# The Stability of Steady Finite Amplitude Convection in a Rotating Fluid Layer

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ABSTRACT:

The stability of the finite amplitude convective flow in an infinite, horizontal fluid layer rotating rigidly about a normal axis is investigated for the case of rigid boundary conditions and finite Prandtl numbers. Solutions of the non-linear steady-state equations are derived approximately by an amplitude expansion. It can be shown by a stability calculation that all three-dimensional solutions are unstable, and furthermore for a given value of the Prandtl number there exists a critical value of the Taylor number above which the only stable two-dimensional rolls become unstable, too. This means there is a transition from pure heat conduction to time-dependent convective flow.

## 1. Introduction

In the problem of convection in a horizontal fluid layer rotating rigidly about a normal axis and heated from below, the relevant parameters are the temperature difference $\Delta T$ between the lower and upper boundaries of the layer and the rotation rate. The first one is represented in a dimensionless description by the Rayleigh number $R$, the second one by the Taylor number $\tau^2$. A comprehensive bibliography of the linear properties of convection with and without rotation is given by Chandrasekhor (1961).

Solutions of the nonlinear steady-state equations were derived by Veronis (1958) for the case of free boundary conditions. For this case and for infinite Prandtl number a stability calculation was done by Küppers and Lortz (1969) (hereafter referred to as I) which shows that there exists a critical value of the Taylor number above which no stable steady-state convective flow takes place. This means that for a given supercritical value of the Taylor number there is a transition from pure heat conduction to a time-dependent convective flow if the Rayleigh number increases from a subcritical value.

For rigid boundary conditions and for infinite Prandtl number the existence and the value of the critical Taylor number with these qualities has been shown by Küppers (1969). Therefore, it is of interest to know the stability properties in the more realistic case of finite Prandtl number and rigid boundary conditions.

To solve the nonlinear steady-state equations we expand the variables in powers of the convection amplitude, a method introduced in the nonrotating case by Malkus and Veronis (1958).

It is due to the infinite horizontal extent of the layer that the convective flow is not uniquely determined by the equations of motion and the boundary conditions. To sort the physically realized solutions from the infinite manifold of possible solutions, a stability calculation is necessary. As we consider convection with relatively small amplitude, it is reasonable to solve the stability equations as well by successive approximation as first used by Schlüter, Lortz and Busse (1965).

## 2. Fundamental equations

The well-known form of the fundamental equations yields in the same way as in I the steady-state system

$$RWX + UX = Q(X,X) \tag{2.1}$$

and the linear stability equations for the infinitesimal amplitude perturbation $\tilde{X}$ with time dependence $e^{\sigma t}$

$$R W \tilde{X} + U \tilde{X} = \sigma V \tilde{X} + Q(X, \tilde{X}) + Q(\tilde{X}, X) \qquad (2.2)$$

$$X = \begin{pmatrix} v \\ w \\ \theta \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Delta_2 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} P^{-1}\Delta\Delta_2 & 0 & 0 \\ 0 & P^{-1}\Delta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$Q(X', X) = \begin{pmatrix} \delta_i [(\delta_j v' + \varepsilon_j w') \partial_j (\delta_i v + \varepsilon_i w)] \\ \varepsilon_i [(\delta_j v' + \varepsilon_j w') \partial_j (\delta_i v + \varepsilon_i w)] \\ (\delta_j v' + \varepsilon_j w') \partial_j \theta \end{pmatrix}, \quad U = \begin{pmatrix} \Delta^2\Delta_2 & \tau\partial_z\Delta_2 & -\Delta_2 \\ -\tau\partial_z\Delta_2 & \Delta\Delta_2 & 0 \\ 0 & 0 & \Delta \end{pmatrix}$$

$$\partial_i = \frac{\partial}{\partial x_i} \ (i = 1,2,3), \quad \delta_i = \partial_j \partial_k \lambda_k - \lambda_i \Delta, \quad \varepsilon_i = \varepsilon_{ijk} \lambda_j \partial_k,$$
$$\Delta_2 = \Delta - \partial_{jj}^2 \lambda_j, \quad u_i = \delta_i v + \varepsilon_i w$$

where $\lambda_i$ is the unit vector opposite to the force of gravity. $u_i$ is the vector of velocity, $v$ and $w$ are scalar position functions. $\theta$ denotes the deviation from the linear temperature distribution of the static state. In this dimensionless form of the fundamental equations the following dimensionless parameters appear:

$$R = g\alpha\Delta T d^3/\varkappa\nu, \quad \tau = 2\Omega_i\lambda_i d^2/\nu, \quad P = \nu/\varkappa$$

where $g$ is the acceleration of gravity, $\alpha$ the expansion coefficient, $d$ the depth of the layer, $\nu$ the kinematic viscosity, $\varkappa$ the thermometric conductivity and $\Omega_i$ the angular velocity. $P$ is the Prandtl number. Note that the summation convention is used.

As in I, solutions of (2.1) and (2.2) are sought by expanding the variables with respect to an amplitude parameter as follows:

$$X = \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3 + \cdots$$
$$R = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots$$
$$\tilde{X} = \tilde{X}_1 + \varepsilon \tilde{X}_2 + \varepsilon^2 \tilde{X}_3 + \cdots \qquad (2.3)$$
$$\sigma = \sigma_0 + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \cdots$$

Substituting these in the systems (2.1) and (2.2) respectively yields in the first case a set of inhomogeneous equations and in the second case a hierarchy of equations analogous to the stationary case. The solubility conditions determine the $R_n$ and $\sigma_n$.

## 3. Boundary conditions

The layer is of infinite horizontal extent, and all functions occurring are everywhere bounded. Suppose that the layer is bounded at z = ± 1/2 by a perfectly conducting medium. This corresponds to the boundary condition for the temperature

$$\Theta = 0, \quad z = \pm 1/2 \qquad (3.1)$$

Since a viscous fluid with rigid boundaries is involved, all components of the velocity must vanish, and this yields the following dynamic boundary conditions:

$$v = \partial_z v = w = 0, \quad z = \pm 1/2 \qquad (3.2)$$

## 4. Solution of the linear problem

$$R_0 W X_1 + \mathcal{U} X_1 = 0 \tag{4.1}$$

$$R_0 W \tilde{X}_1 + \mathcal{U} \tilde{X}_1 = \sigma_0 V \tilde{X}_1 \tag{4.2}$$

With $\sigma_0 = 0$ (4.2) is the same as (4.1), and so only the more general equation (4.2) is discussed.

First we define a weighted scalar product:

$$\langle X' , X \rangle = R_0 (v'v)_a + R_0 (w'w)_a + (\theta'\theta)_a$$

with $X'$ and $X$ satisfying the boundary conditions (3.1) and (3.2). $(\ )_a$ denotes averaging over the layer. Then the matrix differential operator $\mathcal{U} + R_0 W$ is Hermitian. Because $V$ is, in general, not definite, it does not follow from the hermiticity of $\mathcal{U} + R_0 W$ and from equations (4.2) that $\sigma_0$ is real. Overstability can occur.

It is well-known that the vertical and horizontal dependences of the solutions of the linear equation (4.2) can be separated by assuming that

$$\Delta_2 \tilde{X}_1 + a^2 \tilde{X}_1 = 0$$

where $a$ is the wave number. The neutral curve $\sigma_0 = 0$ divides the $(R_0, a^2)$ -plane into a stable and an unstable region and on this curve there is a minimum value $R_0 = R_c$ at $a = a_c$. In the following we put $a=a_c$ because this is the only case where (4.2) yields no instability. Then the most critical disturbance has $\sigma_0=0$, and (4.2) is identical with (4.1).

Solution of the linear system (4.1) with the boundary conditions (3.1) and (3.2) can be written in the form

$$X_1 = \begin{pmatrix} f(z) \\ g(z) \\ h(z) \end{pmatrix} \sum_{m=-N}^{+N} c_m \omega_m \tag{4.3}$$

$$\tilde{X}_1 = \begin{pmatrix} f(z) \\ g(z) \\ h(z) \end{pmatrix} \sum_{m=-\infty}^{+\infty} \tilde{c}_m \omega_m \tag{4.4}$$

$$\omega_m = \exp(i\vec{k}_m\cdot\vec{r}),\quad |\vec{k}_m|^2 = a^2,\quad \vec{k}_{-m} = -\vec{k}_m,\quad c_{-m} = c_m^{*}$$

$\vec{r}$ is a horizontal position vector and $\vec{k}_m$ a horizontal wave number vector with overall wave number $a$.

To determine the $z$-dependence of $v, w$ and $\Theta$ we eliminate in (4.1) $w$ and $\Theta$ and obtain an equation for $f$.

$$\left[(\partial_z^2 - a^2)^3 + \tau^2\partial_z^2 + R_0 a^2\right] f(z) = 0$$

Because we are interested in the lowest value of $R_0$, we consider only the lowest mode, and therefore the solution of this equation is:

$$f(z) = \sum_{j=1}^{3} B_j \cosh q_j z$$

where $q_j$ is a solution of

$$(q^2-a^2)^3 + \tau^2 q^2 + R_0 a^2 = 0$$

$h(z)$ and $g(z)$ can be determined by the second and third equations of (4.1).

$$h(z) = -R_0 a^2 \sum_{j=1}^{3} \frac{B_j}{q_j^2-a^2} \cosh q_j z + B_4 a\tau^2 \cosh az$$

$$g(z) = \tau\left( \sum_{j=1}^{3} \frac{q_j B_j}{q_j^2-a^2} \sinh q_j z + B_4 \sinh az\right)$$

The boundary conditions for $f$ and $h$ $(f = f' = h = 0,\ z = \pm 1/2)$ determine $B_j,\ j = 1,2,3$ and $B_4$ is determined by the boundary condition for $g$ $(g = 0,\ z = \pm 1/2)$.

$$B_4 = \frac{-\sum_{j=1}^{3} \frac{q_j B_j}{q_j^2-a^2} \sinh q_j/2}{\sinh a/2}$$

The homogeneous equations for $B_1, B_2$, and $B_3$ are

$$\sum_{j=1}^{3} B_j \cosh q_j/2 = 0$$

$$\sum_{j=1}^{3} B_j q_j \sinh q_j/2 = 0$$

$$\sum_{j=1}^{3} \frac{B_j}{q_j^2-a^2} \cosh q_j/2 = 0$$

The determinant of the coefficient matrix must vanish, and this yields an equation to determine $R_c$ and $a_c$.

## 5. Second-order solutions

The second-order equation

$$R_0 W X_2 + U X_2 = Q(x_1, x_1) + R_1 W X_1 \qquad (5.1)$$

yields the solubility condition

$$0 = \langle x_1', Q(x_1, x_1)\rangle - R_1 \langle x_1' W x_1 \rangle$$

$$X_1' = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \omega_n \,, \quad n = -\infty \cdots\cdots -1, 1 \cdots\cdots +\infty$$

Because of the symmetry of the first-order functions, it is readily seen that the term $\langle X_1', Q(X_1, X_1)\rangle$ is equal to zero. This gives $R_1 = 0$ because

$$\langle X_1', W X_1 \rangle = \left( (\partial_j \Theta_1') \partial_j \Theta_1 \right)_a / R_0 \neq 0$$

For $R_1 = 0$ the stability equations in this order are

$$R_0 W \tilde{X}_2 + U \tilde{x}_2 = Q(\tilde{X}_1, X_1) + Q(X_1, \tilde{X}_1) + \sigma_1 V \tilde{X}_1 \qquad (5.2)$$

with the solubility condition

$$0 = \langle X_1', Q(\tilde{X}_1, X_1) + Q(X_1, \tilde{X}_1)\rangle + \sigma_1 \langle X_1' V \tilde{X}_1 \rangle$$

The first term on the right-hand side is again zero, and it follows that $\sigma_1 = 0$.

To determine $R_2$ and $\sigma_2$ we have to calculate the solutions $X_2$ and $\tilde{X}_2$. We try to find $X_2$ in the form

$$X_2 = \sum_{k,l=-N}^{+N} \begin{pmatrix} F(\varphi_{kl}, z) \\ G(\varphi_{kl}, z) \\ H(\varphi_{kl}, z) \end{pmatrix} c_k c_l \omega_k \omega_l$$

$$\varphi_{kl} = \vec{k}_k \cdot \vec{k}_l / a^2$$

Because of

$$\sum_{k,l} F(c_k \tilde{c}_l + \tilde{c}_k c_l) \omega_k \omega_l = 2 \sum_{k,l} F c_k \tilde{c}_l \omega_k \omega_l \quad \text{etc.}$$

the analogous form of $\tilde{X}_2$ is

$$\tilde{X}_2 = 2 \sum_{k=-N}^{+N} \sum_{l=-\infty}^{+\infty} \begin{pmatrix} F \\ G \\ H \end{pmatrix} c_k \tilde{c}_l \omega_k \omega_l$$

Note that from the structure of the inhomogeneities it follows that F, G and H are not only functions depending on the scalar product of the two k-vectors, but also are functions of the z-component of the vector product of the two k-vectors. But in the formulas for $X_2$ and $\tilde{X}_2$ the summation runs over $k$ and $l$, and therefore antisymmetric parts vanish.

To determine F, G and H one can do the same as in the linear problem, but this yields incomparably more complicated formulas than in the linear case. Therefore it is much more convenient to integrate the inhomogeneous second-order equations numerically.

## 6. The order $\varepsilon^3$

The equations are

$$R_0 W X_3 + U X_3 = Q(X_1, X_2) + Q(X_2, X_1) - R_2 W X_1 \tag{6.1}$$

$$\begin{aligned} R_0 W \tilde{X}_3 + U \tilde{X}_3 = {} & Q(\tilde{X}_1, X_2) + Q(X_1, \tilde{X}_2) + Q(\tilde{X}_2, X_1) + Q(X_2, \tilde{X}_1) \\ & - R_2 W \tilde{X}_1 + \sigma_2 V \tilde{X}_1 \end{aligned} \tag{6.2}$$

The solubility conditions are of the form

$$0 = \langle X_1', Q(X_1, X_2) + Q(X_2, X_1) \rangle - R_2 \langle X_1', W X_1 \rangle$$

$$\begin{aligned} 0 = {} & \langle X_1', Q(\tilde{X}_1, X_2) + Q(X_1, \tilde{X}_2) + Q(\tilde{X}_2, X_1) + Q(X_2, X_1) \rangle \\ & - R_2 \langle X_1', W \tilde{X}_1 \rangle + \sigma_2 \langle X_1', V \tilde{X}_1 \rangle \end{aligned}$$

The expressions for $R_2$ and $\sigma_2$ are obtained from these solubility conditions in the same way as in I, and we content ourselves with giving the corresponding formulas. For more detail see I section 7, 8 and 9.

$$R_2 K = \sum_{k=1}^{N} T_{ik} |c_k|^2 \qquad i = 1 \cdots\cdots N$$

$$K = a^2 (f, h)_a \tag{6.3}$$

$$T_{ik} = \begin{cases} -2L_{i,-i,i,i} - L_{i,i,-i,i} & i = \pm k \\ -2\left(L_{i,-k,k,i} + L_{i,k,-k,i} + L_{-i,i,k,k}\right) & \text{otherwise} \end{cases}$$

$$T_{ik} = T_{-ik} = T_{i-k}\,, \quad T_{ik} \neq T_{ki}$$

$$\begin{aligned} L_{k,l,m,n} = \Big( & H(\varphi_{kl}, z) \cdot a^2 \left( f h' + \varphi_{mn} f' h + \psi_{mn} h g \right) - G(\varphi_{kl}, z) a^2 \psi_{mn} h h \\ & - P^{-1} R_0 \Big\{ F(\varphi_{kl}, z) \cdot a^4 \big( -2\psi_{mn}^2 d' g - 2\varphi_{mn}^2 f'' f + \varphi_{mn} f' f'' + \\ & 2a^2 \varphi_{mn}^2 f' f + f f''' - \varphi_{mn} f f''' + f' f'' - 2a^2 f f' - \psi_{mn} \big( 2\varphi_{mn} f'' g \\ & + 2\varphi_{mn} g f' + 2a^2 g f - 2a^2 \varphi_{mn} g f + f g'' - g f'' \big) \big) \\ & + G(\varphi_{kl}, z) \cdot a^4 \big( (1 - \varphi_{mn}) \cdot (4\varphi_{mn} + 3) \cdot f' g + (1 - \varphi_{mn}) f g' - \\ & \psi_{mn} \left( -2\varphi_{mn} f' f' + 2\varphi_{mn} g g - g g - f f'' - a^2 f f \right) \big) \Big\} \Big)_a \end{aligned}$$

$$\psi_{mn} = \left( \vec{k}_m \times \vec{k}_n \right) \cdot \vec{\lambda} / a^2$$

$$' = \frac{\partial}{\partial z}$$

For disturbances with $k$-vectors which are coincident with the basic vectors of the cellpattern, one gets in the case of rolls the only eigenvalue

$$\sigma_2 = -T_{ii}/M \tag{6.4}$$

$$M = \left(h^2 - P^{-1}a^2 f(\partial_z^2 - a^2) f - P^{-1} a^2 gg\right)_a$$

Disturbances the $k$-vectors of which are hot coincident with those of the steady-state pattern yield the continuous eigenvalue

$$\sigma_2 M = -\frac{1}{2}\left(T_{r1} - T_{11}\right) \tag{6.5}$$

## 7. Numerical results

Now we discuss these formulas and the numerical results. Together with the normalization condition

$$\sum_{k=1}^{N} |c_k|^2 = 1/2$$

(6.3) is a system of $N+1$ inhomogeneous equations which - determines $R_2$ and $|c_k|^2$ and which restricts the manifold of first-order solutions. For negative $R_2$ steady state convection at subcritical Rayleigh numbers can occur. If $T_{ik}$ is positive for all values of the angle $\varphi$ between the two $k$-vectors $\vec{k}_i$ and $\vec{k}_k$, $R_2 > 0$ is surely valid for all cell pattern. In Fig.1 the curve $Min_\varphi(T_{ik}) = 0$ is plotted. $Min_\varphi(T_{ik})$ denotes the minimum of $T_{ik}$ with respect to $\varphi$. Above this curve $T_{ik}$ is positive definite, and therefore $R_2$ is greater than zero. This means that in this region subcritical steady-state small amplitude convection does not exist.

In order to consider the stability of two-dimensional rolls we have to discuss (6.4) and (6.5).

In Fig.1 the curve $T_{ii} = 0$ is plotted. In the region above this curve $T_{ii}$ and $M$+) are greater than zero, and rolls in this region are stable with respect to disturbances coincident with the pattern. For disturbances not coincident with the pattern it follows from (6.5) that for a given value of $P$ there exists a neutral curve $\tau$ versus $\varphi$, where $\varphi$ is the angle between the k-vector $\vec{k}_\varphi$ of the disturbance and the basic steady-state k-vector $\vec{k}_1$. On this curve there exists a minimum value of the Taylor number, the so-called critical Taylor number $\tau_c^2$, at $\varphi = \varphi_c$. In Fig.1 $(P, \tau_c)$ is plottet. On the left-hand side of this curve rolls are stable.

In I it is shown that from the condition

$$S_{ik} - T_{ii} > 0 \tag{7.1}$$

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+) The curve $M = 0$ in the $(P, \tau)$-plane is in the region in which convection begins as overstability.

where $S_{ik}$ is the symmetric part of $T_{ik}$ , it follows that all three-dimensional flows are unstable. $S_{ik}-T_{ii}$ can be aranged in powers of the Prandtl number, and in Fig.2a-2c the coefficients of $P^2$, $P^1$ and $P^0$ are plotted as functions of the angle between the $k$-vectors $\vec{k}_i$ and $\vec{k}_k$ for some representative Taylor numbers. From these diagrams it can be seen that for $P \geq 1$, $S_{ik}-T_{ii}$ is always positive and for $P<1$ the minimum of $S_{ik}-T_{ii}$ is at $\varphi=0$ . For $\varphi=0$, $S_{ik}-T_{ii}$ becomes equal to $T_{ii}$ . The curve $T_{ii}=0$ in Fig.1 divides the $(P,\tau)$ -plane into two regions. In the region above the curve all three-dimensional solutions are unstable because (7.1) is satisfied, and in the other region all steady-state solutions are unstable because the trace of the stability matrix is negative.

## 8. Conclusions

In the case of rigid boundary conditions it has been shown that for slightly supercritical Rayleigh numbers all three--dimensional convective flows are unstable and that for a given value of the Prandtl number rolls are stable provided that the Taylor number is smaller than a critical value $\tau_c^2$ . Therefore, for supercritical Taylor numbers - the critical value depends on the Prandtl number - no steady-state convective flow exists if the Rayleigh number is slightly supercritical. All flows are necessarily time-dependent.

For values $P$ and $\tau$ for which $R_2 > 0$ holds for all possible convective flows, there is no small amplitude convection at subcritical Rayleigh numbers and therefore in this region there is no steady-state convection at all, provided $\tau > \tau_c(P)$. This means there is a transition from pure heat conduction to time-dependent convective flow if the Rayleigh number increases from a subcritical value.

It is well known that in the rotating case overstability can occur if the Prandtl number is below a certain value. Preliminary investigations are done to study this effect.

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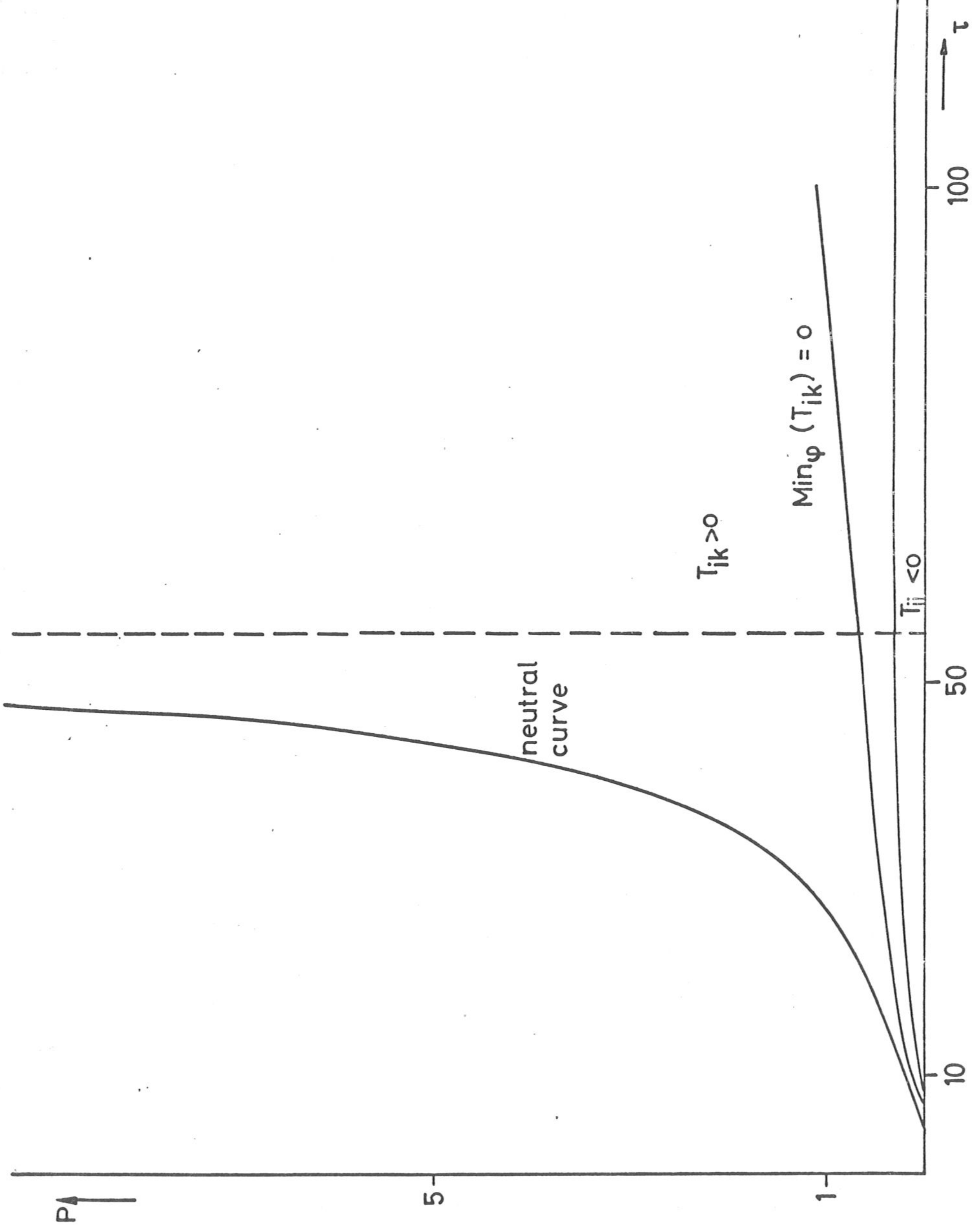


Fig.1 ρ, Prandtl number; $\tau^2$, Taylor number; on the left-hand side of the neutral curve rolls are stable; in the region $T_{ik}>o$ steady-state convection at subcritical Rayleigh numbers can not occur.

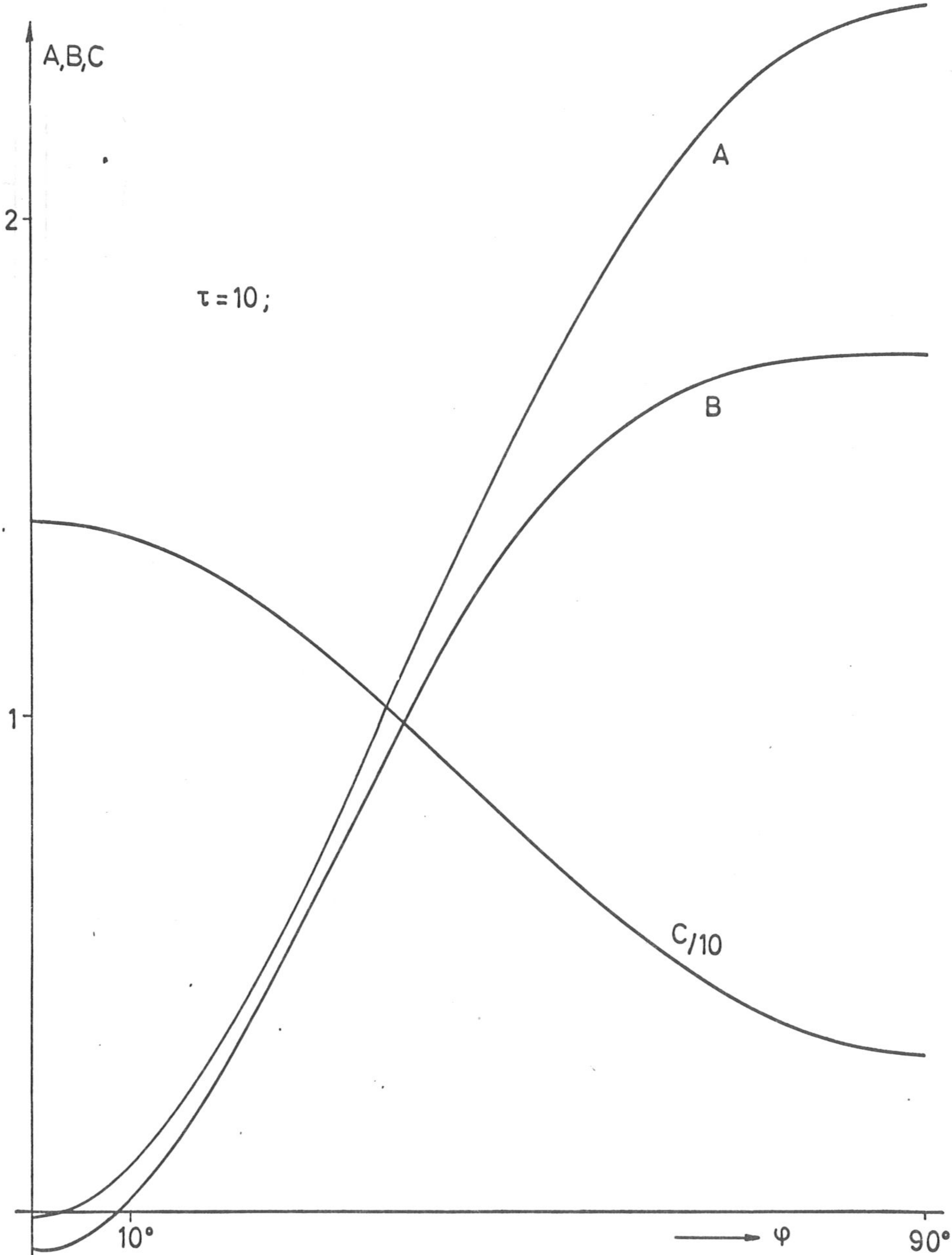


Figure 2a A,B,C coefficients of $p^0, p^1, p^2$ in the expression $S_{ik} - T_{ii}$; φ, angle between the two k-vectors; τ, square-root of the Taylor number.

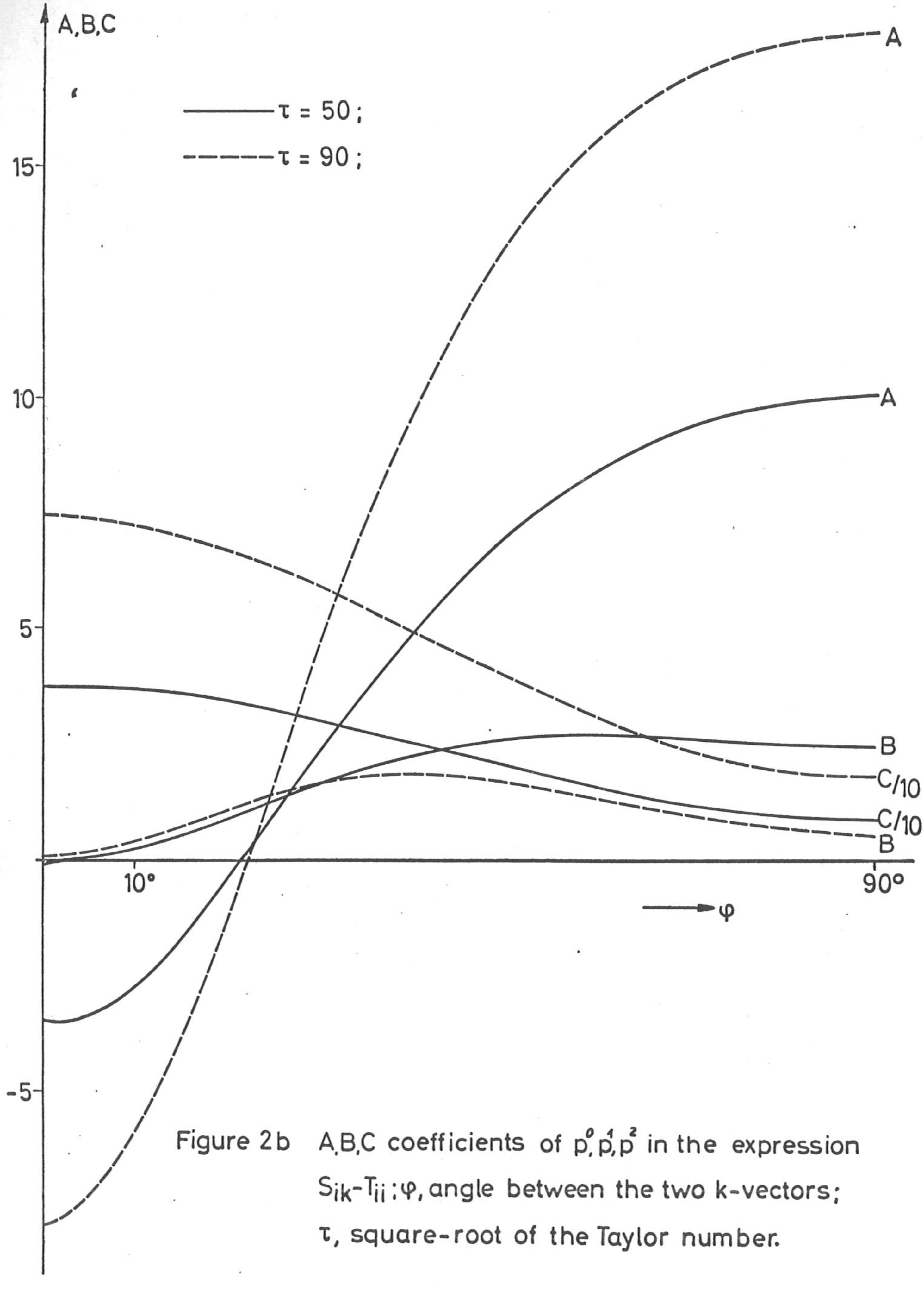


Figure 2b A,B,C coefficients of $p^0, p^1, p^2$ in the expression $S_{ik}-T_{ii}$; φ, angle between the two k-vectors; τ, square-root of the Taylor number.

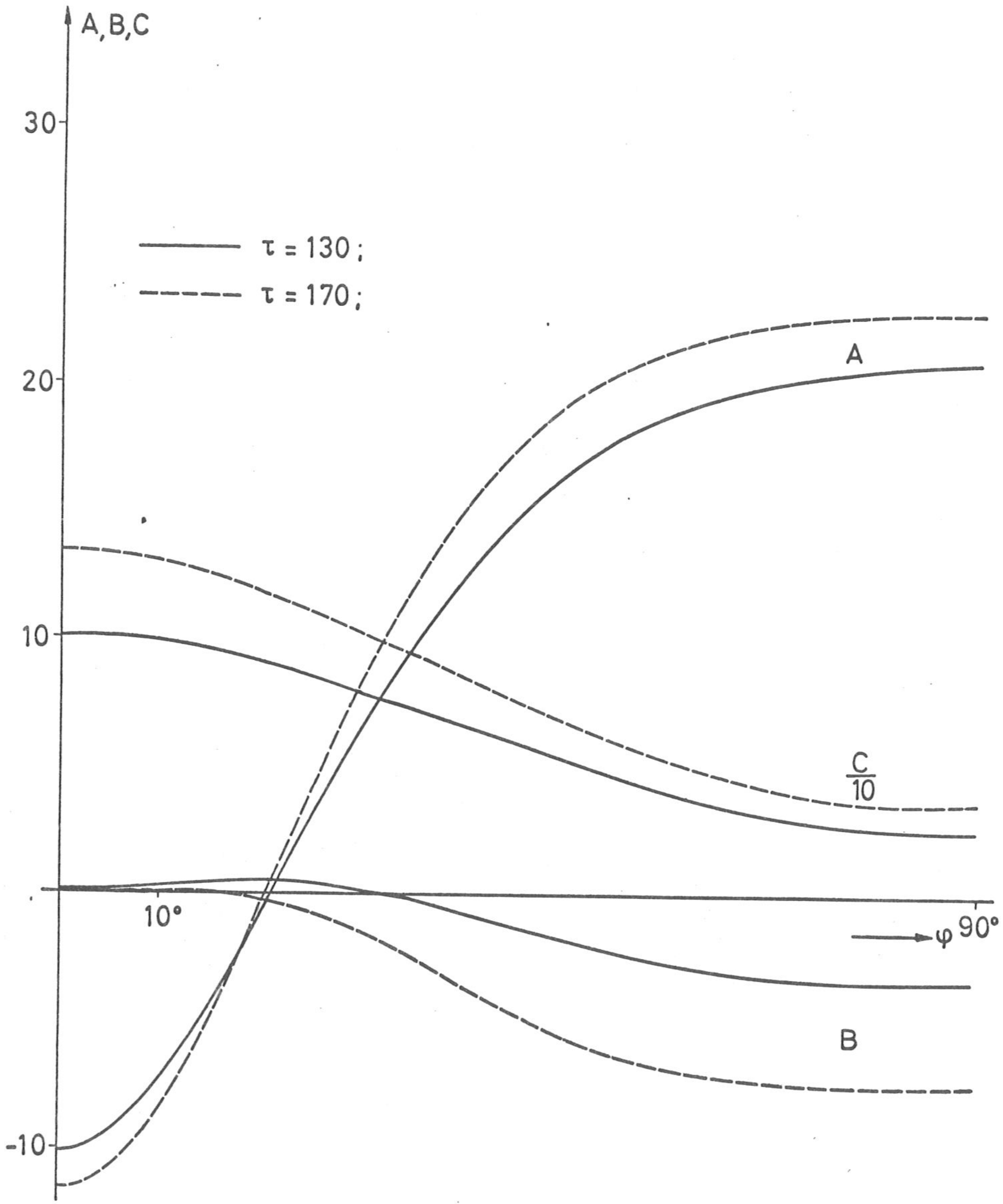


Figure 2c A,B,C coefficients of $p^0, p^1, p^2$ in the expression $S_{ik} - T_{ii}$; φ, angle between the two k-vectors; τ, square-root of the Taylor number.