, and using (26) to (30) we obtain

$$\dot{\Lambda} I - \Lambda \dot{I} = \frac{U}{\eta_{\parallel}} \Lambda \dot{\Omega} \tag{31}$$

The integrability condition for equation(22)is nothing other than the integrated form of the continuity equation (5).We introduce M,the mass flow rate through a magnetic surface:

$$M := \int_{G(x)=G} \rho \underline{v} \cdot d\underline{S} = \int_{G(x)\leq G} Q \, d^3\tau \tag{32}$$

Equations (1), (4) and (16) give

$$-\underline{v}\cdot\nabla p = \underline{j}\cdot(\underline{v}\times\underline{B}) = \eta : \underline{j}\,\underline{j} + \underline{j}\cdot\nabla\Phi \tag{33}$$

The left hand side of (32) can be written as

$$M = -\rho|\dot{p}|^{-1}\int \underline{v}\cdot\nabla p \frac{dS}{|\nabla G|} = -\rho|\dot{p}|^{-1}\frac{d}{dG}\int \underline{v}\cdot\nabla p \, d^3\tau \tag{34}$$

which with (33), (29) and (18) leads to

$$M = \rho|\dot{p}|^{-1}\left\{\frac{d}{dG}\int \eta : \underline{j}\,\underline{j} \, d^3\tau + U\dot{I}\right\} \tag{35}$$

We wish to re-express this result for M in a form where it is easy to identify the different contributions involving

(I) $\underline{j}_\perp$, the so called "Classical Diffusion" term

(II) the correction due to toroidicity, first derived by Pfirsch and Schlüter [5].

(III) a new term, involving the ring voltage.

For this we need the differential generalisation of the Bennett-Pinch relation to toroidal geometry. The corresponding equation can be found by multiplying the MHD equilibrium condition (14) by $1/|\nabla G|$ and integrating over a magnetic surface. With the use of (27) the result is

$$\dot{I} + \frac{1}{\mu_0 \iota}\dot{\Lambda} + \dot{p}\dot{V} = 0 \tag{36}$$

where V is the volume of a magnetic surface.

We now take $\underline{j}_\perp$ from (1), $\underline{j}_\parallel$ from (26), use (14) and eliminate derivatives of Λ and I by means of equations (31) and (36). This gives for M

$$M = \varrho \left\{ \eta_\perp \int_{F(x)=F} \frac{|\nabla p|}{B^2} dS - \eta_\parallel \frac{p'\Lambda^2}{\iota^2} \left(\int_{F(x)=F} \frac{1}{B^2} \frac{dS}{|\nabla F|} - \frac{V'^2}{\Lambda + \mu_o \iota I} \right) + \frac{\mu_o U I V'}{\Lambda + \mu_o \iota I} \right\} \quad (37)$$

A prime denotes differentiation with respect to F. (37) has the form required for an identification of terms. The first term is the classical diffusion term, the second is the Pfirsch-Schlüter correction, and the last term is the one containing the effects of induced current. It it of interest to note that the second term can quite generally be shown to be always positive. Calling this term P, using (28) and

$$\frac{d}{dF} \int B^2 d^3\tau = \Lambda + \mu_o \iota I \quad (38)$$

(see [3]), we get

$$P = - \frac{\eta_\parallel \frac{dp}{dF}}{\frac{d}{dF} \int B^2 d^3\tau} \left\{ \left(\int B^2 \frac{dS}{|\nabla F|} \right) \left(\int B^{-2} \frac{dS}{|\nabla F|} \right) - \left(\int \frac{dS}{|\nabla F|} \right)^2 \right\} \quad (39)$$

which by Schwarz's inequality and for outwardly decreasing pressure profiles is always positive definite, except the trivial case B^2= const.

We can recast the expression for M in a simpler form, which is more convenient for theoretical considerations.

$$M = \varrho \left(U\dot{V} - \frac{\eta_\perp}{\mu_o} \Lambda \dot{\Lambda} \dot{V} - \eta_\perp \dot{p} \dot{\Sigma} + (\eta_\perp - \eta_\parallel)(\dot{p} \dot{\Xi} + \frac{1}{\mu_o} \Lambda \dot{\Lambda} \dot{V}) \right) \quad (40)$$

$$\Sigma := \int (2\pi R)^2 d^3\tau \qquad \Xi := \int \frac{\Lambda^2}{B^2} d^3\tau \quad (41)$$

Finally, we must discuss the energy equation. After a volume integration of (8) and differentiation with respect to G, we obtain

$$\frac{d}{dG}\left\{\int e\varrho\underline{v}\cdot d\underline{S} + \int p \,\mathrm{div}\,\underline{v}\, d^3\tau\right\} = \frac{d}{dG}\left\{\int(\boldsymbol{\eta} : \underline{j}\,\underline{j} + Q_E)\, d^3\tau + \int \kappa \nabla T \cdot d\underline{S}\right\} \quad (42)$$

From (33) we have

$$-\frac{d}{dG}\int \underline{v}\cdot\nabla p \, d^3\tau = \frac{d}{dG}\int \boldsymbol{\eta} : \underline{j}\,\underline{j}\, d^3\tau + U\dot{I} \quad (43)$$

Now for an ideal gas the specific internal energy is given by

$$e = \frac{kT}{m(\gamma-1)} \quad (44)$$

where γ is the ratio of the specific heats. Using (42) and (43) we find the following form of the energy equation:

$$\frac{k\gamma}{m(\gamma-1)}(TM)^{\cdot} + U\dot{I} = \frac{d}{dG}\left(\dot{T}K + \int Q_E \, d^3\tau\right) \quad (45)$$

$$K := \int \kappa |\nabla G|^2 d^3\tau \quad (46)$$

We now collect the six equations which determine G, Λ , I, p, T and ϱ :

$$\mathrm{div}\frac{\nabla G}{R^2} + \frac{\Lambda\dot{\Lambda}}{R^2} + 4\pi^2\mu_0\dot{p} = 0 \quad (47)$$

$$\dot{\Lambda}I - \Lambda\dot{I} = \frac{U}{\eta_{\parallel}}\Lambda\dot{\Omega} \quad (48)$$

$$\dot{I} + \frac{1}{\mu_0}\Lambda\dot{\Lambda}\dot{\Omega} + \dot{p}\dot{V} = 0 \quad (49)$$

$$M \equiv \varrho\left(U\dot{V} - \frac{\eta_\perp}{\mu_0}\Lambda\dot{\Lambda}\dot{V} - \eta_\perp\dot{p}\dot{\Sigma} + (\eta_\perp - \eta_\parallel)\left(\dot{p}\dot{\Xi} + \frac{1}{\mu_0}\Lambda\dot{\Lambda}\dot{V}\right)\right) = \int Q \, d^3\tau \quad (50)$$

$$p = \varrho\frac{kT}{m} \quad (51)$$

$$\frac{k\gamma}{m(\gamma-1)}(TM)^{\cdot} + U\dot{I} = \frac{d}{dG}\left(\dot{T}K + \int Q_E \, d^3\tau\right) \quad (52)$$

An alternative set of equations could have been obtained by not introducing I. This set is obtained from (47) to (52) by dropping (49), which is a weighted average of (47), and inserting the expression (27) for I in (48) and (52). For reasons of greater convenience we have chosen to solve the system as shown above, together with the constraint (27) for G.

V. Equations for Currents, Pressure and Temperature

Once G is known as a function in space, and the mass density is replaced in (50) by (51), equations (48),(49),(50) and (52) represent four ordinary differential equations for the G-dependence of currents, pressure and temperature. These equations can be explicitly solved with respect to the derivatives. The corresponding formulae are relatively simple when we introduce the following dimensionless functions

$$\tau := (z-1)(\dot{\Sigma}-\dot{\Xi})\frac{\dot{\Omega}}{\dot{V}^2} + \frac{\dot{\Sigma}\dot{\Omega}}{\dot{V}^2} - 1 \tag{53}$$

$$\varepsilon := \frac{\mu_0 I}{\Lambda^2 \Omega} = \frac{\iota I}{J_A - J} \tag{54}$$

$$z := \frac{\eta_\perp}{\eta_\parallel} \tag{55}$$

$\sqrt{\varepsilon}$ is roughly the ratio of the meridional to the toroidal component of magnetic field. For isotropic resistivity (z=1) τ vanishes together with the toroidicity; otherwise the first term in τ is related to ε , because

$$\dot{\Sigma} - \dot{\Xi} = \int (2\pi R)^2 \left(1 - \frac{1}{1 + B_M^2/B_T^2}\right) \frac{dS}{|\nabla G|} \tag{56}$$

In what follows we restrict our considerations to the case where the source terms in (50) and (52) do not contribute. The logical development of the solulution procedure is not altered for any other choice of source functions.

Solving explicitly for the derivatives we find

$$\dot{I} = -\frac{U\dot{\Omega}}{\eta_{\parallel}} \frac{\varepsilon+\tau}{\varepsilon+(1+\varepsilon)\tau} \tag{57}$$

$$\dot{\Lambda} = \frac{\mu_0 U}{\eta_{\parallel}\Lambda} \frac{\tau}{\varepsilon+(1+\varepsilon)\tau} \tag{58}$$

$$\dot{p} = \frac{U\dot{\Omega}}{\eta_{\parallel}\dot{V}} \frac{\varepsilon}{\varepsilon+(1+\varepsilon)\tau} \tag{59}$$

$$\dot{T} = \frac{UI}{\int \kappa |\nabla G| \, dS} \tag{60}$$

Assuming for the moment that G as a function in space is given, then these equations can be solved for I, Λ, p and T. Related to the calculation of G is the Virial theorem.

VI. The Virial Equation

With the spatial part of the energy-momentum tensor

$$\mathsf{T} = -\left(\frac{1}{2\mu_0}B^2 + p\right)\mathbf{1} + \frac{1}{\mu_0}\underline{B}\,\underline{B} \tag{61}$$

and the position vector $\underline{x}$, and the understanding that (2) and (3) are satisfied, $\operatorname{div}\mathsf{T} = 0$ is equivalent to (1). We have

$$\operatorname{div}(\underline{x}\cdot\mathsf{T}) = \operatorname{Tr}\mathsf{T} = -\frac{1}{2\mu_0}B^2 - 3p \tag{62}$$

Calculation of $\underline{x}\cdot\mathsf{T}$ from (61) and integration of (62) over the interior of a magnetic surface yield

$$\int\left(\frac{1}{2\mu_o}B^2+p\right)\underline{x}\cdot d\underline{S} = \int\left(\frac{1}{2\mu_o}B^2+3p\right)d^3\tau \tag{63}$$

Differentiation of this equation with respect to G, use of (9) for $\underline{B}$ for the left, and (38) for the right hand side give together with (30) and (36)

$$\frac{1}{2\mu_o}\frac{d}{dG}\int B_M^2\,\underline{x}\cdot d\underline{S} = \dot{p}\left(\frac{\dot{V}\Omega}{\dot{\Omega}}-3V\right)+\frac{1}{2}I+\frac{\Omega}{\dot{\Omega}}\dot{I} \tag{64}$$

Note that (64), like the differential Bennett-Pinch relation (49), is a weighted average of the equilibrium condition over a magnetic surface, but with a different weight function. In this way, (49) as (64) are independent parts of the information contained in the equilibrium equation (47). In the next section we will treat the case of a small inverse aspect-ratio torus in which it turns out that no further information from (47) other than that contained in (49) and (64), is required.

VII. Approximate Determination of the Equilibrium

The results of Section V make it appear desirable to retain G as an independent variable for a treatment of the complete system of equations (47) to (52). Unfortunately, such an attempt seems to be in conflict with the basic equilibrium equation (47), which usually is considered as an equation for G.

Taking for guaranteed the existence of a reasonable equilibrium solution with nested level surfaces of G we can clarify the situation as follows:

We introduce an orthogonal, for the present unknown curvilinear co-ordinate system (G,φ,ξ). The co-ordinate surfaces are represented by the magnetic surfaces, the rotation surfaces of the orthogonal trajectories to the latter, and by the meridional planes ζ = const. Equation (47) connects the geometry with physical quantities, because in the system (G,φ,ζ) the relation

$$\operatorname{div}\frac{\nabla G}{R^2} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial G}\left(\sqrt{g}\,R^{-2}g^{GG}\right) \quad , \qquad R^{-2} = g^{\zeta\zeta} \tag{65}$$

holds. (We have introduced the metric tensor $g_{\alpha\beta}$ $(\alpha,\beta = G,\varphi,\zeta)$ and the volume element $\sqrt{g}\;dG\,d\varphi\,d\zeta$.)

Because a solution is assumed to exist, there is a metric tensor such that Riemann's curvature tensor vanishes. The corresponding differential equations for the co-ordinates of the metric tensor, our system of equations and differential equations connecting cartesian and the curvilinear co-ordinates under consideration, allow us in principle to find the equilibrium solution and the vectorial relation $\underline{x} = \underline{x}(G,\varphi,\zeta)$. In this way one could retain G as an independent variable.

Neither will we outline here such a programme in detail, nor will we discuss how useful such an approach to the problem in general might be. Rather we wish to use the convenience of a properly chosen co-ordinate system for treating the equations approximately.

The general idea we will profit by is to start with an analytically given coordinate system, which anticipates the expected geometry of the magnetic surfaces and which is provided with largely arbitrary built-in functions. We can treat then equations (57) to (60), and get solutions which depend on the built-in functions. After this we can use the latter to modify shape and position of the magnetic surfaces in such a manner that the approximation with respect to equation (47) is as good as possible.

A procedure which we can imagine reasonable for a small inverse aspect-ratio torus , would consist of the initial assumption, that the magnetic surfaces form a family of nonconcentric toroids of ellipsoidal meridional cross-section. Shape-functions in this case would be the center-shift toward the container-wall and the eccentricity of the ellipsoidal cross-sections.

Although we are preparing a treatment of this case, we restrict ourselve here to the simplest situation showing up all essential features of the equilibrium, where one works with nonconcentric circular cross-sections, so that only one shape-function enters. This is the displacement Δ , introduced by Shafranov [6] and defined below.

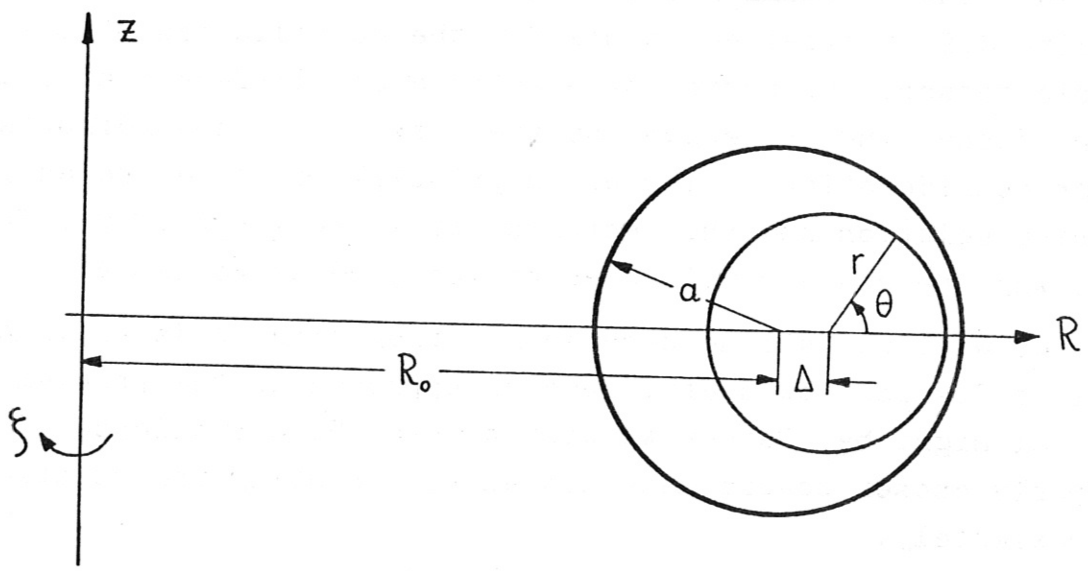

Fig.2

With respect to boundary conditions our results will be valid in the sense of either of the following interpretations:

(1) The container is made of ideally conducting material and has circular cross-section of sufficiently large aspect-ratio, so that all inner magnetic surfaces are circular too.
(2) We disregard boundary conditions and restrict our results to a sufficiently small neighborhood of the magnetic axis, such that the outermost magnetic surface to a good approximation is circular.

We will use the nonorthogonal coordinate system given by

$$\underline{x} = R(r,\theta)(\cos\xi\,\underline{e}_x - \sin\xi\,\underline{e}_y) + r\sin\theta\,\underline{e}_z \tag{66}$$

$$R(r,\theta) = R_0 + \Delta(r) + r\cos\theta \tag{67}$$

R_0, $\Delta = \Delta(r)$, r and θ are defined in fig.2. The metric tensor, its inverse and its determinant g are calculated as

$$g = \begin{pmatrix} 1+2\Delta'\cos\theta+\Delta'^2 & -r\Delta'\sin\theta & 0 \\ -r\Delta'\sin\theta & r^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix} \tag{68}$$

$$g^{-1} = \begin{pmatrix} \dfrac{1}{(1+\Delta'\cos\theta)^2} & \dfrac{\Delta'\sin\theta}{r(1+\Delta'\cos\theta)^2} & 0 \\ \dfrac{\Delta'\sin\theta}{r(1+\Delta'\cos\theta)^2} & \dfrac{1+2\Delta'\cos\theta+\Delta'^2}{r^2(1+\Delta'\cos\theta)^2} & 0 \\ 0 & 0 & \dfrac{1}{R^2} \end{pmatrix} \tag{69}$$

$$\sqrt{g} = rR(1+\Delta'\cos\theta) \tag{70}$$

G in this approximation and geometry is a function of r only and is related to I and Δ by (27) or (29) (see the remark after equation (52))

$$I = \frac{1}{\mu_0}\frac{r}{R_0}\frac{dG}{dr}\left\langle\frac{1}{ND}\right\rangle \tag{71}$$

$$D := 1+\Delta'\cos\theta \quad , \quad \Delta' := \frac{d\Delta}{dr} \tag{72}$$

$$N := \frac{R}{R_0} = 1 + \frac{\Delta}{R_0} + \frac{r}{R_0}\cos\theta \tag{73}$$

$$\left\langle\frac{1}{ND}\right\rangle := \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{ND} \tag{74}$$

As we have pointed out earlier, satisfying (49) is equivalent to the fulfillment of (47) in a restricted sense, which, as can be seen, is

$$\langle \sqrt{g}\, L \rangle = 0 \tag{75}$$

where with L we have represented the left hand side of (47). Now the function Δ (as yet unspecified) enables us in addition to guarantee that

$$\langle \sqrt{g}\, L \cos\theta \rangle = 0 \tag{76}$$

and will lead to a differential equation for Δ . It can be shown that if (75) and (76) are satisfied then L is at least of order $(r/R_o)^2$. With the assumption that this will lead to an equally good approximation of the exact poloidal flux function by G(r), we conclude that in the sense of our approximation (76) replaces equation (47). (76) is exactly the Virial equation (64).

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