

I N S T I T U T F Ü R P L A S M A P H Y S I K
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Parametric Motion and its Effects on
Particle Diffusion and the Plasma
Dielectric Constant

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Abstract

A uniform electric field transverse to a spatially non-uniform magnetic field produces particle diffusion even when the electric field spectrum contains no dc component. This diffusion is additional to the usual $E \times B$ diffusion and occurs isotropically in the plane transverse to the magnetic field direction.

The dielectric constant derived for the case of a uniform magnetic field is shown to be quite inappropriate for a plasma in a spatially non-uniform magnetic field.

I. Introduction

Study of particle motion in spatially uniform transverse electric and magnetic fields reveals the presence of an $E \times B$ drift. This drift is due to the pole at $\omega = 0$ when the position of the particle in the $E \times B$ direction is expressed as a function of the electric field. Spitzer /1/ pointed out that if the electric field is fluctuating in time, particle diffusion would take place instead of the dc drift. The diffusion rate is proportional to the value of the dc component of the electric field spectrum /2/. In Refs. 3 and 4 these results are generalized to the case when the electric field frequency is no longer small compared to the particle gyro-frequency.

Such results have led to the generally accepted conclusion that only the dc component of the electric field spectrum in the particle's frame of motion could produce a diffusion transverse to the magnetic field /5/.

In Sec. III of this paper it is shown that the parametric resonances due to the longitudinal motion of the particle in a non-uniform magnetic field cause the appearance of additional poles in the equation relating the particle position and the electric field. The resultant drift and diffusion occur isotropically in the plane transverse to the magnetic field direction.

Sec. IV deals with the effect of the parametric motion on the plasma dielectric constant. It is shown that in a non-uniform magnetic field the local approximation of the dielectric constant derived for the case of a uniform magnetic field is seldom valid.

II. Particle Motion in an Ideal Parabolic Mirror

Let $B_x(x, y, z)$, $B_y(x, y, z)$ and $B_z(x, y, z)$ be the magnetic field of the axially symmetric magnetic mirror used to contain the particle. Let $B_x[x(t), y(t), z(t)]$, $B_y[x(t), y(t), z(t)]$ and $B_z[x(t), y(t), z(t)]$, henceforth referred to as $B_x(t)$, $B_y(t)$ and

$B_z(t)$, be the magnetic field at the instantaneous position of the particle. Then, the non-relativistic equations of motion of the particle of charge q and mass m are

$$\dot{v}_r(t) = -j \omega_c(t) v_r(t) + j \omega_{cr}(t) v_z(t) + \eta E(t) \quad (1)$$

and
$$\dot{v}_z(t) = \eta [B_y(t) v_x(t) - B_x(t) v_y(t)]$$

where $v_r = v_x + jv_y$, $\eta = q/m$, $\omega_c(t) = \eta B_z(t)$, $\omega_{cr}(t) = \eta [B_x(t) + j B_y(t)]$ and $E(t)$ is the spatially uniform electric field in the x direction.

We consider only the paraxial particles with $r = x + jy \rightarrow 0$. At the axis $\omega_{cr} \rightarrow 0$, so that the term containing ω_{cr} in Eq. (1) can be neglected. Next we consider a parabolic mirror so that in the absence of an electric field /6/

$$\omega_c(t) = \omega_c \left[1 + A \cos (pt + \psi) \right] \quad (2)$$

where $A = (R-1)/(R+1)$, R being the ratio of the maximum to the minimum field seen by the particle, $p/2$ is the frequency of the longitudinal motion of the particle along the mirror axis and depends upon the initial position of the particle. Finally we assume that the electric field $E(t)$ is vanishingly small and leaves Eq. (2) unaffected. We shall neglect the effects due to the finite wavelength of the electric field as well as due to the magnetic field associated with the time variation of the electric field. In Sec. V we shall return to examine some of the implications of these assumptions. The solution of Eq. (1) may be written as

$$v_r(t) = \exp \left[-j \omega_c t - j \beta \sin (pt + \psi) \right] \cdot \left[\sum_n \eta \int_{-\infty}^{\infty} \tilde{E}(\omega) d\omega J_n(\beta) \left\{ \frac{\exp [j(\omega + \omega_c + np)t] - 1}{j(\omega + \omega_c + np)} \right\} \exp(jn\psi) \right. \\ \left. + v_r(0) \exp(j\beta \sin \psi) \right] \quad (3)$$

where $\beta = (\omega_c A)/p$, J_n is the Bessel function of first kind and order n and $\tilde{E}(\omega)$, the Fourier-transform of $E(t)$ is defined as

$$\tilde{E}(\omega) = \frac{1}{2\pi} \int_0^t E(t) \exp(-j\omega t) dt$$

and

$$E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega) \exp(j\omega t) d\omega.$$

All summations extend from $-\infty$ to $+\infty$. Note that the single resonance at the gyrofrequency for the case of a uniform magnetic field has been replaced by an infinite set of parametric resonances. If the electric field is stationary random in time, applying Fermi's-golden-rule (see Appendix A) to Eq. (3) gives

$$d v^2(t)/dt = 2\pi \eta^2 \sum_n J_n^2(\beta) \Phi(\omega_c + n\omega_p) \quad (4)$$

where the power spectrum $\Phi(\omega)$ and the associated autocorrelation function $\varphi(\tau)$ are defined as

$$\varphi(\tau) = \frac{1}{2T} \int_{-T}^T E(t) E^*(t+\tau) dt, \quad T \rightarrow \infty$$

and

$$\Phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\tau) \exp(-j\omega\tau) d\tau.$$

Equation (4) is valid only when $t \rightarrow \infty$ and $v(t)$ is the particle velocity at asymptotic times. A similar result valid for all times would be obtained by considering an average of $v^2(t)$ over an infinite ensemble of electric fields of finite duration. In Eq. (4) $v^2(t)$ would be replaced by $\langle v^2(t) \rangle$ where the angle brackets denote an ensemble average. Equation (4) reproduces the results obtained in Refs. 4 and 7 using different mathematical techniques. Integrating Eq. (3) we obtain

$$\begin{aligned} v(t) = & \eta \sum_s \sum_m J_m(\beta) J_{m-s}(\beta) \frac{\exp(-js\psi)}{\omega_c + m\omega_p} \int_{-\infty}^{\infty} \tilde{E}(\omega) d\omega \cdot \\ & \left[- \frac{\exp\{j(\omega - s\omega_p)t\} - 1}{\omega - s\omega_p} + \exp\{-j(\omega_c + m\omega_p)t\} \frac{\exp\{j(\omega + \omega_c + m\omega_p - s\omega_p)t\} - 1}{\omega + \omega_c + m\omega_p - s\omega_p} \right] \\ & + j v_r(c) \exp(j\beta \sin \psi) \sum_m J_m(\beta) \exp(-jm\psi) \frac{\exp\{j(\omega_c + m\omega_p)t\} - 1}{\omega_c + m\omega_p} + v(c) \end{aligned}$$

(5)

In the next section we study the implications of this equation.

III. Diffusion Caused by the Electric Field

A. Analytical Solution

Of the two sets of poles in Eq. (5), the first set at $\omega = sp$ gives rise to guiding center drift. Application of l'Hôpital rule to one of these poles gives

$$r(t) = \eta \sum_m J_m(\beta) J_{m-s}(\beta) \frac{\exp(-js\psi)}{\omega_c + mp} \int_{-\infty}^{\infty} \tilde{E}(\omega) d\omega (-j^s t) \quad (6 a)$$

Since ψ can assume arbitrary values between 0 and 2π , the drift produced is isotropic for the case $\omega = sp \neq 0$. For the case $\omega = sp = 0$, however, $r(t)$ is purely imaginary and the drift occurs in the y direction only, corresponding to the $E \times B$ drift in a uniform magnetic field. This is to be expected since in a dc electric field no particle drift can occur in the direction of the electric field without exchanging energy with the field, this possibility being excluded due to the absence of a corresponding pole in the velocity Eq. (3). For $\omega \neq 0$ no such restriction exists because particle drift is conceivable without exchanging energy with the electric field.

The second set of poles in Eq. (5) at $\omega = \omega_c + np$, where $n = m-s$, represent the effect of expanding particle orbits due to the corresponding poles in the velocity Eq. (3) and do not contribute to guiding center motion. This is readily verified by applying once again l'Hôpital rule to one of these poles,

$$r(t) = \eta \sum_m J_m(\beta) J_{m-s}(\beta) \frac{\exp(-js\psi)}{\omega_c + mp} \int_{-\infty}^{\infty} \tilde{E}(\omega) d\omega \exp[-j(\omega_c + mp)t] j^s t \quad (6 b)$$

The oscillating factor $\exp[-j(\omega_c + mp)t]$ gives rise to a spiraling motion to $|r(t)|$ which increases linearly in time. There is one exception, however, when ω_c/p has the integer value $-m$, in which case Eqs. (6a) and (6b) combine to give on applying l'Hôpital rule,

$$r(t) = -\eta J_{-\omega_c/p}(\beta) J_{-(\omega_c/p)-s}(\beta) \exp(-js\psi) \int_{-\infty}^{\infty} \tilde{E}(\omega) d\omega t^2 \quad (6 c)$$

The reason for such behavior can be appreciated by noting from the term containing $v_r(0)$ in Eq. (5) that even in the complete absence of an electric field a drift is produced when ω_c is an exact multiple of p . The electric field acting on this already resonant system gives rise to a quadratic drift when its frequency ω is also a multiple of p . For a practical plasma confinement system $\omega_c/p \gg \beta$ so that, $J_{-\omega_c/p}(\beta) J_{-(\omega_c/p)-s}(\beta) \rightarrow 0$, and the quadratic diffusion term in Eq. (6 c) is negligibly small. In this paper we shall circumvent this singularity by the purely mathematical artifice of requiring ω_c/p to be a non-integer.

Applying Fermi's-golden-rule to Eq. (5) we obtain

$$\frac{dr_g^2}{dt} = 2\pi\eta^2 \sum_s \left[\sum_m \frac{J_m(\beta) J_{m-s}(\beta)}{\omega_c + mp} \right]^2 \Phi(sp) \quad (7)$$

where the subscript "g" denotes guiding center diffusion. It is to be remembered that the diffusion is isotropic for $s \neq 0$ and in the $E \times B$ direction for $s = 0$.

For the case $s = 0$,

$$\left. \frac{dr_g^2}{dt} \right|_{\omega=c} = 2\pi\eta^2 \left[\sum_m \frac{J_m^2(\beta)}{\omega_c + mp} \right]^2 \Phi(0)$$

Since $J_m(\beta) \rightarrow 0$ if $m \gg \beta$ (or $mp \gg \omega_c A$), for the case of a not very large value of R , we get

$$\left. \frac{dr_g^2}{dt} \right|_{\omega=c} \cong \frac{2\pi\eta^2}{\omega_c^2} \Phi(0)$$

where we have used the identity $\sum_m J_m^2(\beta) = 1$. By comparing the above equation with the results of Refs. 2 to 4 it is seen that the $E \times B$ diffusion produced by the dc component of the electric field spectrum is not materially affected by the magnetic field being non-uniform.

Using the identity $\sum_m J_m(\beta) J_{m-s}(\beta) = J_s(0)$, it is seen from Eq. (7) that for $\omega = sp \neq 0$, the magnitude of the diffusion term is quite small compared to the case $\omega = sp = 0$.

B. The Physical Mechanism

Consider the case $\omega = p$ with the electric field "in phase" with the magnetic field as shown in Fig. 1. The particle sees a stronger

magnetic field during the positive half-cycle and a weaker magnetic field during the negative half-cycle of the electric field. Owing to the different drift rates in the two regions of the magnetic field, there is a net displacement of the guiding center in the y direction during one complete period of the electric field.

Although such a simple explanation is not available for the x -direction drift, some insight may be gained by considering the velocity Eq. (3). The particle velocity oscillates at the frequency ω_p , in addition to the frequencies ω and ω_c to be found for the case of uniform magnetic field. During the positive half-cycle of the electric field of frequency $\omega = \omega_p$, the order of change in the particle velocity is $\eta E/\omega_c$ and implies an x -displacement $\Delta x \approx E/\omega_c B$. Due to the absence of a pole at $\omega = \omega_p$ in Eq. (3), the particle is obliged to return this reactive energy during the subsequent negative half-cycle of the electric field, necessitating a further displacement Δx in the same direction.

The preceding considerations give the impression that the " x " and " y " drifts are caused by apparently unrelated mechanisms. However, the isotropic nature of the drift is more likely to be the manifestation of a yet unknown physical mechanism rather than pure coincidence.

IV. Dielectric Tensor and Electric Field Penetration into the Plasma

Equation of motion of a charged particle is given by the Lorentz-force-law. An ensemble of charged particles (e.g., a plasma) situated in a given externally applied electromagnetic field may exhibit charge separation and bunching in co-ordinate as well as in the velocity space. The total field inside the plasma is then a superposition of the externally applied fields and the fields produced by the particles themselves.

Plasma dielectric constant ϵ is a measure of the fields created by the particles as a reaction to the external electromagnetic fields. From Eqs. (3) and (5) we note that both the velocity and position of the particle in response to an external electric field consist of in-

finite sums of parametric modes. When one considers a group of particles with different values of p , the bunching effects either in velocity or co-ordinate space would tend to disappear. We may, therefore, conjecture that $\epsilon \rightarrow 1$ when considering sufficiently non-uniform magnetic fields and the plasma appears transparent to an externally applied electric field.

Furthermore, this effect plays havoc with the dispersion relations derived using the "local" approximation for the plasma dielectric constant in a non-uniform magnetic field. Also, the results derived for patently mirror microinstabilities (like the loss-cone instability), by treating the problem in a uniform magnetic field, should be employed with caution.

V. Discussion

The principal result obtained in this paper is the presence of an isotropic guiding center drift in the plane perpendicular to the direction of a non-uniform magnetic field. This drift is in addition to the usual $E \times B$ drift and is caused by the non-zero frequency components of the electric field acting transverse to the magnetic field. Compared to the diffusion co-efficient due to the dc component of the electric field spectrum, the diffusion co-efficient associated with this isotropic drift is quite small for comparable values of the electric field strength. However, the contribution to diffusion parallel to the electric field may still be important due to the presence of rather large ambipolar fields in the radial direction.

In an actual experiment, the departure from the parabolic magnetic field variation, the off-axis (non-Paraxial) and finite electric field effects, as well as magnetic field imperfections will all add up to produce complicated variations in $\omega_c(t)$. Though this would tend to smear out the distinct parametric resonances replacing them by a continuous spectrum, it may be seen by comparison with the results obtained in Ref. 7 that the total diffusion produced will not be much altered.

The neglect of the term containing $\omega_{cr}(t)$ in Eq. (1) introduces an error by eliminating the mechanism responsible for energy exchange between the longitudinal and the transverse velocity components. This results in a constant value of $|v_r(t)|$ as a function of time in Eq. (3) in the absence of an electric field, and is obviously incorrect. Hence we refrain from commenting on the significance of the pole at $\omega_c + mp = 0$ in Eq. (5). Whether this is symptomatic of a deeper pathology introduced by the paraxial approximation is hard to say. Thus the quantitative estimates of these phenomena will have to await not only an accurate measurement of the ambipolar fields but also a more precise theoretical model than the one used here.

Appendix A

$\xi(t)$, the response of a linear system to an applied drive $\xi(t)$ may be expressed as

$$\xi(t) = K \int_{-\infty}^{\infty} \tilde{\xi}(\omega) d\omega \frac{\exp[j(\omega - \omega_0)t] - 1}{\omega - \omega_0}$$

where it is assumed that the system has only one simple resonance at $\omega = \omega_0$. On squaring one obtains

$$\xi^2(t) = K^2 \iint_{-\infty}^{\infty} \tilde{\xi}(\omega) \tilde{\xi}^*(\omega') d\omega d\omega' \frac{\exp[j(\omega - \omega_0)t] - 1}{\omega - \omega_0} \frac{\exp[j(\omega' - \omega_0)t] - 1}{\omega' - \omega_0}$$

In the limit $t \rightarrow \infty$, $\tilde{\xi}(\omega) \tilde{\xi}^*(\omega') \rightarrow \Phi(\omega) \delta(\omega - \omega')$, so that

$$\xi^2(t) = K^2 \int_{-\infty}^{\infty} \Phi(\omega) d\omega \frac{4 \sin^2[(\omega - \omega_0)t/2]}{(\omega - \omega_0)^2}$$

But $4 \sin^2[(\omega - \omega_0)t/2] / (\omega - \omega_0)^2 \rightarrow 2\pi t \delta(\omega - \omega_0)$ as $t \rightarrow \infty$ and we finally get

$$\frac{d\xi^2(t)}{dt} = 2\pi K^2 \Phi(\omega_0)$$

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Figure Captions

Fig. 1 Electric and magnetic field variation in time as seen by the particle.

