INSTITUT FÜR PLASMAPHYSIK

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Characteristics of the System of Stationary Plasma Macroscopic Equations

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ABSTRACT

The purpose of this work is to study the character of the system of stationary macroscopic equations including resistivity and viscosity. The system can be split into 5 first-order subsystems and one single equation. The subsystems all have an elliptic character, and the single equation is responsible for the mixed character of the whole system. This mixed character could cause trouble for the boundary value problem of stationary equilibrium even in axisymmetry, for which an example is given. It seems plausible that more refined systems (in example, with thermal conductivity) retain this mixed character.

The problem of the existence of solutions for a system of partial differential equations is related to the nature of the system. For a stationary problem it would be preferable to have an elliptic system. It is very probable that the vanishing denominators found in the M.H.D. equilibria [1] are related to the hyperbolic or mixed character of the magnetohydrostatic equations. One interesting point is to understand the influence of transport coefficients on the character of the macroscopic system. The macroscopic system used here contains resistivity, viscosity in the form and the isothermal assumption.

First let us recall the definition of characteristics [2] of a quasilinear system of first-order partial differential equations:

$$a^{ij,\nu} u^i_{x_\nu} + b^j = 0 \qquad , (j = 1 \cdots k) \; , \; (\nu = 1 \cdots\cdots n)$$

with $a^{ij,\nu}$ and b^j depending on $\vec{x}$ and $\vec{u}$

The system could be formally written:

$$A^\nu u_\nu + B = 0 \qquad \text{where } A^\nu \text{ are } k \times k \text{ matrices}$$

If $\varphi(\vec{x}) = ct$ is a hypersurface with $\vec{\nabla}\varphi \neq 0$, the characteristic matrix is $A = A^\nu \varphi_\nu$ with $\varphi_\nu = \frac{\partial \varphi}{\partial x_\nu}$ and the characteristic equation is:

$$\|A\| = 0$$

- If all the solutions φ_ν of the characteristic equations are real, the system is totally hyperbolic.
- If all φ_ν are complex, the system is elliptic.

- If some are real and some others complex, the system is called mixed.

The system of stationary macroscopic equations used here for the plasma is:

$$(1)\qquad \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{j}\times\vec{B} - c^2\vec{\nabla}\rho + \mu\Delta\vec{v}$$

$$(2)\qquad \nabla\cdot(\rho\vec{v}) = Q$$

$$(3)\qquad \vec{E} + \vec{v}\times\vec{B} = \eta\vec{j}$$

$$(4)\qquad \nabla\times\vec{E} = 0$$

$$(5)\qquad \nabla\cdot\vec{B} = 0$$

$$(6)\qquad \nabla\times\vec{B} = \vec{j}$$

$\vec{v}$ is the macroscopic velocity, $\vec{B}$ the magnetic field, $\vec{E}$ the electric field, $\vec{j}$ the current density, ρ the mass density, c the sound velocity, and η the resistivity.

The eq. (4) has 3 components which are not completely independent. Assuming the validity of two components of (4) and the validity of the third on a surface, it follows that the third is valid in the volume. The same is true for (6) if one uses instead of (6) two components of (6) and the consistency condition $\nabla\cdot\vec{j} = 0$

If we substitute $\vec{j}$ from (3) in (1), in (6) and in $\nabla\cdot\vec{j} = 0$ we obtain the following new system:

$$(7) \qquad \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{(\vec{E}+\vec{v}\times\vec{B})\times\vec{B}}{\eta} - c^2\vec{\nabla}\rho + \mu\Delta\vec{v}$$

$$(8) \qquad \begin{cases} \text{2 comp.} \left\{ \nabla\times\vec{E} = 0 \right. \\ \nabla\cdot\vec{E} = \dfrac{\vec{E}\cdot\vec{v}}{\eta} - \vec{B}\cdot\nabla\times\vec{v} \end{cases}$$

$$(9) \qquad \begin{cases} \text{2 comp.} \left\{ \nabla\times\vec{B} = \dfrac{\vec{E}+\vec{v}\times\vec{B}}{\eta} \right. \\ \nabla\cdot\vec{B} = 0 \end{cases}$$

$$(10) \qquad \nabla\cdot(\rho\vec{v}) = Q$$

In order to obtain a system of first-order quasilinear equations, we have to transform (7) to an equivalent system. Every component of (7) can be replaced[3] by a system of 3 first-order differential equations where the components of the $\vec{\nabla}\vec{v}$ tensor are the 9 unknowns.

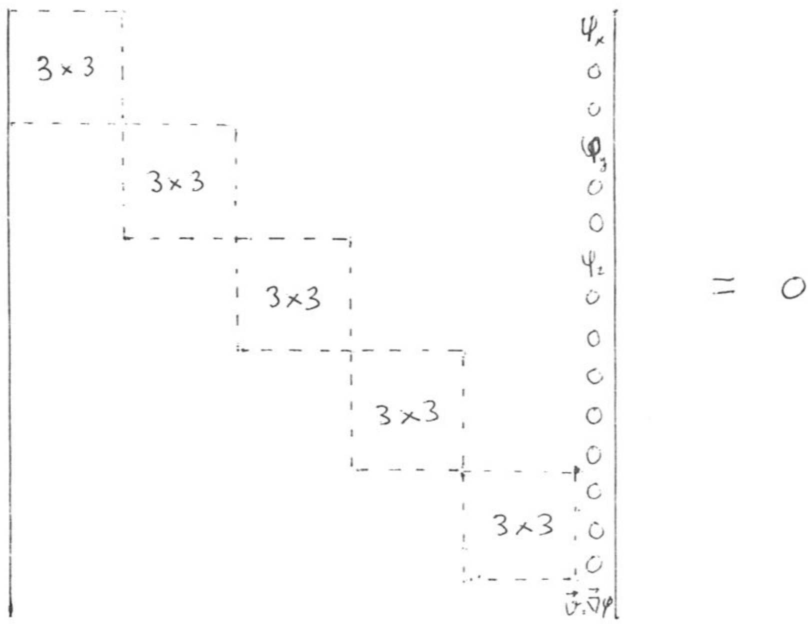
The final system is then composed as follows:

Equation (7) gives 3 subsystems each of 3 first-order differential equations;

eqs. (8) and (9) are 2 subsystems of 3 differential equations each;

and eq. (10) is a single equation.

Note that each subsystem of (7), (8) and (9) is of the elliptic type. Equation (10) remains alone and is hyperbolic. The total characteristic determinant is as follows:



All 3x3 determinants give an elliptic contribution; there remains one real characteristic $\vec{v} \cdot \vec{\nabla} \varphi = 0$ for which the continuity equation is responsible. The total system is of the mixed type: 10 complex characteristics and 1 real.

The mixed character of the system could cause trouble for toroidal stationary equilibria without symmetry, analogous to the difficulties already encountered in the literature [1]. If the torus is axisymmetric, the real characteristic is also axisymmetric and could be adapted to the boundary condition unless it degenerates to a cylinder.

Let us give an example of such a situation: The equation for axisymmetric magnetostatic equilibria is:

$$(11) \quad j_\varphi \equiv \frac{1}{r}\left[\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right] = - r\frac{dP}{d\psi} - \frac{1}{r} T \frac{dT}{d\psi}$$

where ψ is the stream function (the magnetic flux), P the pressure, T the meridional current flux and j_φ the azimuthal current. If $P(\psi)$ and $T(\psi)$ are given, there is in general a solution when the domain is small enough. But if j_φ is given (which is experimentally likely the case, for example in a Tokomak with an externally induced electric field), then the equation (11) is no longer a single equation, and one has to satisfy (11) and (12)

(12) $$j_\varphi(r,z) = -r\frac{dP}{d\psi} - \frac{1}{2r}\frac{d(T^2)}{d\psi}$$ where P and T are arbitrary functions of ψ only.

Suppose $j_\varphi(r,z) = f(r,\psi)$ then in general it follows from (12) that $\psi = F(r)$

We know that the magnetic surfaces are the real characteristics of the magnetostatic problem $\vec{j} \times \vec{B} = \vec{\nabla} p$ and $\nabla \cdot \vec{B} = 0$ and we see that the problem (even in axisymmetry) of having toroidal characteristics is still open.

Macroscopic systems with fewer transport coefficients

If the viscosity is suppressed, simple calculation of the characteristic determinant yields the following:

a) If the flow is below the velocity of sound, there are 2 real characteristics and 4 complex. The 2 real are the flow surfaces counted twice.

b) If the flow is above the velocity of sound, there are 4 real and 2 complex characteristics.

Macroscopic systems with more transport coefficients

No calculations of characteristics exist for more refined systems containing more transport coefficients and more equations. But it seems plausible that one should be able to separate the continuity equation, and we would presume that the more refined systems retain their mixed character and will possess at least one real characteristic. This is suggested by the mechnical nature of the problem.

References

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2 Courant, Hilbert — Vol. II Methods of Mathematical Physics p. 173

3 Courant, Hilbert — Vol. II Methods of Mathematical Physics p. 180

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