Radial Oscillations of Pinched Plasma Columns

G. Lehner

IPP 1/104 November 1969

INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Radial Oscillations of
Pinched Plasma Columns

G. Lehner

IPP 1/104 November 1969

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Institut für Plasmaphysik GmbH und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

IPP 1/104

G. Lehner

Radial Oscillations of Pinched Plasma Columns

November 1969 (in English)

Abstract

The following report gives a discussion of the radial oscillations of pinched plasma columns (z-pinches, θ-pinches and antipinches). Both the basic mode and its higher harmonics are included. In stability theory these radial oscillations would be discribed as $m = 0$, $k = 0$ modes. They are, however, not described by the usual pinch dispersion formula in its limit $k \Rightarrow 0$. They need a separate treatment, therefore.

1. Introduction

For several reasons it is interesting to know the eigenfrequencies of pinched plasma columns, both of the fundamental mode and its harmonics. They may be of importance in connection with, for instance, problems of dynamic stabilization and parametric resonances, which may possibly be excited by high frequency stabilisation fields. Radial oscillations are also of some diagnostic interest. Their diagnostic value is limited, however, because they depend only weakly on most parameters.

Using the conventional notation of stability theory, we may describe our radial oscillations as $m = 0$, $k = 0$ modes. They are, however, not included in the usual pinch dispersion formula, which is not continuous in the limit $k \Rightarrow 0$. Taking this limit of the usual formula we obtain a result in which one term is missing. Let us explain the physical reason for this behaviour. Consider an infinitely long pinch column along which an $m = 0$, $k = 0$ mode is excited.

The inductance of the plasma column together with the outside conductor, which is propotional to $\ln R/r_c$ (see figure 1), i.e. the inductance of an equivalent coaxial cable, does not change in first order. This is a consequence of periodicity along the pinch and does not depend on the wave number considered even if it is becoming arbitrarily small.

If $k = 0$, on the other hand, an inductive term, i.e. the above mentioned logarithm, appears, and so the case $m = 0$, $k = 0$ has to be treated separately. For actual pinches of finite length the additional term has to be taken into account as soon as the wavelength is larger than the length of the pinch.

The derivation of the m = 0, k = 0 dispersion relation follows the well known procedures of normal mode analysis and does not have to be given in detail. Actually, it can be found in reference /1/, where the author discusses the behaviour of a tubular pinch column. This is a rather general case from which one can obtain the results of interest in this report by considering appropriate limits. Here we want to discuss the Z-pinch, the θ-pinch, and also the antipinch. The θ-pinch is a special case of the Z-pinch, which we consider first.

2. The Z-pinch and the θ-pinch

We take the Z-pinch with surface currents, uniform density ϱ and pressure p, and with superimposed longitudinal magnetic fields B_{zp} and B_{zV} in the plasma and vacuum region. If we introduce the dimensionless stabilising fields

$$\alpha_V = \frac{B_{zV}}{B_{\varphi o}} \quad , \quad \alpha_p = \frac{B_{zp}}{B_{\varphi o}} \tag{1}$$

where $B_{\varphi o}$ is the azimuthal field at the plasma boundary, the equation of equilibrium

$$1 + \alpha_V^2 = 8\pi p + \alpha_p^2 \tag{2}$$

Assuming an m = 0, k = 0 perturbation which behaves like $e^{i\omega t}$ (i.e. imaginary values of ω correspond to instability), we can derive the following dispersion relation

$$F(x) \equiv \frac{x I_0(x)}{I_0'(x)} = C \tag{3}$$

where I_0 is a modified Bessel function, x a dimensionless growth rate (or frequency),

$$x = \frac{i\omega r_0}{\sqrt{h^2 + s^2}} \tag{4}$$

and C a constant

$$C = \frac{h^2}{\alpha_p^2 (h^2 + s^2)} \left[1 - \frac{2\alpha_v^2 r_o^2}{R^2 - r_o^2} - \frac{1}{\ln \frac{R}{r_o}} \right] \tag{5}$$

h is the Alfven velocity

$$h^2 = \frac{B_{zp}^{\;2}}{4\pi \rho} \tag{6}$$

and s the velocity of sound

$$s^2 = \frac{\gamma p}{\rho} \tag{7}$$

It is the last term of the constant C, equation (5), which would be lost by simply taking the limit $k \Rightarrow 0$ in the usual pinch dispersion relation.

It is easy to show that the relation (3) does not yield unstable solutions. For real positive values of x the function $F(x)$ increases monotonically and is always larger than $F(0) = 2$. On the other hand, C can never become larger than 2, because

$$C \leq \frac{h^2}{\alpha_p^2 (h^2 + s^2)} = \frac{B_{\varphi o}^{\;2}}{4\pi \gamma p + B_{zp}^2} \leq \frac{2}{\gamma} \tag{8}$$

So $C \leq 1$ for $\gamma = 2$, $C \leq 1.2$ for $\gamma = \frac{5}{3}$, and $C \leq 2$ for the extreme case $\gamma = 1$.
On the other hand, Bernstein et al./2/ have shown that overstability cannot occur, i.e. ω^2 is real and ω is real or purely imaginary. Thus, x has to be purely imaginary, too, and we introduce a new dimensionless frequency

$$y = -ix = \frac{\omega r_o}{\sqrt{h^2 + s^2}} = \omega \sqrt{\frac{4M}{B_{zp}^2 + 4\pi \gamma p}} \tag{9}$$

which is real now and obeys the relation

$$\frac{y\, J_0(y)}{J_1(y)} = C \tag{10}$$

M is the mass per unit length of the pinch,

$$M = r_0^2 \pi \rho \tag{11}$$

which is sometimes called the line density.
Figure 3 shows the solutions of this equation (1o). It gives the dimensionless frequencies for the first four modes if the constant C is given. If, for instance, C = O , these are just the zeros of the Bessel function J_0 , i.e. 2.40; 5.52; 8.65; 11.79 etc. In the limit $C \Rightarrow -\infty$ the zeros of J_1 are 3.83; 7.02; 10.17; 13.32 etc. For large mode numbers the differences between successive values approach π in both cases, i.e. if the radial wavelength is small compared with the plasma radius, the waves behave more and more like plane waves.

If $\alpha_v = \alpha_p = 0$ and $\gamma \Rightarrow 1$ we have

$$C = 2\left[1 - \frac{1}{\ln\frac{R}{r_0}}\right] \tag{12}$$

If now $R \Rightarrow \infty$, the basic mode approaches $\omega = 0$, i.e. it becomes marginal. The return conductor exercises a stabilising influence on the plasma column and makes it oscillate. This is due to the following mechanism. Let us assume that the plasma column starts a radial motion.As a consequence the inductance changes and this leads to an opposite change of the total longitudinal current. The azimuthal magnetic field at the surface then varies as

$$\frac{dB_\varphi}{B_\varphi} \approx -\frac{dr}{r_0} + \frac{1}{\ln\frac{R}{r_0}}\frac{dr}{r_0} \tag{13}$$

where the first term comes from the motion of the plasma surface for constant current and second term from the current change. The plasma pressure changes as

$$\frac{dp}{p} \approx -2\gamma\frac{dr}{r_0} \tag{14}$$

The pressure difference in the direction of increasing r is thus

$$d\left(p-\frac{B_\varphi^2}{8\pi}\right) \approx -2p\left(\frac{1}{\ln\frac{R}{r_0}}+\gamma-1\right)\frac{dr}{r_0} \tag{15}$$

This restores the initial situation because

$$\frac{1}{\ln R/r_0}+\gamma-1 \quad > 0$$

for any conceivable value of γ. This explains the formal results given above.

The θ-pinch case is obtained by assuming $B_{\varphi o} = 0$:

$$C = -\frac{2B_{zv}^2 r_0^2}{(4\pi\gamma p + B_{zp}^2)(R^2-r_0^2)} \tag{16}$$

The equilibrium condition is

$$B_{zv}^2 = 8\pi p + B_{zp}^2 \tag{17}$$

so that for γ = 2 we simply get

$$C = -\frac{2}{\left(\frac{R}{r_0}\right)^2 - 1} \tag{18}$$

For large compression ratios R/r_0, C is close to zero and the dimensionless frequencies are practically the zeros of J_0.

Equation (16) agrees with the results which Taylor /3/ obtained for the θ-pinch. He also discussed the influence of non-uniform density profiles and of nonlinearity in the case of large amplitude oscillations. It appears that these effects have little influence in general. The may become important in extreme cases only. θ-pinch oscillations have also been discussed by Niblett and Green /4/, for example. Using the snowplough model they find an analytic solution for a nonlinear oscillation with a frequency not depending on the amplitude. This result again suggests that nonlinearity is not important for our problem.

2. The Antipinch

The results for the antipinch depend on the boundary condition applied at the outside boundary at the radius $r = R_e$ (see figure 4). We may assume, for instance, a rigid wall at $r = R_e$ so that the radial velocity has to be zero there. Or we may assume an ideally flexible wall which would make the pressure variation zero there. In any case we get a dispersion relation of the form

$$G(x,\alpha) = D \qquad (19)$$

where x defined as in equation (4), and D is obtained from C by exchanging R and R_i, i.e.

$$D = \frac{h^2}{\alpha_p^2 (h^2 + s^2)} \left[1 + \frac{2\alpha_v^2 r_o^2}{r_o^2 - R_i^2} + \frac{1}{\ln \frac{r_o}{R_i}} \right] \geq 1 \qquad (20)$$

There is an additional parameter appearing in equation (19),

$$\alpha = \frac{R_e}{r_o} \qquad (21)$$

which describes the effect of the outside plasma boundary. The form of $G(x,\alpha)$ depends on the boundary condition used at this boundary.

As before one can show that there are no unstable solutions. The reason for this is that D is always positive. We use y again, therefore, to obtain

$$G_1(y,\alpha) = y \cdot \frac{J_1(\alpha y) N_0(y) - J_0(y) N_1(\alpha y)}{J_1(\alpha y) N_1(y) - J_1(y) N_1(\alpha y)} = D \qquad (22)$$

for a regid wall, and

$$G_2(y,\alpha) = y \cdot \frac{J_0(\alpha y) N_0(y) - J_0(y) N_0(\alpha y)}{J_0(\alpha y) N_1(y) - J_1(y) N_0(\alpha y)} = D \qquad (23)$$

for an ideally flexible wall.

Figures 5 and 6 give examples for the solutions of both equations with $\alpha = 2$. For different values of α the situation is similar.

For $y = 0$ we get

$$G_1(0,\alpha) = -\frac{2}{\alpha^2 - 1} \tag{24}$$

i.e.

$$0 > G_1(0,\alpha) > -\infty \tag{25}$$

and

$$G_2(0,\alpha) = 0 \tag{26}$$

The differences of y between the zeros and between the singularities of both G_1 and G_2 also approach π .

References:

/1/ G. Lehner, Z.f.Naturf. 16a, (1961) 7oo
(see section 2 c, page 7o8).

/2/ J.B.Bernstein et al., Proc.Roy.Soc. A 244 (1958) 17

/3/ J.B. Taylor, Gatlinburg-Conference 1959,
TID-7582 (ORNL-28o5) 26

/4/ G.B.F.Niblett and T.S.Green
Proc.Phys.Soc. 74, (1959) 737

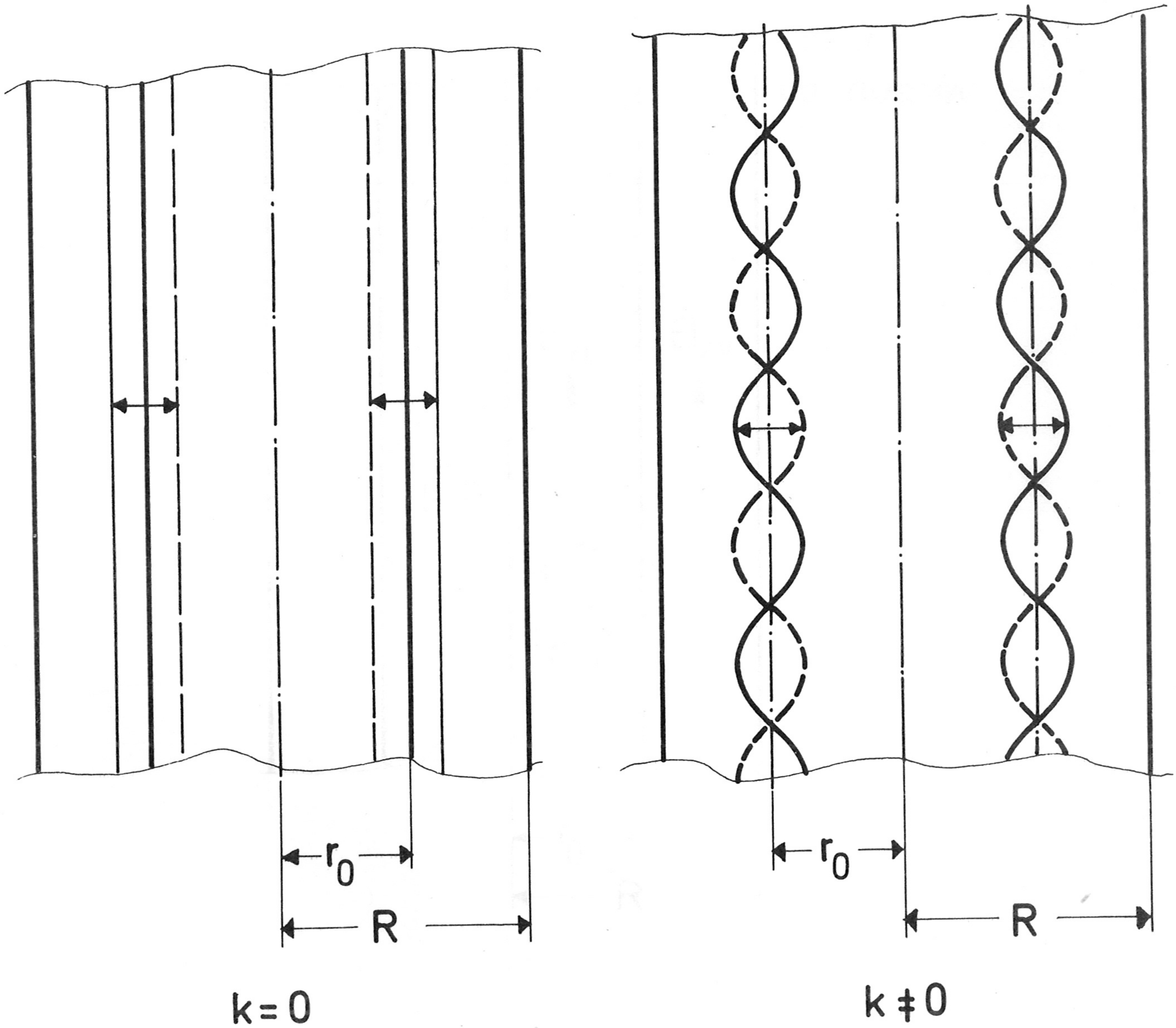

Figure 1: $m = 0$ mode for $k = 0$ and $k \neq 0$

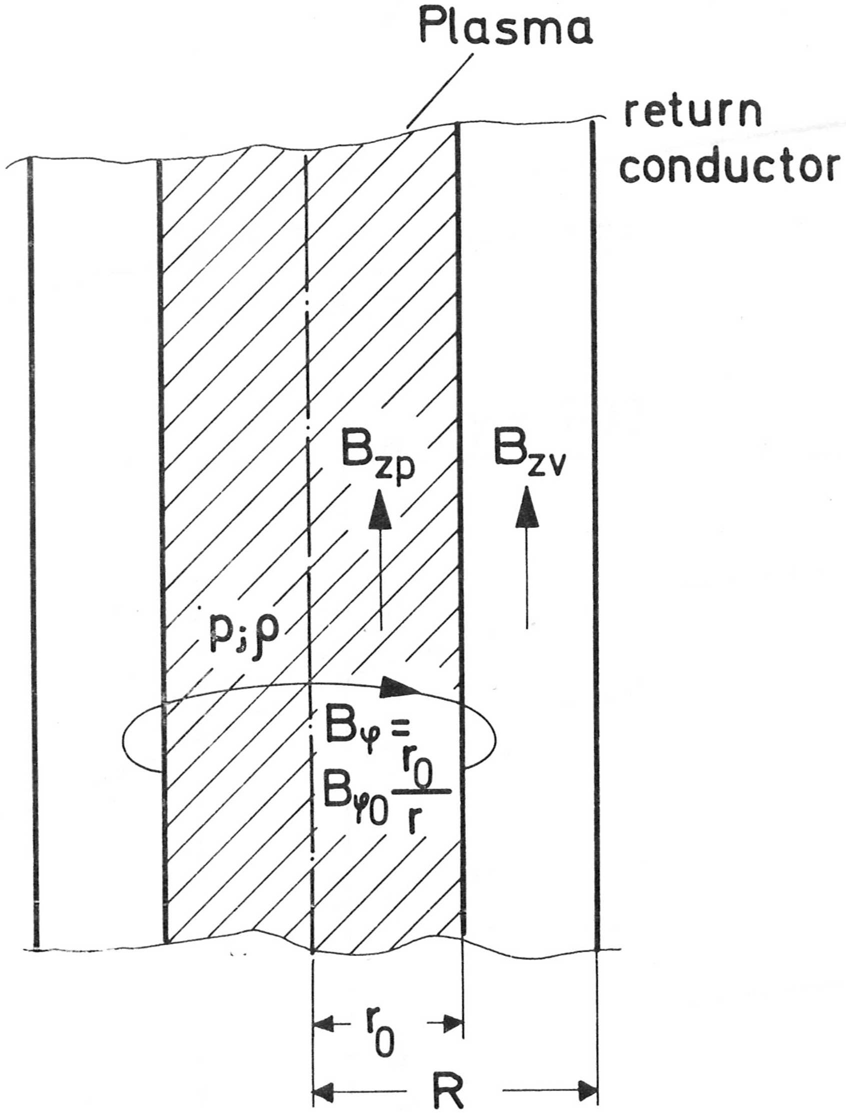

Figure 2: Z- pinch

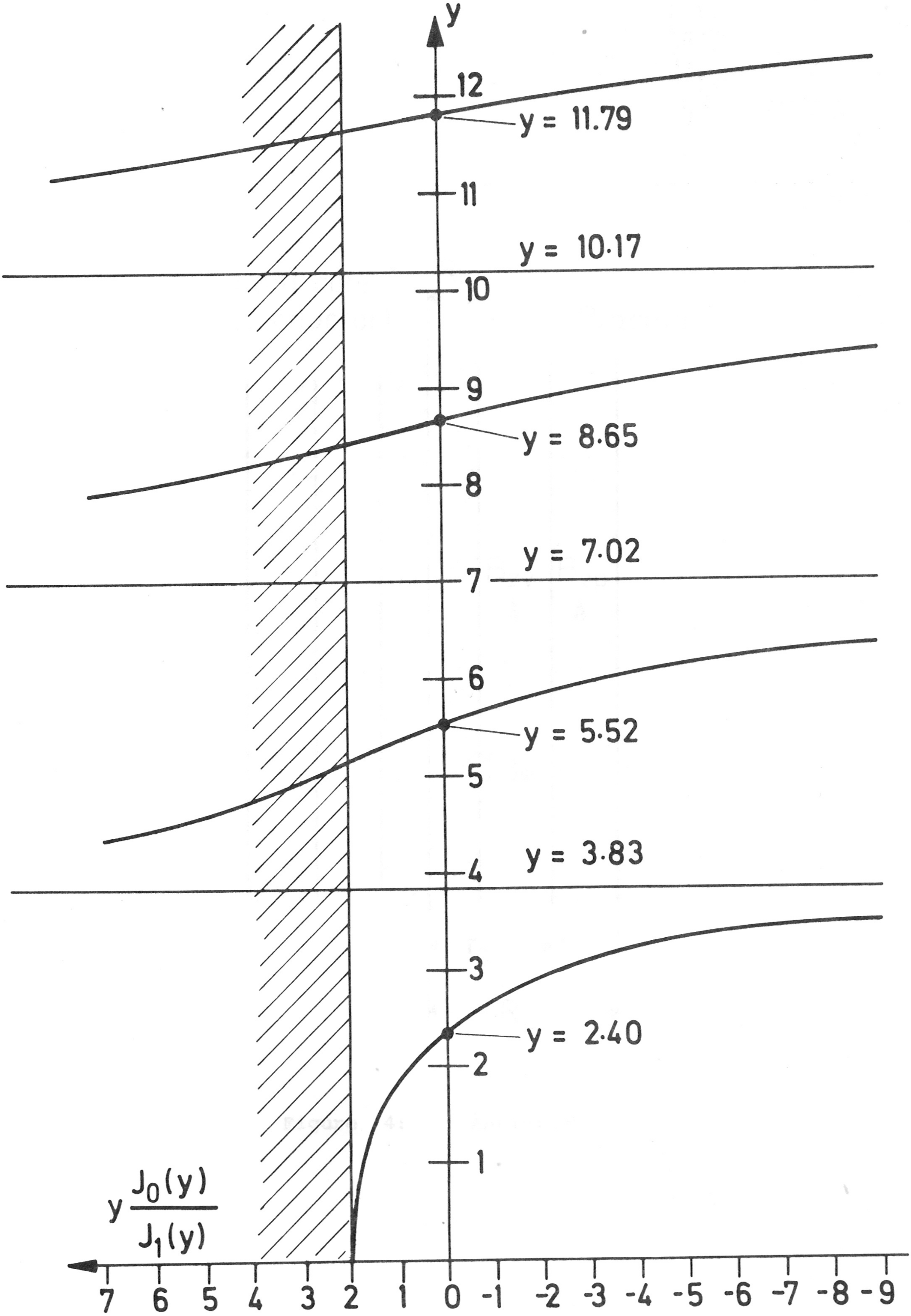

Figure 3: Dispersion relation for the Z-pinch radial oscillations

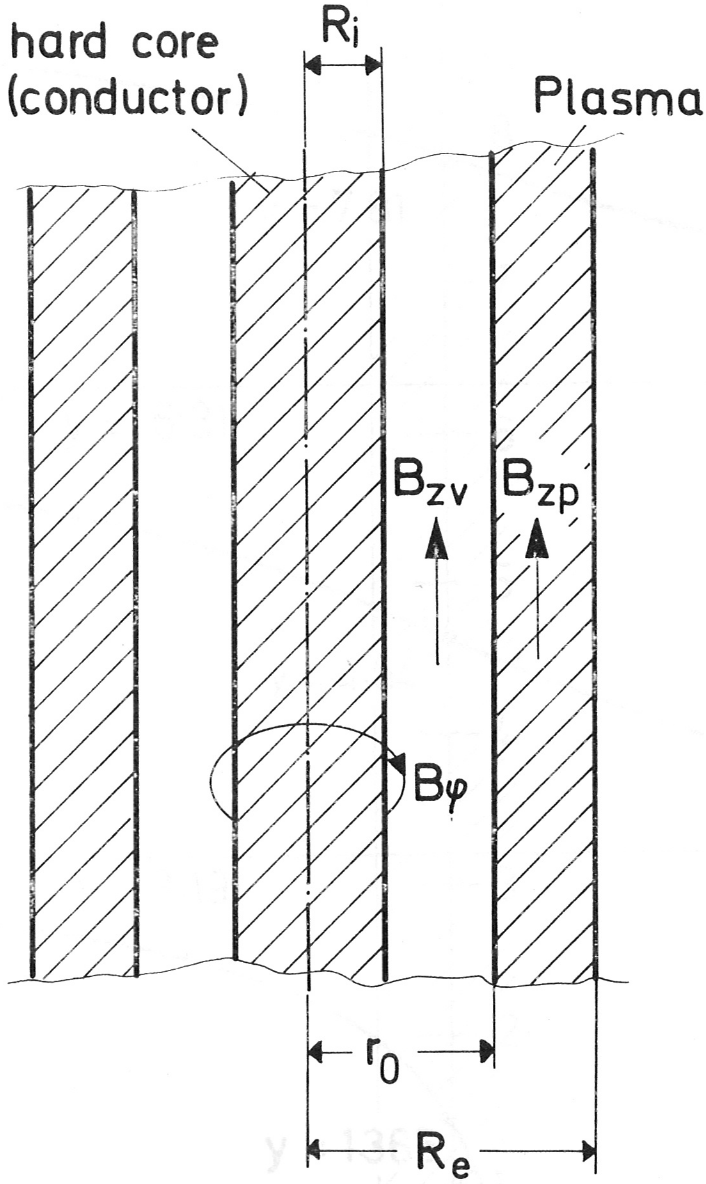

Figure 4: Antipinch

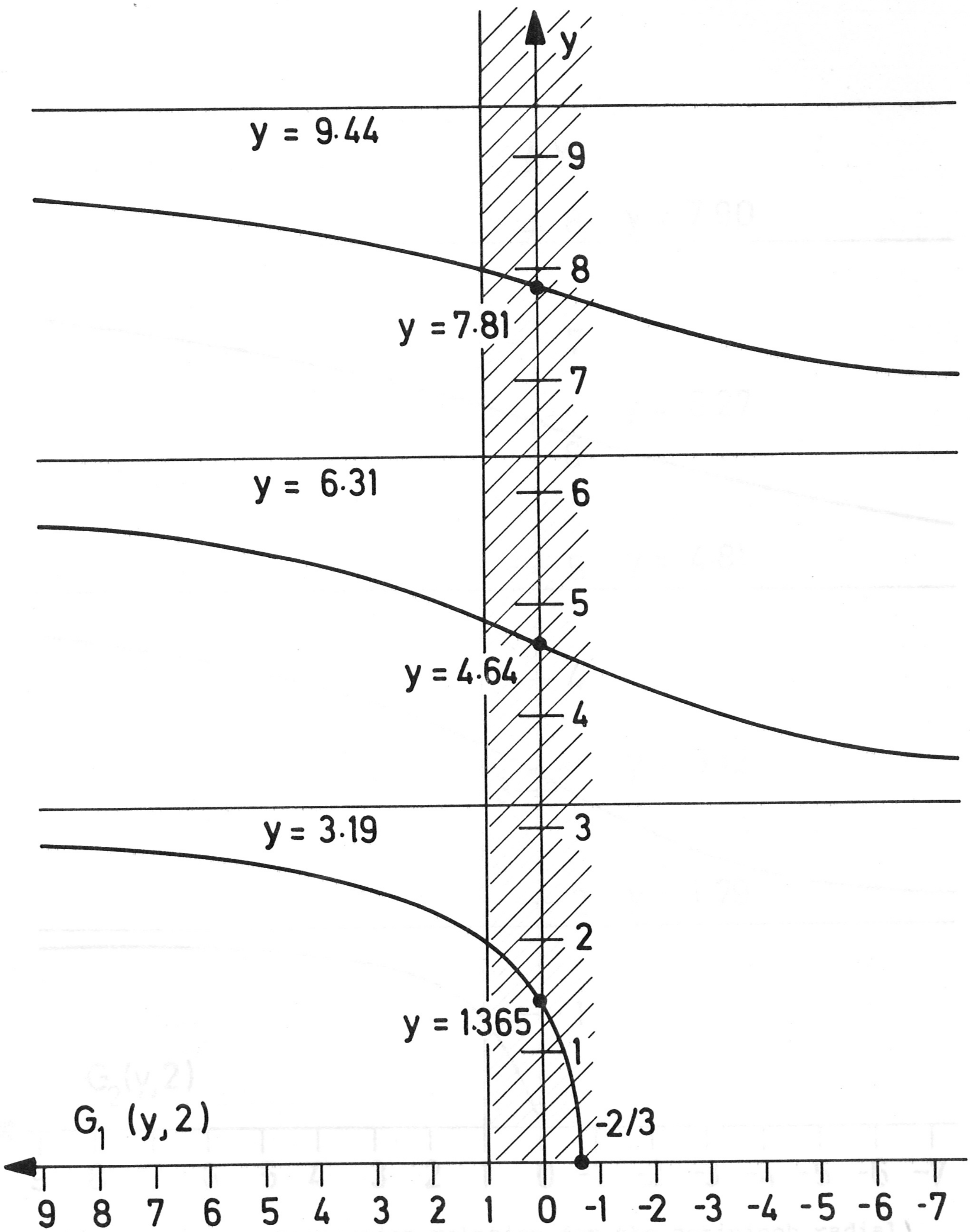

Figure 5: Dispersion relation for the antipinch radial oscillations:
Case of rigid wall, $\alpha = 2$.

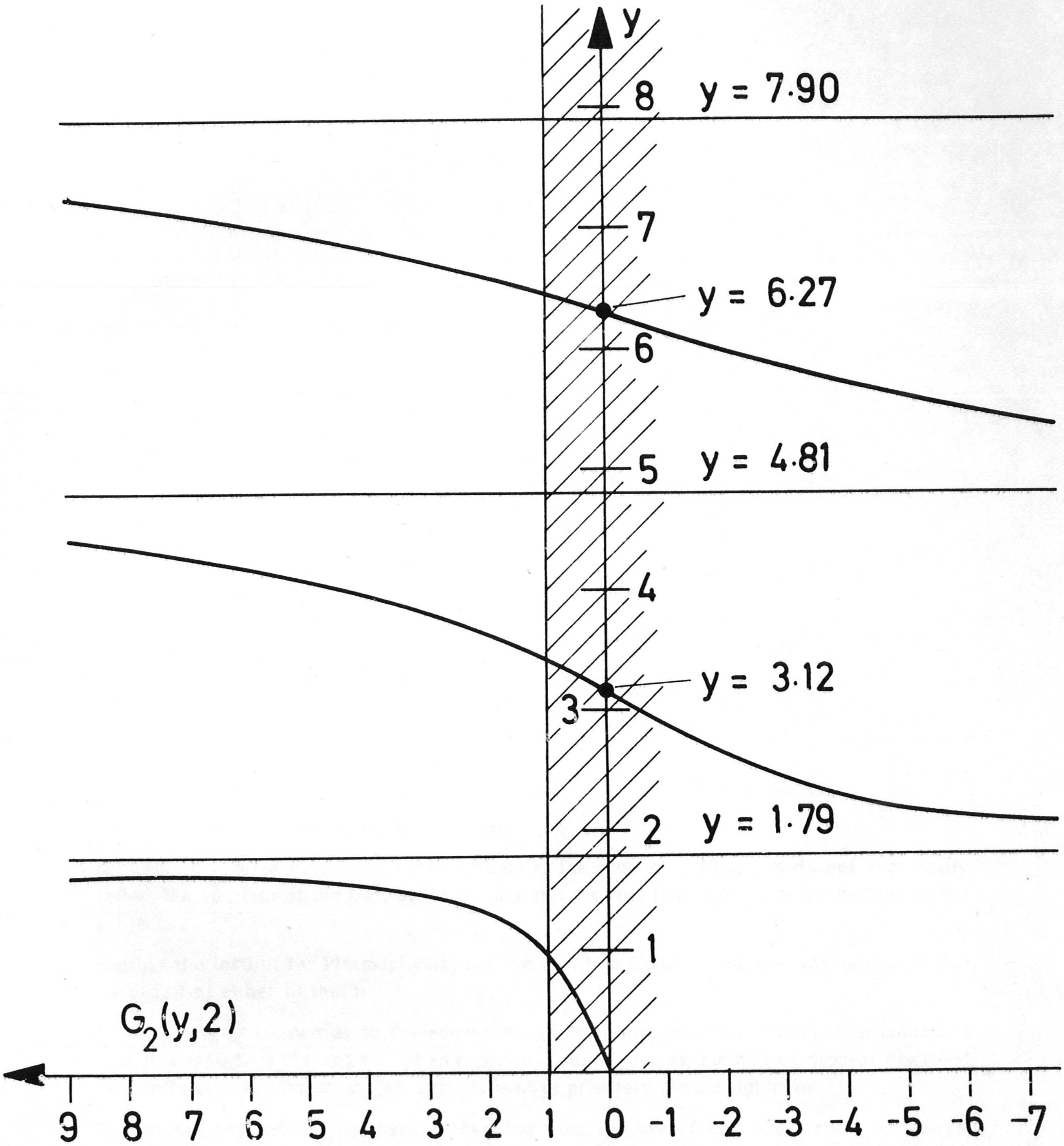

Figure 6: Dispersion relation for the antipinch radial oscillations:
Case of ideally flexible wall, $\alpha = 2$.