Calculation of the Current Flow Directions in a
Channel with Staggered Electrodes under MHD
Generator Conditions

M.G. Croci, R. Gorenflo, F. Hertweck

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ABSTRACT:

With suitable simplifications it is possible by complex analysis to calculate the
strength and direction of the electric current flow in MHD generator configurations.
In the model described the electric current forms a plane potential flow for which
an explicit expression can be given.

ZUSAMMENFASSUNG:

Bei geeigneten Vereinfachungen ist es möglich, Stärke und Richtung der in MHD-
Generator-Anordnungen fließenden elektrischen Ströme mittels komplexer Analysis
zu berechnen. In dem beschriebenen Modell bildet der elektrische Strom eine
ebene Potentialströmung, die explizit angebbar ist.

§ O. Purpose of the investigation

Since the temperature in MHD generator configurations depends on the electric currents, it is important to know their strength and direction. The purpose of this paper is to show that with suitable simplifications methods of complex analysis are a powerful tool for directly calculating the current field. We choose a plane model in which the current vector $\vec{j}$ itself can be considered as an analytic function of the space variable $z = x + iy$.

Methods of complex analysis have also been applied by, for example, FISHMAN, SCHULTZ-GRUNOW and DENZEL, and WITALIS (see "References"), but these authors treat the potential of the electric field as a harmonic function of the two space variables and solve the arising boundary value problems by conformal mapping.

Our investigation was performed as part of an investigation of actual MHD generator configurations reported by BREDERLOW and HODGSON. Our interest here is mainly mathematical. We do not go into the physical details but take the model and its (simplified) equations as given. A sketch of the geometrical configuration of the model is shown in Fig.1.

Between two parallel insulating walls perpendicular to the (x, y) - plane there is a stationary homogeneous flow of plasma in the x- direction. There are a positive and a negative electrode with zero cross sections (infinitely thin straight wires) perpendicular to the (x, y) - plane between these two walls. An electric current

flows from the positive electrode to the negative electrode, and the flow pattern of this current has to be found.
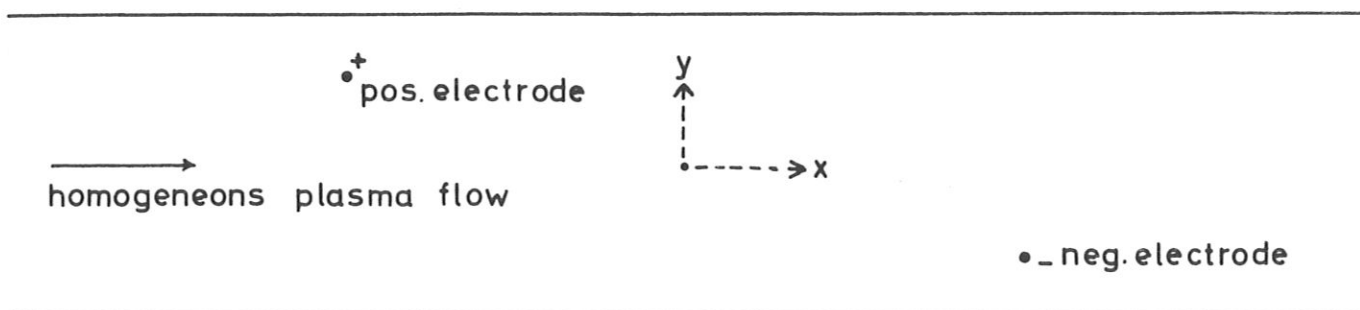

.+
pos. electrode

homogeneons plasma flow

y

-----→x

• _ neg. electrode

Fig. 1

As by-results we obtain detailed discussions of potential flows resulting from two opposite source - vortices between impenetrable parallel walls.

§ 1. Notations and mathematical models.

As usual, we introduce $z = x + iy = re^{i\varphi}$ and transform accordingly the basic equations and boundary conditions of our problem. We idealize the electrodes as infinitely thin straight wires, i.e. as having zero cross sections.

The electrodes then intersect the z- plane at the points $z = z_1$ and $z = z_2$ and are extended perpendicularly to this plane. The electrode at z_1 is positive, the other at z_2 is negative. We have a homogeneous plasma flow with velocity $\vec{v}$ in the x - direction and a homogeneous magnetic field $\vec{B}$ perpendicular to the z- plane. Introducing unit vectors

$\vec{n}_1 =$ unit vector in the x - direction,

$\vec{n}_2 =$ unit vector in the y - direction,

$\vec{n}_3 = \vec{n}_1 \times \vec{n}_2$,

we have

$$\vec{B} = \beta A \vec{n}_3 , \qquad \vec{v} = |\vec{v}| \vec{n}_1 , \qquad |\vec{v}| = const > 0 ,$$

where $A > 0$ and $\beta > 0$ are suitable constants. This, of course, is a simplification whose physical significance we do not want to discuss.

We treat two cases separately, (i) flow without walls,

(ii) flow between parallel insulating walls at $z = x \pm i \frac{\pi}{2}$, $-\infty < x < +\infty$.

We treat case (i) in order to get an insight into the mathemactical structure of the problem. The distance π of the walls in (ii) is not an essential restriction, it just means an appropriate choice of length unit.

The basic equation is the generalized OHM'S law,

(0) $\quad \vec{j}^* = \sigma_0 \left\{ \vec{E} + \vec{v} \times \vec{B} \right\} - \frac{1}{A} \vec{j}^* \times \vec{B}$,

where we assume σ_0 to be constant. See, for example, [7].

Because there is no dependence on time, $\vec{E}$ is irrotational. For convenience, we set $\vec{j} = \vec{j}^* / \sigma_0$ and take into account that $\vec{v} \times \vec{B} = -\alpha \vec{n}_2$ with $\alpha = |\vec{v}| \beta A$. This gives us

(1) $\quad \vec{j} = \vec{E} - \alpha \vec{n}_2 - \beta \vec{j} \times \vec{n}_3$, $\quad \vec{E} = -\nabla \phi$,

as basic equations, in which $\phi, \vec{j}$ and $\vec{E}$ and, in particular, the $\vec{j}$ - lines are to be determined. $\vec{E}$ and $\vec{j}$ are vectors in the (x,y) - plane.

The following conditions must be satisfied.

In case (i): $\quad \vec{j} \to \vec{0} \quad$ for $\quad |z| \to \infty \quad$ (regularity at infinity).

In case (ii): $\quad \vec{j} \to \vec{0} \quad$ for $\quad |x| \to \infty \quad$ (regularity),

$$\vec{j} = j_x(z) \vec{n}_1 \qquad for \qquad y = \pm \pi/2 ,$$

where $j_x (x \pm i\pi/2)$ is real (boundary condition).

Outside the electrodes $\vec{E}$ and $\vec{j}$ are harmonic vector fields and therefore have scalar potentials. It is clear that $\nabla \cdot \vec{j} = 0$ and $\nabla \times \vec{E} = \vec{0}$. We have to prove that $\nabla \times \vec{j} = \vec{0}$ and $\nabla \cdot \vec{E} = 0$. To do this we apply $\nabla \times$ and $\nabla \cdot$ to (1) and obtain.

$$-\frac{1}{\beta} \nabla \times \vec{j} = \nabla \times \left(\vec{j} \times \vec{n_3} \right) = \vec{j} \left(\nabla \cdot \vec{n_3} \right) + \left(\vec{n_3} \nabla \cdot \right) \vec{j} - \vec{n_3} \left(\nabla \cdot \vec{j} \right) - \left(\vec{j} \cdot \nabla \right) \vec{n_3}$$

$$= \left(\vec{n_3} \nabla \cdot \right) \vec{j} = \vec{0},$$

because $\vec{j}$ has no component in the $\vec{n_3}$ - direction. We further obtain

$$0 = \nabla \cdot \vec{j} = \nabla \cdot \vec{E} - \beta \nabla \cdot \left(\vec{j} \times \vec{n_3} \right)$$

$$= \nabla \vec{E} - \beta \left\{ \vec{n_3} \left(\nabla \times \vec{j} \right) - \vec{j} \left(\nabla \times n_3 \right) \right\} = \nabla \cdot \vec{E}.$$

We now represent our problem in complex notation. Instead of $\vec{j} = j_x \vec{n_1} + j_y \vec{n_2}$ we take $j = j_x + i j_y$, and correspondingly for the other vectors. The conjugate complex to $a = a_x + i a_y$ is denoted by $\bar{a} = a_x - i a_y$. In particular, $\vec{n_1}$ corresponds to 1, $\vec{n_2}$ to i, and $\vec{j} \times \vec{n_3}$ to $-i j$. We note that E and j have complex potentials $W(z)$ and $F(z)$, respectively, which are analytic functions,

$$(3) \qquad E = - \overline{W'(z)}, \qquad j = - \overline{F'(z)}.$$

From (1) we obtain the corresponding equation

$$(4) \qquad j = E - \alpha i + i \beta j$$

or the equivalent relations

$$(5) \qquad j = \frac{E - i\alpha}{1 - i\beta}, \qquad F(z) = \frac{W(z) - i\alpha}{1 + i\beta}$$

For convenience, we define

$$(6) \qquad f(z) := \overline{j(z)} = - F'(z)$$

and note that $f(z)$ is an analytic function.

The streamlines of the j - field are now given by

$$(7) \qquad S(z) = const$$

where

$$(8) \qquad S(z) := - \text{Im } F(z)$$

is the flux-function of the electric current.

We first investigate the configuration corresponding to a single electrode at $z = 0$. If there were no magnetic field present, we should have a single source with j streaming radially outward. The resulting radial lines are distorted by the magnetic field (Hall effect) to logarithmic spirals, all j - vectors being turned through the same angle, depending on the strength of B. We therefore have superposition of a source and a vortex. This is generally the local structure of the $\dot{j}$ - field near any electrode. If there is a configuration of more than one electrode, we simply have to construct the whole $\dot{j}$-field from such source (or sink) - vortex elements, thereby using the fact that the E - field does not have any vortices.

With the notations

$$Q = \oint_{\Gamma} \vec{j}\, \vec{n}\; ds \qquad \text{(source strength)},$$

$$P = \oint_{\Gamma} \vec{j}\, \vec{t}\; ds \qquad \text{(circulation)},$$

where the integration is to be made counterclockwise round a closed smooth JORDAN curve with unit tangent vector $\vec{t}$ and outward unit normal vector $\vec{n}$, we have
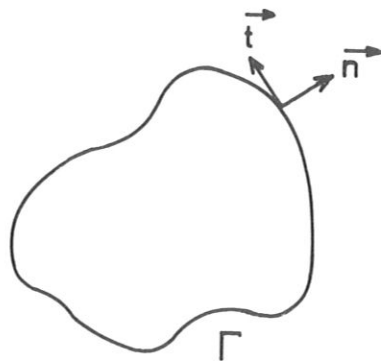
$$(3) \qquad P + iQ = \oint_{\Gamma} f(z)\, dz$$

Fig. 2

A source of strength Q in $z=0$ yields

$$f(z) = \frac{Q}{2\pi} \frac{1}{z} \quad , \qquad F(z) = -\frac{Q}{2\pi} \log z ,$$

and a vortex of circulation P yields

$$f(z) = -\frac{P}{2\pi} \frac{i}{z} \quad , \qquad F(z) = \frac{P}{2\pi} i \log z .$$

The general source-vortex at $z=0$ is represented by

$$(10) \quad f(z) = \frac{Q - iP}{2\pi} \frac{1}{z} \quad , \qquad F(z) = \frac{-Q + iP}{2\pi} \log z ,$$

and its streamlines are logarithmic spirals

$$\text{Im } F(z) = \frac{P}{2\pi} \log r - \frac{Q}{2\pi} \varphi = \text{const}$$

or

$$r = e^C \exp \left(Q \varphi / P \right) .$$

We now look for a generally valid relation between P and Q. Near any electrode which we may imagine as lying at $z = 0$ we have locally

$$j(z) \sim \frac{Q + iP}{2\pi} \frac{1}{\bar{z}}$$

whereas (see (5))

$$\overline{E}(z) = \left(1 - i\beta \right) j(z) + i\alpha \sim \frac{1}{2\pi} \left(1 - i\beta \right) \left(Q + i P \right) \frac{1}{\bar{z}}$$

must be a pure source field. This means

$$\text{Im} \left(1 - i\beta \right) \left(Q + i P \right) = -\beta Q + P = 0, \qquad\qquad \text{or}$$

$$(11) \qquad\qquad P = \beta Q .$$

This relation is valid for every electrode. Given Q, corresponding to the strength of the electrode $\left(Q_2 = - Q_n \right)$, we obtain P by (11) and then the whole configuration by superposing either the fields corresponding to each of the electrodes in case (i) or the fields corresponding to infinitely many electrodes obtained by reflecting the electrodes, the walls and the individual fields of the electrode at the walls in order to fulfill the boundary

conditions in case (ii) . The j-field is independent of α , that is independent of the velocity of the streaming plasma. But, according to (5) , we have

$$(12)\quad E(z) = \left(1 - i\beta\right) j(z) + i\alpha = E_o(z) + i\alpha ,$$

which means that the E-field does depend on the velocity, namely on α. This can easily be explained as the influence of an additional field

$$i\alpha = - \vec{\jmath} \times \vec{B} .$$

In case (ii) this means that the walls are charged accordingly.

The situation is much more complicated for electrodes of non-zero cross sections, for then $\vec{\jmath}$ can no longer be homogeneous. But our theory should correspond closely to reality if these cross sections are sufficiently small circular areas, because the equipotential lines of our E-field are infinitesimal circles near the electrodes. If other kinds of singularities (multipoles $A_{\nu,n}(z-z_\nu)^{-n}$, $n \geq 2$) were superposed, the equipotential lines of the E-field would no longer be infinitesimal circles. It is therefore natural to take only source-vortices at the points $z = z_1$ and $z = z_2$.

With this restriction (infinitesimal circles as equipotential lines of ϕ , the potential of E) the solution to our problem is unique for a given value of $Q = Q_1$. In case (i) $f(z) \to 0$ for $z \to \infty$, and for any other solution $f^*(z)$ we should have $f^*(z) \to 0$ as well, therefore $g(z) := f(z) - f^*(z)$ should be an entire function tending to zero for $z \to \infty$, from which it follows that $g(z) \equiv 0$. In case (ii) the same argument is valid if $z \to \infty$ is replaced by $|x| \to \infty$. In this case $f(z)$ and $f^*(z)$ are both periodic with period $2\pi i$ and have the same singularities and residuals.

We now proceed to construct in detail the solutions for the two cases. For simplicity, we always assume $Q_1 = 2\pi$, which only means a special choice of units.

§ 2. Flow without walls

Because we are mainly interested in the j-field, which is independent of α, we introduce the positive number

$$a = \frac{1}{2}\left(z_1 - z_2\right) e^{-i\,\arg\,(z_1 - z_2)}$$

and rotate the plane so that the electrodes intersect the plane at a and $-a$. This is achieved by

$$z^* = \left(z - \frac{z_1 + z_2}{2}\right) e^{-i\,\arg\,(z_1 - z_2)},$$

Finally, we again write z instead of z^*. We thus have

$$z_1 \longleftrightarrow a, \quad z_2 \longleftrightarrow -a, \quad a > 0.$$

We then get

$$f(z) = \frac{1 - i\beta}{z - a} - \frac{1 - i\beta}{z + a} = \overline{j(z)}, \quad F(z) = -\left(1 - i\beta\right) \log \frac{z - a}{z + a},$$

$$(13) \quad -\operatorname{Re} F(z) = \log\left|\frac{z - a}{z + a}\right| + \beta \arg \frac{z - a}{z + a},$$

$$S(z) := -\operatorname{Im} F(z) = -\beta \log\left|\frac{z - a}{z + a}\right| + \arg \frac{z - a}{z + a}.$$

We shall derive an explicit representation of the j - lines. By introducing the variable $\zeta = \frac{z - a}{z + a}$, i.e. $z = a \frac{1 + \zeta}{1 - \zeta}$, the singular points $z = \pm a$ are mapped into $\zeta_1 = \zeta(a) = 0$ and $\zeta_2 = \zeta(-a) = \infty$. The complex potential is now

$$G(\zeta) := F(z(\zeta)) = -\left(1 - i\beta\right) \log \zeta.$$

Introduction of polar coordinates ϱ, Θ with $\zeta = \varrho e^{i\Theta}$ yields

$$G(\zeta) = -\left(\log \varrho + \beta\Theta\right) - i\left(-\beta \log \varrho + \Theta\right),$$

$$S^*(\zeta) := S(z) = \Theta - \beta \log \varrho.$$

The equation of the j-lines is now

$$\Theta - \beta \log s = C \quad \text{or} \quad s = \exp\left(\frac{\Theta - C}{\beta}\right) \quad \text{or}$$

$$\zeta = e^{it} \exp\left(\frac{t-C}{\beta}\right), \quad -\infty < t < +\infty,$$

where C denotes the current flux measured from the line $S = 0$.

This is equivalent to

$$(14) \quad z(t; C) = z(t) = a \frac{1 + e^{it} \exp\left((t-C)/\beta\right)}{1 - e^{it} \exp\left((t-C)/\beta\right)} .$$

All j- lines are obtained if C varies in $0 \le C < 2\pi$.

We note that

$$z(t) \rightarrow -a \quad \text{for} \quad t \rightarrow +\infty, \quad z(t) \rightarrow \infty \quad \text{for} \quad t \rightarrow 0.$$

The j-lines can easily be computed and plotted. They are the conformal images of logarithmic spirals. Some special questions may now be asked.

The zero line $C = 0$ is the line through $z = \infty$. Its equation is

$$z(t) = a \frac{1 + e^{it} \exp\left(t/\beta\right)}{1 - e^{it} \exp\left(t/\beta\right)} .$$
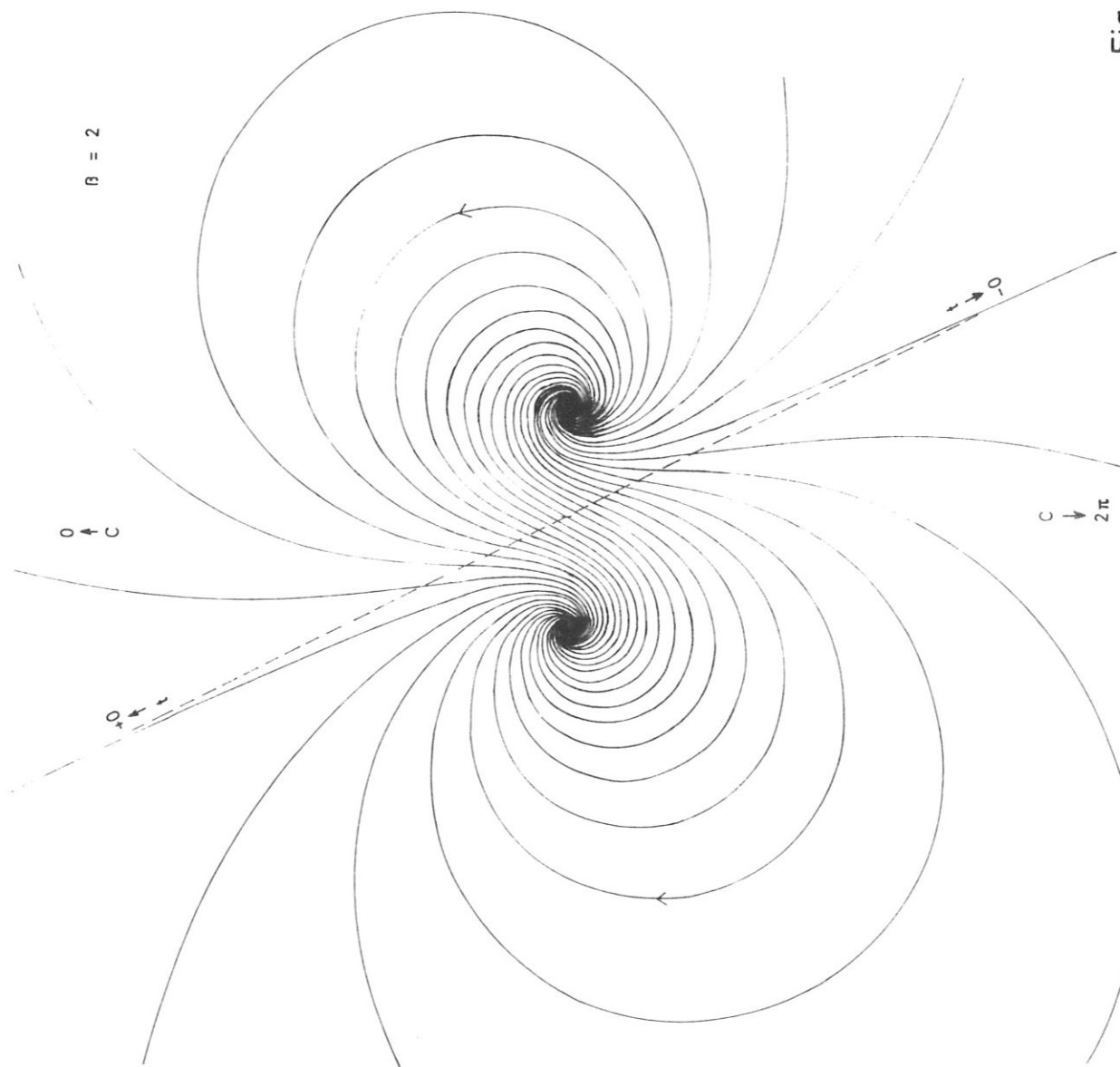
Fig. 3

We are interested in the asymptotic behaviour for $t \to \pm 0$, particularly in the asymptotic slope. We find

$$z(t) = a \; \frac{1 + \exp\left(t\left(i + \frac{1}{\beta}\right)\right)}{1 - \exp\left(t\left(i + \frac{1}{\beta}\right)\right)} \sim \frac{2a\beta}{1+\beta^2}\left(-1 + \beta\, i\right) t^{-1},$$

$$\lim_{t \to +0} \arg z(t) = - \operatorname{arctg} \beta + \pi,$$

$$\lim_{t \to -0} \arg z(t) = - \operatorname{arctg} \beta,$$

where the values of the arctg - function are restricted as usual to the interval $\left(-\pi/2, \, \pi/2\right)$. By a sharper analysis we obtain

$$z(t) = \frac{2a\beta}{1+\beta^2}\left(-1 + i\beta\right) t^{-1} - \frac{a}{6\beta}\left(1 + i\beta\right) t + \ldots ,$$

so that the straight line

$$z = \frac{2a\beta}{1+\beta^2}\left(-1 + i\beta\right)/t$$

or $\quad y = -\beta x \quad$ is an asymptotic line for $z\left(t; \, 0\right)$.

We now consider the behaviour of the j-field on the symmetry line $x = 0$. From (13) we calculate

$$j\left(iy\right) = - \frac{2a}{a^2 + y^2}\left(1 + iy\right) \quad , \text{ which implies}$$

$$j_x\left(iy\right) < 0 \quad \text{ and } \quad j_y\left(iy\right) < 0 \text{ for } \quad -\infty < y < +\infty.$$

Equation (13) is now used again to give

$$S\left(iy\right) = \arg \frac{iy - a}{iy + a} = \arg\left(iy - a\right) - \arg\left(iy + a\right).$$

According to our agreement $0 \leqq C = S(z) < 2\pi$ we take $S(0) = \arg\left(-1\right) = \pi$. As y increases from $-\infty$ via 0 to $+\infty$, $\arg\left(iy - a\right)$ decreases from

$-\frac{3}{2}\tilde{\pi}$ via $\tilde{\pi}$ to $\tilde{\pi}/2$, whereas arg $\left(iy+a\right)$
increases from $-\pi/2$ via 0 to $\tilde{\pi}/2$. We see that $S\left(iy\right)$
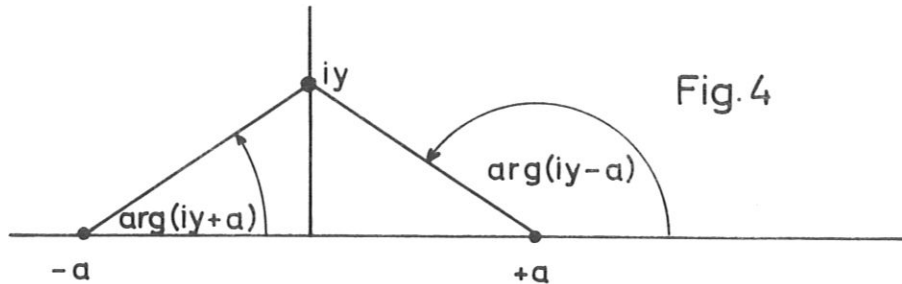
Fig. 4

decreases from $2\tilde{\pi}$ via $\tilde{\pi}$ to 0. This corresponds to the fact that the
source at $z=a$ with $Q=2\tilde{\pi}$ lies to the right of the y -axis. A simple
consequence is that every j-line, with the exception of the line $C=0$,
crosses the axis $x\neq 0$ exactly once. The line $C=0$ does not meet the axis
$x=0$,

We note that

$S\left(iy\right)$ is monotonic, $S\left(iy\right)\to 0$ for $y\to\infty$,
$S\left(iy\right)\to 2\tilde{\pi}$ for $y\to -\infty$, $S(0)=\tilde{\pi}$.

The equation $S\left(iy\right)=C$ can be explicitly solved for $y=\gamma\left(C\right)$
so that it is possible to find the points on the y-axis which correspond to given
values of the current flux. From

$C=\arg\dfrac{iy-a}{iy+a}$ and $\left|\dfrac{iy-a}{iy+a}\right|=1$ it follows
that $e^{iC}=\dfrac{iy-a}{iy+a}$, whence $y=ia\dfrac{e^{iC}+1}{e^{iC}-1}=a\ \mathrm{ctg}\dfrac{C}{2}$.

For $y=a\ \mathrm{ctg}\dfrac{C}{2}$ we have $S\left(iy\right)=C$.

The corresponding value of t for the j-line

$$S(z) = C \quad \text{is} \quad t = C, \text{ that is we have}$$

$$z(C; C) = a \frac{1 + e^{iC}}{1 - e^{iC}} = i a \, ctg \frac{C}{2} \, .$$

§ 3 . Flow between parallel insulating walls

We treat this case by conformally mapping the strip

$$-\pi/2 < y < \pi/2 \qquad \qquad \text{onto the}$$

upper half-plane $Im \ t > 0$ with $t = i e^z$.

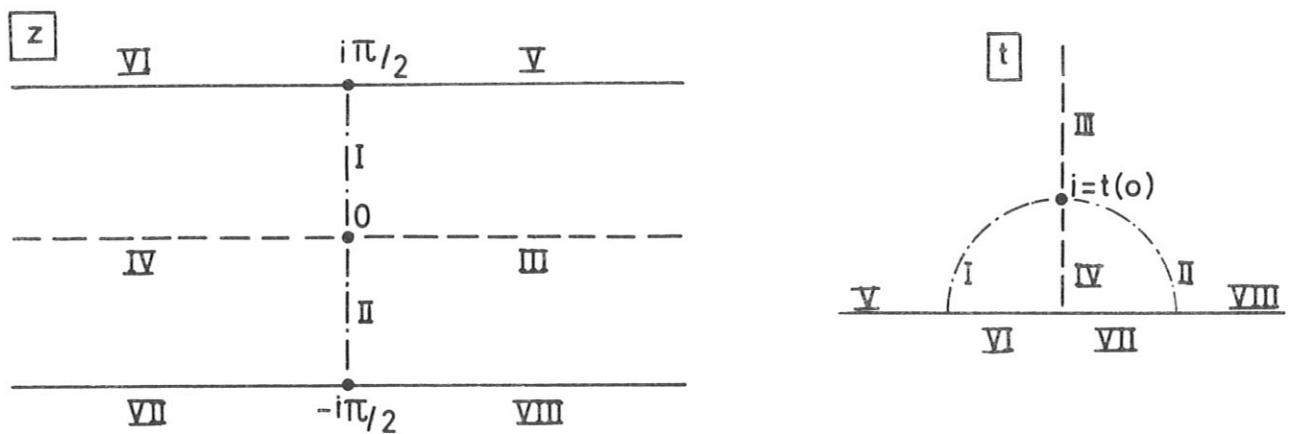The singular points (the electrodes)z_λ are mapped onto $t_\lambda = i e^{z_\lambda}$

Fig. 5

Source – strengths and circulations are invariant to conformal mapping:

$$\int f(z)\, dz = -\int d\, F(z) = -\int d\, F(z(t)).$$

As boundary condition we now have that in the t – plane the flow must be tangential to the real axis. In the t – plane we introduce the flow vector

$\overline{g(t)}$ and the notations

$$(15)\quad G(t) := F(z) = -\int^{t} g(t)\, dt = -\int^{z} f(z)\, dz.$$

By the method of reflection we find

$$(16)\quad g(t) = \left(\frac{1-i\beta}{t-t_1} + \frac{1+i\beta}{t-\overline{t_1}}\right) - \left(\frac{1-i\beta}{t-t_2} + \frac{1+i\beta}{t-\overline{t_2}}\right).$$
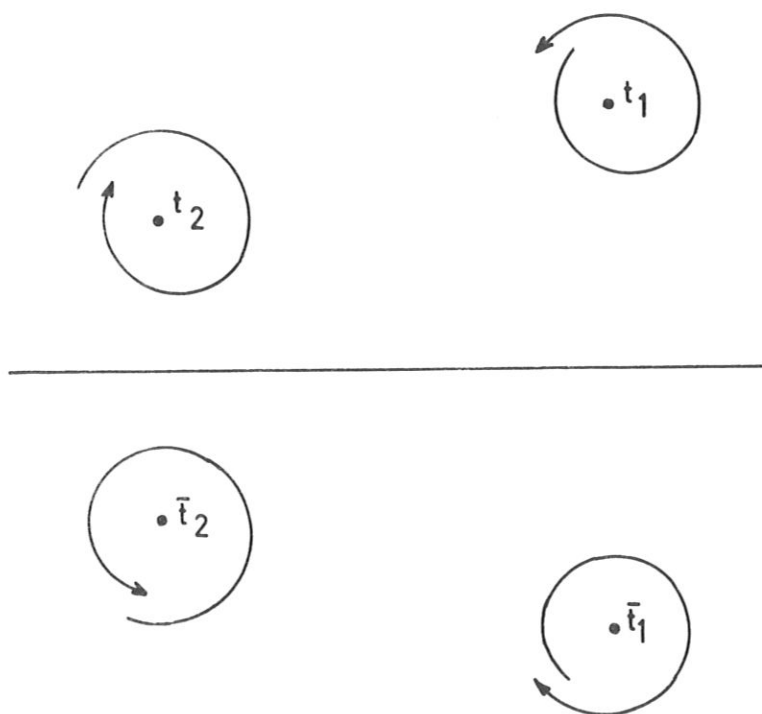
Fig. 6

$(17) \quad -G(t) = (1 - i\beta)\log(t - t_1) + (1 + i\beta)\log(t - \bar{t}_1) - $

$$\left\{(1 - i\beta)\log(t - t_2) + (1 + i\beta)\log(t - \bar{t}_2)\right\},$$

$(18) \quad j(z) = \overline{f(z)}$ with $f(z) = -F'(z) = -G'(t)\dfrac{dt}{dz} = ie^z g(ie^z),$

$(19) \quad f(z) = t\,g(t)$ with $t = ie^z$.

The boundary condition is, of course, now satisfied. The

regularity condition $f(z) \to 0$ for $x \to \pm\infty$ is transformed into

$t\,g(t) \to 0$ for $t \to \infty$ and $t \to 0$. We find, indeed,

$$t\,g(t) = \frac{1 - i\beta}{1 - \dfrac{t_1}{t}} + \frac{1 + i\beta}{1 - \dfrac{\bar{t}_1}{t}} - \left\{\frac{1 - i\beta}{1 - \dfrac{t_2}{t}} + \frac{1 + i\beta}{1 - \dfrac{\bar{t}_2}{t}}\right\} = \mathcal{O}(t^{-1}).$$

From now on we restrict our attention to the particular case of

symmetry, that is to the case

$$(20) \quad z_1 = a + ib, \quad a > 0, \quad -\pi/2 < b < \pi/2, \quad z_2 = -z_1,$$

and investigate the qualitative behaviour of the j-flow. The local

structure near the electrodes is evident (source-vortex). We note that the

whole configuration of j-lines is symmetric to $z = 0$ (because the singularities

possess this property), any j-line is mapped onto another j-line by replacing

z by $-z$, and the direction of flow has also to be inverted. The

j-line passing through $z = 0$ is symmetric to itself.

We shall prove that there are two neutral points (stagnation points)

in which $j = 0$, one on each boundary component, if $0 < \beta < \infty$

and $\operatorname{tg}\beta \neq \beta$ th a.

In order to determine neutral points, we have to solve $f(z) = 0$

or $g(t) = 0$. The latter equation, multiplied by

$$(t - t_1)(t - \bar{t}_1)(t - t_2)(t - \bar{t}_2),$$

leads to

$$(21) \quad A t^2 + B t + C = 0$$

where

$$A = t_1 + \bar{t}_1 - (t_2 + \bar{t}_2) - i\beta\left(t_1 - \bar{t}_1 - t_2 + \bar{t}_2\right),$$

$$B = 2\left(|t_2|^2 - |t_1|^2\right) + 2 i\beta\left(t_1 \bar{t}_2 - \bar{t}_1 t_2\right),$$

$$C = -\left(t_1 + \bar{t}_1\right)|t_2|^2 + \left(t_2 + \bar{t}_2\right)|t_1|^2 + i\beta\left\{(\bar{t}_1 - t_1)|t_2|^2 + (t_2 - \bar{t}_2)|t_1|^2\right\}$$

Obviously, A, B, C are real.

Setting $z_\lambda = x_\lambda + i y_\lambda$ yields

$$(22) \quad \begin{cases} A/2 = e^{x_1}(-\sin y_1 + \beta \cos y_1) + e^{x_2}(\sin y_2 - \beta \cos y_2) \\ B/2 = e^{2x_2} - e^{2x_1} - 2\beta \sin(y_1 - y_2) \\ C/2 = e^{2x_2 + x_1}(\sin y_1 + \beta \cos y_1) - e^{2x_1 + x_2}(\sin y_2 + \beta \cos y_2). \end{cases}$$

By (20) we find

$$(23) \quad \begin{cases} A/4 = - \operatorname{ch} a \sin b + \beta \operatorname{sh} a \cos b \\ B/4 = - (\operatorname{sh}(2a) + \beta \sin(2b)) \\ C/4 = \operatorname{ch} a \sin b - \beta \operatorname{sh} a \cos b = - A/4. \end{cases}$$

If $A \neq 0$, we have from (21)

$$(24) \quad t^2 + \frac{B}{A} t - 1 = 0, \quad t = t_\pm = \frac{1}{2A}\left(-B \mp \sqrt{B^2 + 4A^2}\right),$$

whence $t_+ t_- = -1$ for the solutions $t = t_\pm$

and, because A and B are real, both values $t_\pm$ are real, e.g.

$$t_+ < 0, \quad t_- > 0.$$

It follows

$$(25) \quad z_+ = - z_- = \frac{i\tilde{\pi}}{2} + \log(-t_+).$$

On each of the boundary components $y = \pm\tilde{\pi}/2$ there is exactly one neutral point. (This is no longer generally valid in the general case $z_2 \neq -z_1$ since in that case A/C need not be < 0, numerical test computations showed that $A/C > 0$ may happen.)

If $A = 0$, we have the unique solution $t = 0$, $z = \infty$. There is no finite neutral point in this case. $A = 0$ happens if

$$(26) \quad tg\, b = \beta\, th\, a\,,$$

and this splits up into three cases

$(.)\quad \beta = 0, \quad b = 0, \quad a > 0.$
Source and sink. See Fig. 7.[+]

$(..)\quad \beta = \infty, \quad a = 0, \quad b \neq 0, \; \left\{ th\, a = \frac{1}{\beta}\, tg\, b \right\}.$
Vortex pair. See Fig. 8.

$(...)\quad 0 < \beta < \infty, \quad a > 0, \quad 0 < b = arctg\left(\beta\, th\, a\right) < \pi/2.$
Source-vortex and sink-vortex. See Fig. 9.

+ The field lines in this and subsequent plots were obtained by numerical integration of the j-field.

a = π   b = 0
β = 0
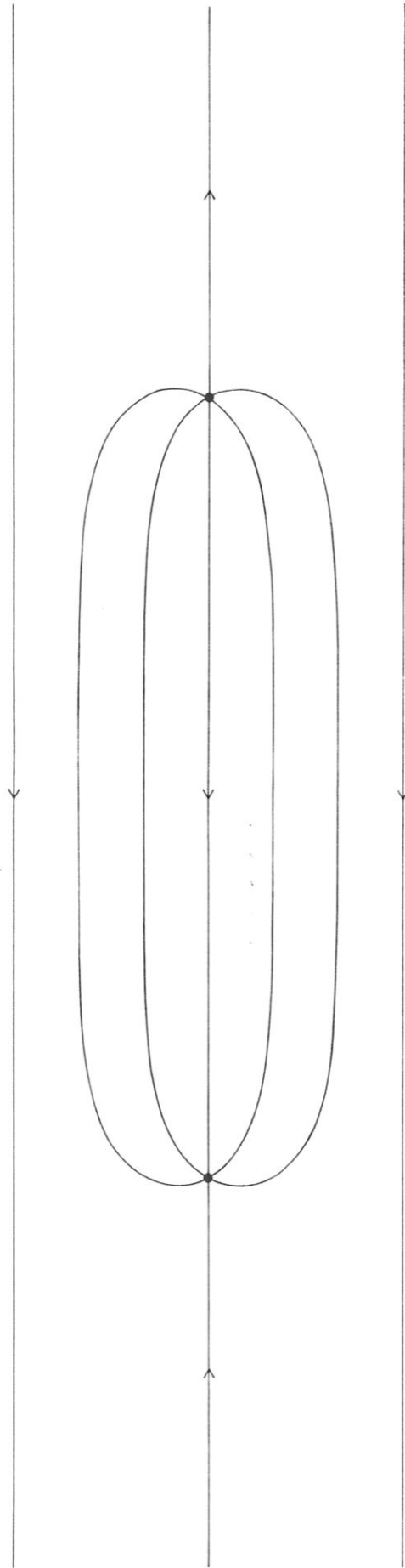
Fig.7

Fig. 8

a = 0   b = π/4

β = ∞

a = π   b = arctan   (β tha)

β = 1

Fig. 9

ß = 3

a = 2.749   b = - 0.6239

Fig. 10

a = 0   b = π / 4

β = 2
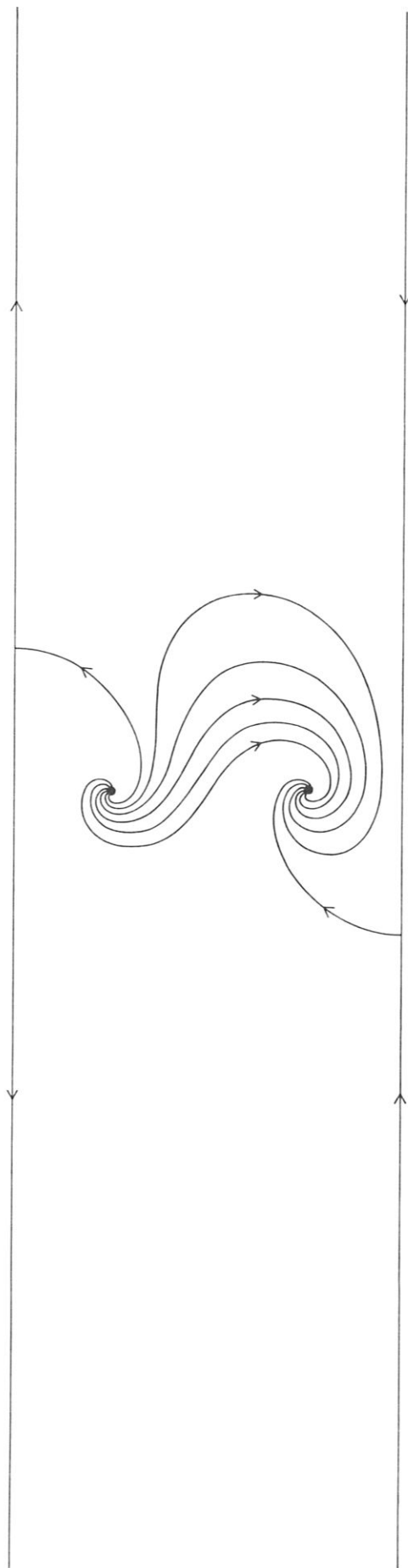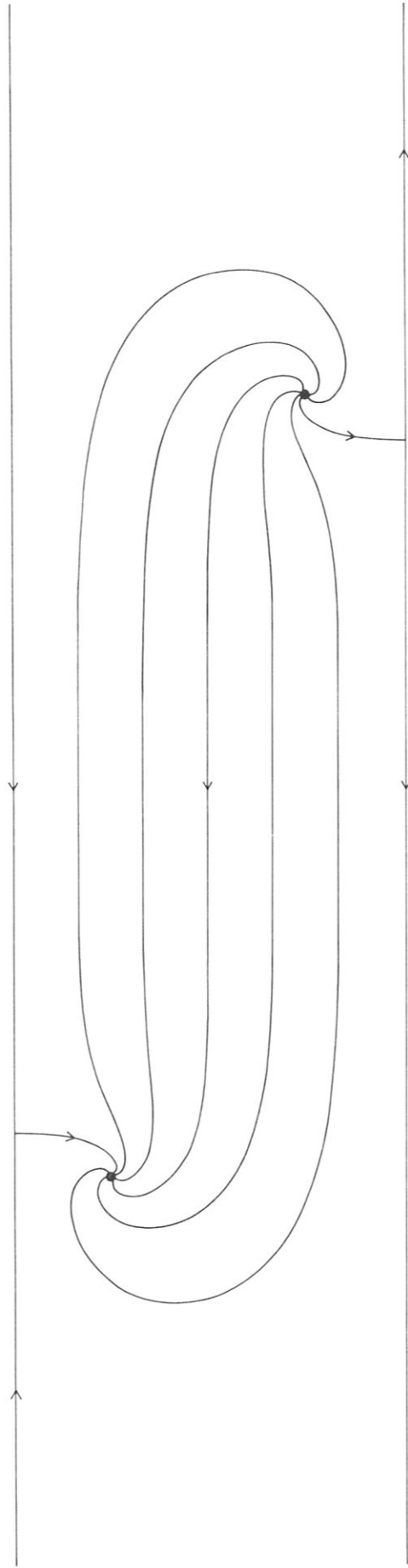
Fig. 11

a = π   b = -arctan (βtha)
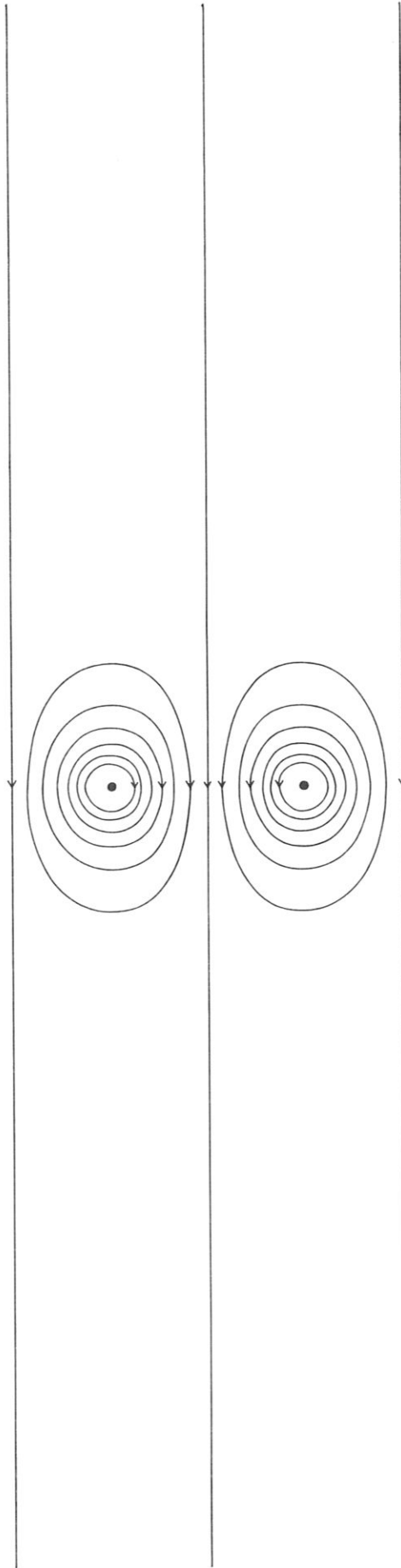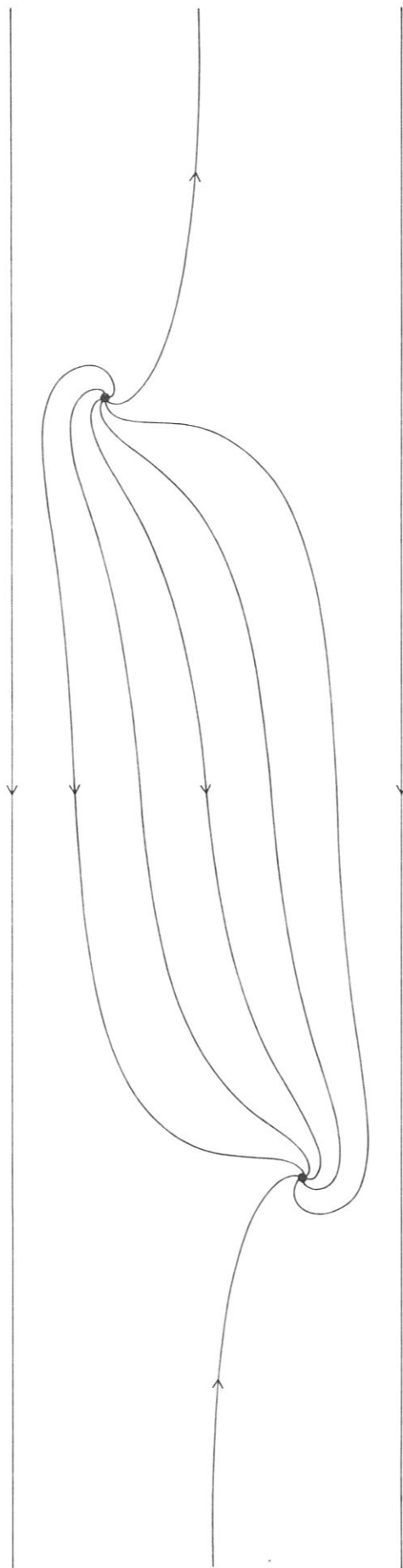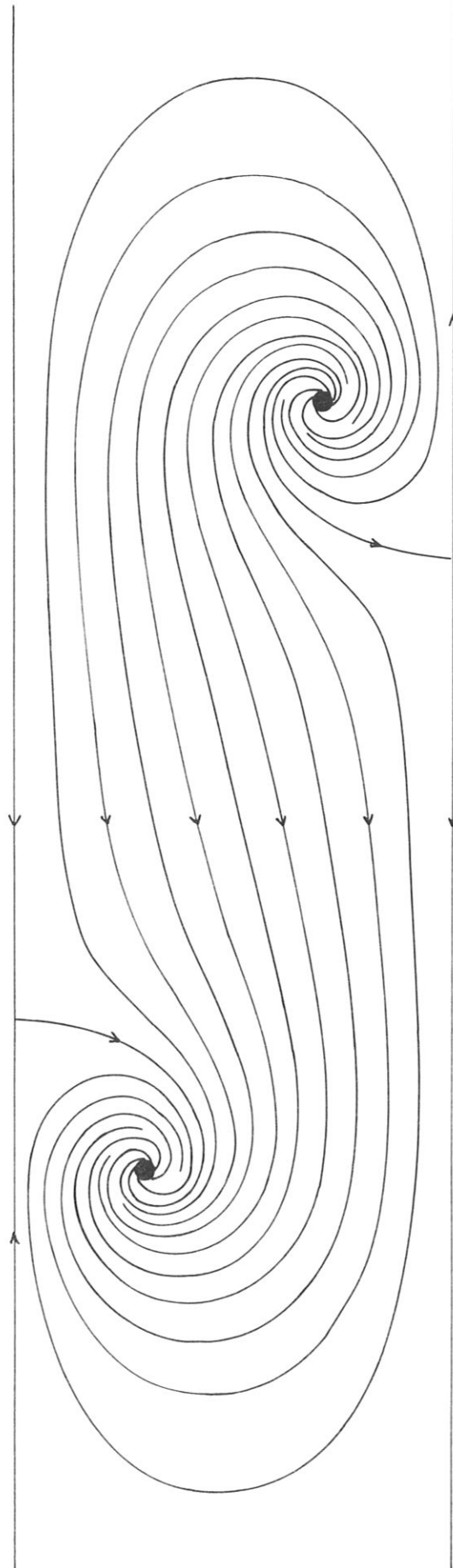
β = 1

Fig. 12

The structure of the j-field for the case of two finite neutral points can easily be sketched (see Fig. 10). $S(z)$ is a single - valued function if we make cuts along the j-lines beginning or ending in the neutral points.

We cannot expect that $S(iy)$, the flux on the line of symmetry, generally behaves as regularly as in the case (i) , where there are no walls. If the distance of the electrodes from the line of symmetry is sufficiently small compared with their distance from the walls, we must expect that there are j - lines which intersect the line $x = 0$ more than once. $S(iy)$ is not a monotonic function in this case. See Fig. 11. Nevertheless, if $a > 0$, we may fix the constant of integration in

$$S(iy) = \int^y j_x(iy)\, dy$$

in such a way that

$$S\left(-i\pi/2\right) = 2\pi \quad \text{and} \quad S\left(i\pi/2\right) = 0.$$

Compared with case $(.)$ we see that we can expect monotonicity of $S(iy)$ if $|b| << a$ and β is not too great. Observing the special case (33) (to be treated later in this §), we see that smallness of β is not so important if $|b| << a$.

A property worth having for physical reasons (see $[1]$) is that the j - flow near $x = 0$ should be nearly homogeneous. It can be hoped that this is the case if $j(0)$ is real (and < 0), that is if

$$j_y(0) = 0.$$

Noting $z = 0 \curvearrowright t = i$ and

$$\left\{z = iy,\ -\pi/2 < y < \pi/2\right\} \curvearrowright \left\{|t| = 1,\ \text{Im}\ t > 0\right\}$$

we see that because of conformity we must have

$$(27) \qquad \text{Re } g(i) = 0.$$

This condition can always be satisfied if $a > 0$ and $b = b(a, \beta)$ is suitably taken. Defining

$$(28) \qquad \gamma(b) = \gamma(a, \beta; b) := \text{Re } g(i)$$

we obtain, after a lengthy but quite elementary calculation,

$$(29) \qquad \gamma(a, \beta; b) = \frac{\beta(e^{2a} - 1) + 2e^{a} \sin b}{|e^{a} - e^{ib}|^{2}} + \frac{\beta(-e^{2a} + 1) + 2e^{a} \sin b}{|e^{a} + e^{-ib}|^{2}}$$

whence

$$\gamma(b) \to \frac{4e^{a}}{|e^{a} - i|^{2}} > 0 \qquad \text{for} \qquad b \to \pi/2,$$

$$\gamma(b) \to -\frac{4e^{a}}{|e^{a} + i|^{2}} < 0 \qquad \text{for} \qquad b \to -\pi/2$$

Therefore γ hast at least one zero in $-\pi/2 < b < \pi/2$.

The solution is unique and is given by

$$(29') \qquad b = -\text{arctg}(\beta \text{ th } a).$$

See Fig. 12.

As a final particular case we treat the case

$a > 0, \ b = 0, \ \beta > 0$ in a different manner, namely by hyperbolic functions and without intermediate conformal mapping.
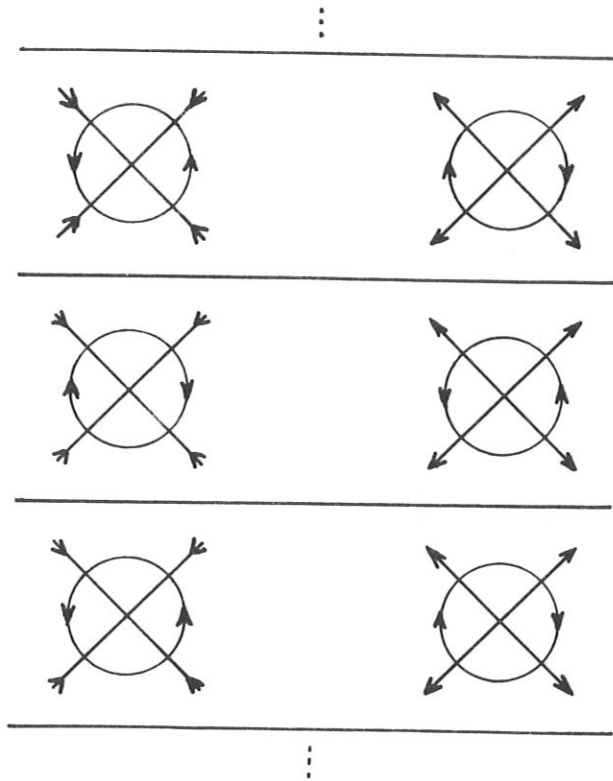
Fig. 13

By superposing infinitely many sources and vortices as indicated in Fig. 13 we satisfy the boundary condition. We have

$$f(z) = f_{Source}(z) + f_{Vortex}(z)$$

where

$$(30) \begin{cases} f_{Source}(z) = ctgh(z-a) - ctgh(z+a) \\ f_{Vortex}(z) = -i\beta \left\{ \dfrac{1}{sh(z-a)} - \dfrac{1}{sh(z+a)} \right\} \end{cases}$$

Therefore

$$(31) \quad f(z) = \operatorname{tgh}(z-a) - \operatorname{tgh}(z+a) - i\beta \left\{ \frac{1}{\operatorname{sh}(z-a)} - \frac{1}{\operatorname{sh}(z+a)} \right\}.$$

On the boundary we have

$$f\left(x \pm \frac{i\pi}{2}\right) = \operatorname{th}(x-a) - \operatorname{th}(x+a) \mp \beta \left\{ \frac{1}{\operatorname{ch}(x-a)} - \frac{1}{\operatorname{ch}(x+a)} \right\}$$

which is real. It can easily be checked that

$$f(z) \to 0 \quad \text{if} \quad |x| \to \infty.$$

The neutral points are obtained from $f(z) = 0$ as $z = z_\pm$,

$$(32) \quad \begin{cases} z_+ = \dfrac{i\pi}{2} - x^* = -z_- \\[2mm] x^* = \log\left\{ \dfrac{\operatorname{ch} a}{\beta} + \sqrt{\left(\dfrac{\operatorname{ch} a}{\beta}\right)^2 + 1} \right\}. \end{cases} \quad \text{where}$$

If β increases from 0 to ∞, x^* decreases from ∞ to 0.

By elementary calculation we find for the symmetry line $x = 0$

$$(33) \quad j(iy) = \overline{f(iy)} = 2\left\{ -\frac{\operatorname{sh}(2a)}{\operatorname{ch}(2a) - \cos(2y)} - 2i\beta \frac{\operatorname{sh} a \; \cos y}{\operatorname{ch}(2a) - \cos 2y} \right\}$$

$$j_x(iy) < 0, \qquad j_y(iy) < 0.$$

It can easily be checked that

$$\int_{-\pi/2}^{\pi/2} \operatorname{Re} \; j(iy)\, dy = -2\pi.$$

We see that $S(iy)$ is monotonically decreasing if y increases from $-\pi/2$ to $\pi/2$. We set $S(-\pi/2) = 2\pi$, $S(\pi/2) = 0$, and have $S(0) = \pi$ for symmetry reasons.

The complex potential is not a single - valued function of z.

We find

$$(34) \quad -F(z) = \log \frac{sh(z-a)}{sh(z+a)} - i\beta \log \frac{th \cdot \frac{z-a}{2}}{th \frac{z+a}{2}} \quad)$$

$$(35) \quad S(z) = \arg \frac{sh(z-a)}{sh(z+a)} - \beta \log \left| \frac{th \frac{z-a}{2}}{th \frac{z+a}{2}} \right| .$$

Again, $S(z)$ can be made single – valued by cutting the plane along the ¡ – lines beginning or ending at the neutral points.

Finally, we sketch the structure of the three ¡ – fields for the cases

(*) $\beta = 0$, source and sink, as in $\left(.\right)$. See Fig. 14.

(**) $0 < \beta < \infty$, normal case. See Fig. 15.

(***) $\beta = \infty$, vortex pair. See Fig. 16.

$a = \pi \quad b = 0$

$\beta = 0$

Fig. 14

b = 0
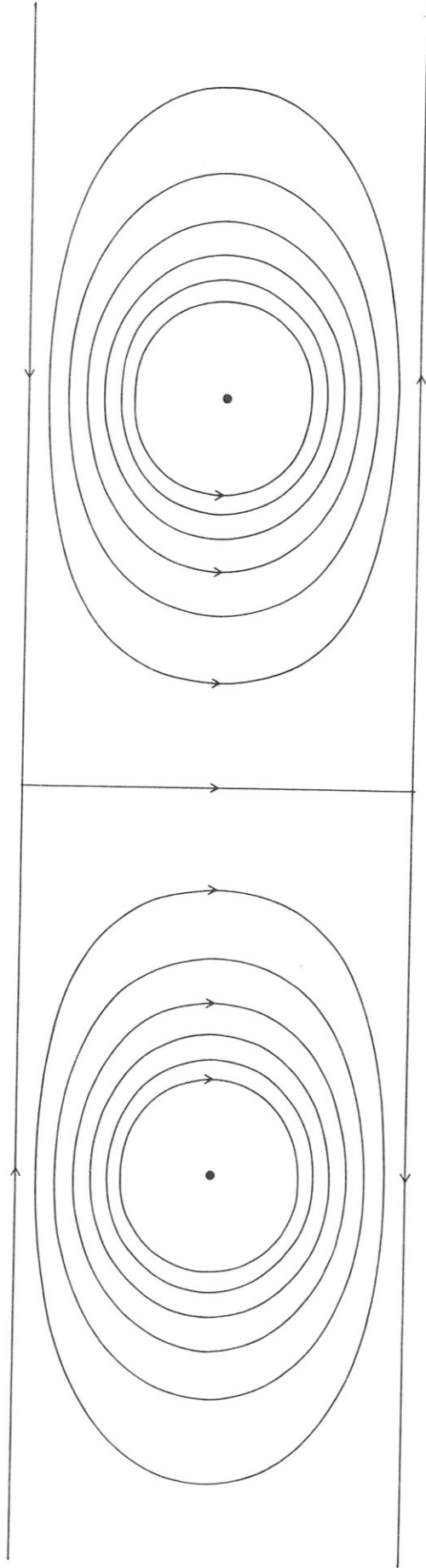
ß = 3

Fig. 15

α = π   b = 0

β = ∞
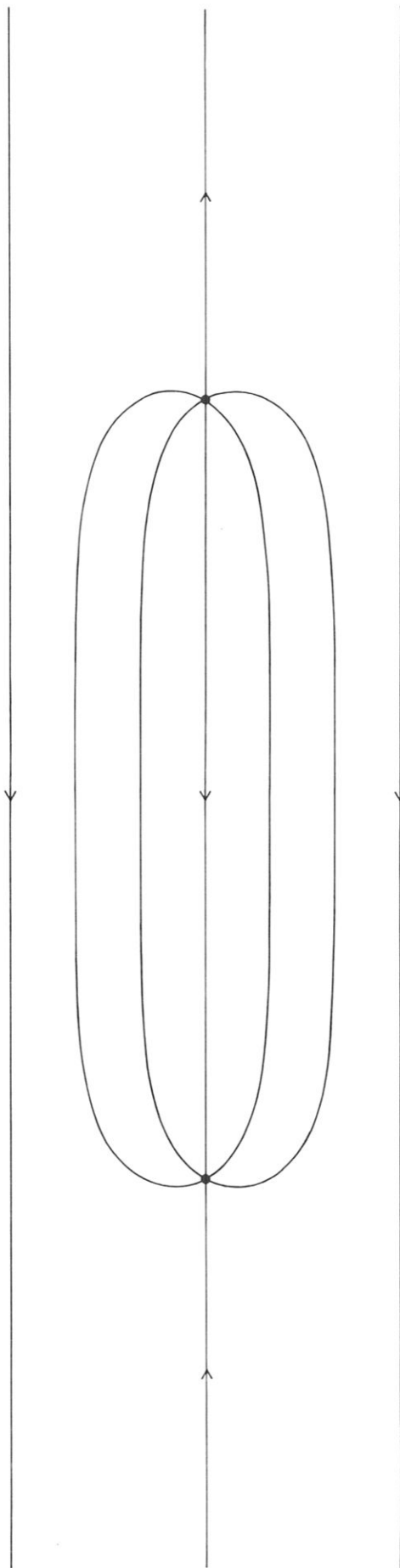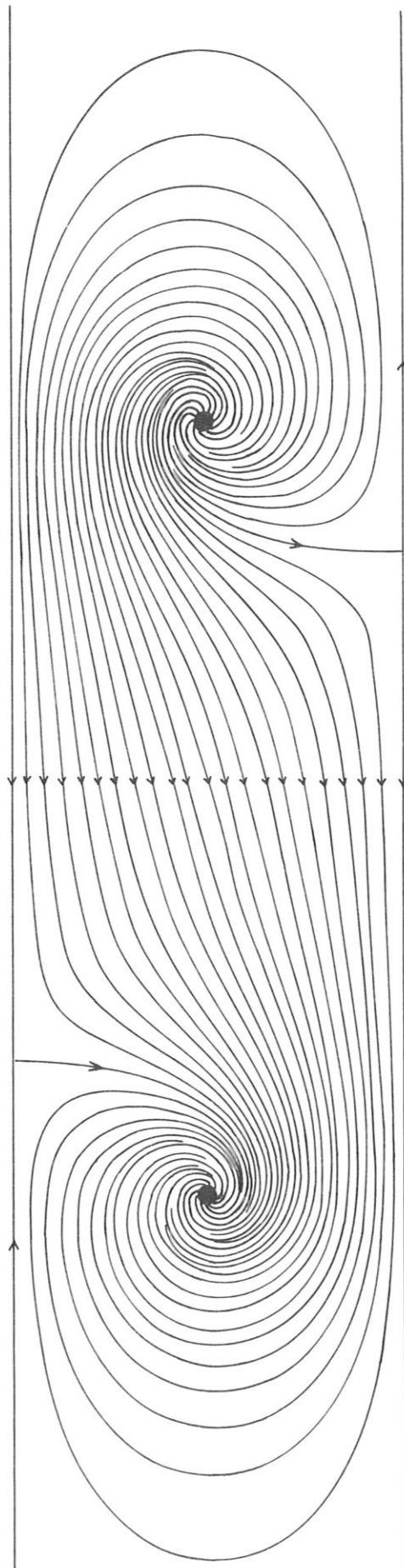
Fig. 16

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