Propagation and Fresnel Effect of Microwaves in a Waveguide Partially Filled with a Cold Plasma Dielectric

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Abstract Page

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Abstract

The propagation of electromagnetic waves through a circular waveguide partially filled with a homogeneous, lossless and nonmagnetic dielectric rod has been studied. The characteristic equation is a transcendental equation where the unknown wavenumbers appear in the arguments of Bessel and Neumann functions. Approximate analytical solutions are found by appropriate approximations of these functions for the case where the dielectric discontinuity is small and the roots of the equation lie in the neighbourhood of the roots for the empty guide.

The results are used to give a rough theory for the Fresnel dragging effect on microwaves by the positive column of a glow discharge that burns inside a waveguide. It is shown that symmetric modes of the H (or TE) type should show no effect. Of all the other modes only the three lowest modes are treated in detail. These are the E_{01}, the H_{11}-like and the E_{11}-like modes. For low plasma densities the effect is linear for the electric and of higher order for the magnetic modes in plasma density.

Comparison with measurements is made and rough agreement is obtained. Some details of the experimental curves however cannot be explained. At the present stage the remaining discrepancies seem to be more likely due to end and finite length effects than due to hot plasma properties.

I. The solution of Maxwell's equations for cylindrical boundary conditions

This fundamental procedure is described in text books of which we only give two references[1)2)]. We first shortly repeat the principal analysis and then specify to our problem.

Wave equation. Maxwell's equation for a homogeneous dielectric (ε, μ):

$$\nabla \cdot \mathcal{E} = 0 \qquad \nabla \times \mathcal{E} + \frac{1}{c}\frac{\partial \mathcal{B}}{\partial t} = 0$$
$$\nabla \mathcal{B} = 0 \qquad \nabla \times \mathcal{B} - \frac{\mu\varepsilon}{c}\frac{\partial \mathcal{E}}{\partial t} = 0 \tag{1}$$

are easily combined to the wave equation:

$$\left(\nabla^2 \frac{\mu\varepsilon}{c^2} \cdot \frac{\partial^2}{\partial t^2}\right) F_k = 0$$

in which F_k denotes any of the cartesian components of $\mathcal{E}$ or $\mathcal{B}$. Supposing now fields of periodic time and z-dependence:

$$F_k = F_{tk}(x,y)\, e^{-i\omega t + i k_z z}$$

we obtain for F_{tk} the equation

$$(\nabla_t^2 + k^2) F_{tk} = 0 \qquad \text{where} \quad \nabla_t^2 = \nabla^2 - \partial^2/\partial z^2$$
$$\text{and} \quad k^2 = \mu\varepsilon k_0^2 - k_z^2 \tag{2}$$
$$k_0^2 = \omega^2 c^{-2}$$

∇_t^2 is the transverse part of the ∇^2-operator, k_o^2 is the vacuum wavenumber and k^2 the transverse wavenumber in the medium.

Transverse and longitudinal components. The fields may be seperated into components parallel and perpendicular to the z-axis with help of a unit vector n_3 in the z-direction:

$$\mathcal{F} = \mathcal{F}_z + \mathcal{F}_t \; ; \; \mathcal{F}_z = (n_3 \cdot \mathcal{F})\, n_3 \; ; \; \mathcal{F}_t = (n_3 \times \mathcal{F}) \times n_3$$

Maxwell's curl equations together with the assumed z-dependence lead to the determination of the transverse fields by the axial components:

$$\mathcal{E}_t = i k^{-2}(k_z \nabla_t E_z - k_0 n_3 \times \nabla_t B_z)\; ; \; \mathcal{B}_t = i k^{-2}(k_z \nabla_t B_z + \mu\varepsilon k_0 n_3 \times \nabla_t E_z) \tag{3}$$

Thus for nonvanishing axial components and radial wavenumbers the problem is solved once these components have been determined. For vanishing axial components the approach of the limits $B_z \to 0$ and $E_z \to 0$ has to be considered.

Mode classification. Pure modes are those in which either B_z or E_z or both vanish. We distinguish:

H or TE modes	E or TM modes	TEM modes
$\mathcal{E}_z = 0$	$\mathcal{B}_z = 0$	$\mathcal{E}_z = \mathcal{B}_z = 0$
$\mathcal{B}_t = i k_z k^{-2} \nabla_t B_z$	$\mathcal{E}_t = i k_z k^{-2} \nabla_t E_z$	
$\mathcal{E}_t = -k_0 k_z^{-1} n_3 \times \mathcal{B}_t$	$\mathcal{B}_t = \mu\varepsilon k_0 k_z^{-1} n_3 \mathcal{E}_t$	

(4)

Boundary conditions. At the conductor which we take to be perfect we have the boundary conditions:

$$(n \mathcal{B}) = 0 \qquad \text{and} \qquad (n \times \mathcal{E}) = 0$$

where $\boldsymbol{n}$ is a unit normal at the surface. At the interface between two dielectrics (ε_1, μ_1) and (ε_2, μ_2) the continuity of tangential $\mathcal{E}$ and $\mathcal{H} \equiv \mu^{-1}\mathcal{B}$ is required. Maxwell's divergence equations then imply the continuity of normal $\mathcal{B}$ and $\mathcal{D} \equiv \varepsilon\,\mathcal{E}$.

Circular Waveguide. We now specify to boundaries of rotational symmetry. Introducing

$$\nabla_t^{\,2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

and setting

$$F_{tz}\,(x,y) = F(r)\begin{cases}\sin m\varphi \\ \cos m\varphi\end{cases}$$

we obtain from Eq. (2) for the radial dependence of the cartesian z-component the equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2 - \frac{m^2}{r^2}\right)F(r) = 0$$

which is solved by linear combinations of the form:

$$F(r) = CJ_m(kr) + DN_m(kr) \text{ or shortly } = CJ\,(kr) + DN\,(kr) \tag{5}$$

Starting now from the expressions

$$B_z = B(r)\,\sin m\varphi \quad \text{and} \quad E_z = E(r)\,\cos m\varphi$$

we introduce the unit vectors $\boldsymbol{n}_1$ in the r- and $\boldsymbol{n}_2$ in the φ-direction and write

$$\nabla_t = \frac{\partial}{\partial r}\boldsymbol{n}_1 + \frac{1}{r}\frac{\partial}{\partial \varphi}\boldsymbol{n}_2 \quad .$$

Observing

$$\boldsymbol{n}_3 \times \boldsymbol{n}_1 = \boldsymbol{n}_2 \qquad \text{and} \qquad \boldsymbol{n}_3 \times \boldsymbol{n}_2 = -\boldsymbol{n}_1$$

we then derive the rest of the components from Eq. (4), grouping them into H mode and E mode components.

H components	E components
$B_z = B \sin m\varphi$	$E_z = E \cos m\varphi$
$B_r = \frac{ik_z}{k^2}\frac{\partial B}{\partial r}\sin m\varphi$	$B_r = \frac{i\mu\varepsilon k_0}{k^2}\frac{m}{r}\sin m\varphi$
$B_\varphi = \frac{ik_z}{k^2}\frac{m}{r}B\cos m\varphi$	$B_\varphi = i\mu\varepsilon\frac{k_0}{k^2}\frac{\partial E}{\partial r}\cos m\varphi$
$E_r = \frac{ik_0}{k^2}\frac{m}{r}B\cos m\varphi$	$E_r = i\frac{k_z}{k^2}\frac{\partial E}{\partial r}\cos m\varphi$
$E_\varphi = -\frac{ik_0}{k^2}\cdot\frac{\partial B}{\partial r}\sin m\varphi$	$E_\varphi = -i\frac{k_z}{k^2}\frac{m}{r}E\sin m\varphi$

(6)

The reason why we have chosen the azimuthal dependence of B_z and E_z to be 90° out of phase is that only in this way we get the same angular dependence for H and E components in the transverse direction i.e. the same "mode".

Inserting now the linear combinations given by Eq. (5) with different coefficients for the H and E mode components we combine the two groups to obtain:

$$B_z = [C_H J(kr) + D_H N(kr)]\sin m\varphi\,; \quad E_z = [C_E J(kr) + D_E N(kr)]\cos m\varphi$$

$$B_\varphi = ik^{-2}\cos m\varphi\left\{mk_z r^{-1}[C_H J(kr) + D_H N(kr)] + \varepsilon\mu k_0\frac{\partial}{\partial r}[C_E J(kr) + D_E N(kr)]\right\} \tag{7}$$

$$E_\varphi = -ik^{-2}\sin m\varphi\left\{mk_z r^{-1}[C_E J(kr) + D_E N(kr)] + k_0\frac{\partial}{\partial r}[C_H J(kr) + D_H N(kr)]\right\}$$

As in Eq. (5) the index m which is common to all cylindrical functions has been omitted. We did not write down the radial components since for the boundary conditions it is sufficient to consider the tangential components only.

Boundary conditions for the dielectric coaxial line. Suppose now a circular waveguide of radius b which is filled with dielectric "one" ($\mathcal{E}_1, \mu_1 = 1$) up to a radius r = a and with dielectric "two" ($\mathcal{E}_2, \mu_2 = 1$) for the rest. For each dielectric we have the four constants of Eq. (7), that is eight constants altogether. At the same time we have eight boundary conditions: finiteness of B_z and E_z on the axis, continuity of $B_z = H_z$, E_z, $B_\varphi = H_\varphi$ and E_φ at the interface and vanishing E_z and E_φ at the conductor. The transverse wavenumbers in the two media are named k_1 and k_2. These quantities are unknown and have to be determined from the requirement that the full set of boundary conditions has a nontrivial solution for the coefficients $C^{(1,2)}_{E,H}$ $D^{(1,2)}_{E,H}$.

We start by demanding finiteness of B_z and E_z on the axis, yielding:

$$D_H^{(1)} = D_E^{(1)} = 0.$$

The components in medium "one" are therefore given by

$$\begin{aligned} B_z^{(1)}(r) &= C_H^{(1)} J(k_1 r) \sin m\varphi \quad ; \quad E_z^{(1)}(r) = C_E^{(1)} J(k_1 r) \cos m\varphi \\ B_\varphi^{(1)}(r) &= \frac{i}{k_1} \cos m\varphi \left\{ \frac{m k_z}{k_1 r} C_H^{(1)} J(k_1 r) + \mathcal{E}_1 k_0 C_E^{(1)} J'(k_1 r) \right\} \\ E_\varphi^{(1)}(r) &= -\frac{i}{k_1} \sin m\varphi \left\{ \frac{m k_z}{k_1 r} C_E^{(1)} J(k_1 r) + k_0 C_H^{(1)} J'(k_1 r) \right\} \end{aligned} \tag{8}$$

At the conductor surface we must have $E_z^{(2)} = E_\varphi^{(2)} = 0$. With the abbreviation

$$\frac{J(k_2 b)}{N(k_2 b)} = \vartheta \quad \text{and} \quad \frac{J'(k_2 b)}{N'(k_2 b)} = \vartheta' \tag{9}$$

we get from Eq. (7)

$$D_E^{(2)} = -\vartheta C_E^{(2)} \quad \text{and} \quad D_H^{(2)} = -\vartheta' C_H^{(2)}$$

These components in medium "two" are accordingly given by

$$\begin{aligned} B_z^{(2)}(r) &= C_H^{(2)} [J(k_2 r) - \vartheta' N(k_2 r)] \sin m\varphi ; \quad E_z^{(2)}(r) = C_E^{(2)} [J(k_2 r) - \vartheta N(k_2 r)] \cos m\varphi \\ B_\varphi^{(2)}(r) &= \frac{i}{k_2} \cos m\varphi \left\{ \frac{m k_z}{k_2} C_H^{(2)} [J(k_2 r) - \vartheta' N(k_2 r)] + \mathcal{E}_2 k_0 C_E^{(2)} [J'(k_2 r) + \vartheta N'(k_2 r)] \right\} \\ E_\varphi^{(2)}(r) &= -\frac{i}{k_2} \sin m\varphi \left\{ \frac{m k_z}{k_2} C_E^{(2)} [J(k_2 r) - \vartheta N(k_2 r)] + k_0 C_H^{(2)} [J'(k_2 r) + \vartheta' N'(k_2 r)] \right\} \end{aligned} \tag{10}$$

Having eliminated so far four of the eight original constants the remaining four ones are related by the boundary conditions at the interface. Introducing

$$\begin{aligned} & x = k_1 a, \ \beta = b/a, \ \beta x = \bar{x} \ ; \ y = k_2 a, \ \beta y = \bar{y} \\ & J(y) = J, \ J'(y) = J', \ J(\bar{y}) = \bar{J}, \ J'(\bar{y}) = \bar{J'} \\ & N(y) = N, \ N'(y) = N', \ N(\bar{y}) = \bar{N}, \ N'(\bar{y}) = \bar{N'} \end{aligned} \tag{11}$$

We get a set of 4 linear homogeneous equations:

$$
\begin{aligned}
B_z^{(1)}(a)=B_z^{(2)}(a):&\quad C_H^{(1)}J(x) = C_H^{(2)}(J-\vartheta' N)\\
E_z^{(1)}(a)=E_z^{(2)}(a):&\quad C_E^{(1)}J(x) = C_E^{(2)}(J-\vartheta N)\\
B_\varphi^{(1)}(a)=B_\varphi^{(2)}(a):&\quad m k_z x^{-2} C_H^{(1)}J(x)+\frac{\varepsilon_1 k_0}{x}C_E^{(1)}J'(x) = m k_z y^{-2} C_H^{(2)}(J-\vartheta' N)+\frac{\varepsilon_2 k_0}{y}C_E^{(2)}(J'-\vartheta N')\\
E_\varphi^{(1)}(a)=E_\varphi^{(2)}(a):&\quad m k_z x^{-2} C_E^{(1)}J(x)+\frac{k_0}{x}C_H^{(1)}J'(x) = m k_z y^{-2} C_E^{(2)}(J-\vartheta N)+\frac{k_0}{y}C_H^{(2)}(J'-\vartheta' N')
\end{aligned}
\tag{12}
$$

In order for solutions to exist the determinant of this system has to vanish.

Dispersion equation. After a little rearrangement the latter condition can be written in the form:

$$
\begin{array}{cccc}
C_H^{(1)} & C_E^{(1)} & C_H^{(2)} & C_E^{(2)} \\
\end{array}
$$

$$
\begin{vmatrix}
\frac{m k_z}{k_0}\frac{1}{x^2} & \varepsilon_1 \frac{K}{x} & \frac{m k_z}{k_0}\frac{1}{y^2} & \varepsilon_2 \frac{L}{y} \\
\frac{K}{x} & \frac{m k_z}{k_0}\frac{1}{x^2} & \frac{M}{y} & \frac{m k_z}{k_0}\frac{1}{y^2} \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{vmatrix} = 0 \tag{14}
$$

The heading on each column indicates the unknown coefficient C to which it is related. We also used the symbols

$$
K = \frac{J'(x)}{J(x)} \; ; \; M = \frac{J'-\vartheta' N'}{J-\vartheta' N} = \frac{J'\bar{N}'-\bar{J}'N'}{J\bar{N}'-\bar{J}'N} \; ; \; L = \frac{J'-\vartheta N'}{J-\vartheta N} = \frac{J'\bar{N}-\bar{J}N'}{J\bar{N}-\bar{J}N} \tag{15}
$$

using the definitions given in Eqs. (9) to (11). Evaluating the determinant one obtains:

$$
\frac{(y^2-x^2)^2}{x^4y^4}\cdot\frac{m^2k_z^2}{k_0^2} = \left(\frac{K}{x}-\frac{M}{y}\right)\left(\varepsilon_1\frac{K}{x}-\varepsilon_2\frac{L}{y}\right) \tag{16}
$$

An almost identical equation has first been obtained by Hondros[3] for the case where the outer surface has been removed to infinity and the inner "dielectric" consists of a metallic wire. In his case the linear combination of J and N functions is replaced by Hankel functions. The differences to the present problem are that the existence of $m \neq 0$ modes is restricted by the conductivity of the wire and critical wavelengths exist above which propagation is impeded by radiation. Eq. (16) is also derived in [4].

In order to solve for the wavenumbers it is still necessary to introduce into Eq. (16) the relations that exist between the three wavenumbers k_1, k_2 and k_z. This relation is given by the need that k_z in both media has to be the same.

$$
k_z^2 = \varepsilon_1 k_0^2 - k_1^2 = \varepsilon_2 k_0^2 - k_2^2 \tag{17}
$$

Writing the last equation as

$$
(k_z^2/k_0^2)(y^2-x^2) = \varepsilon_1 y^2 - \varepsilon_2 x^2
$$

we can eliminate k_z^2/k_0^2 from the left side of Eq. (16) and obtain the more symmetrical form[5]

$$m^2\left(\frac{1}{x^2}-\frac{1}{y^2}\right)\left(\frac{\varepsilon_1}{x^2}-\frac{\varepsilon_2}{y^2}\right) = \left(\frac{xK}{x^2}-\frac{yL}{y^2}\right)\left(\varepsilon_1\frac{xK}{x^2}-\varepsilon_2\frac{yL}{y^2}\right) \qquad (16a)$$

For the following it is convenient to use as a reciprocal unit length the quantity

$$k_p = k_o(\varepsilon_2-\varepsilon_1)^{1/2} = k_o(\Delta\varepsilon)^{1/2} = k_o\,\varepsilon_2^{1/2}\delta^{1/2} \qquad (18)$$

with

$$\delta = (\varepsilon_2-\varepsilon_1)/\varepsilon_2$$

The δ introduced in the last equation is a measure for the magnitude of the dielectric discontinuity. The index p is reminding of the case $\varepsilon_1 = 1 - \omega_p^2/\omega^2$, $\varepsilon_2 = 1$ in which k_p is the vacuum wavenumber at $\omega = \omega_p$. For the two radii a and b occurring in our geometry we write

$$a k_p = A \quad \text{and} \quad b k_p = B \qquad (19)$$

(There is no danger to confuse B with B = B(r) used earlier).

With these definitions and the definitions (11) the relation (17) becomes

$$y^2 - x^2 = A^2 = a^2 k_o^2 \varepsilon_2 \delta \qquad (20)$$

whereas for the ratio k_z^2/k_o^2 we get

$$k_z^2/k_o^2 = \varepsilon_2 - y^2 k_o^{-2} a^{-2} \qquad (21)$$

In view of Eqs. (20) and (21) we rewrite Eq. (16) after multiplying by x^4y^4:

$$m^2(1-y^2A^{-2}\delta)A^4 = [y^2xK - x^2yM][(1-\delta)y^2xK - x^2yL] \qquad (22)$$

and we write this equations symbolically:

$$[Z] = [M]\cdot[L] \qquad (22)$$

Before trying to solve this equation some remarks may be justified: For m = o i.e. rotational symmetric modes, Eq. (22) can be satisfied by equating to zero either of the two brackets on the right hand side. Tracing back these factors to their origin we find that the one containing the ratio of the dielectric constants stems from E mode components whereas the other one is due to H mode components. Thus we have mode separation where each kind of mode has its own dispersion equation which is

for magnetic modes		for electric modes	
$[M] = 0$	(22a)	$[L] = 0$	(22b)

For $m \neq 0$ there is no such separation and neither the E nor the B field is purely transversal. At $k_z = 0$ (cut-off) the left side again vanishes (see Eq. 16).

Solutions of the dispersion equation. The dispersion equation as given by Eq. (22) and implying Eq. (20) is a transcendental equation for the unknown wavenumber, say y, which through Eq. (21) determines the more significant axial wavenumber k_z. The cylindrical functions contained in $[K]$ $[L]$ and $[M]$ are oscillating functions for real arguments and therefore an infinite series of solutions is to be expected. Solutions could in principle be obtained by numerical calculations. We consider it however more instructive to attempt analytical solutions by using approximations of the transcendental functions in regions where we have reason to expect solutions to exist. For the following we confine ourselves to the case of rotational symmetry (m = 0) and the dipole case (m = 1). The following idea can guide us in the discovery of solutions: We know that for δ = 0 we have the well-known modes of a homogeneously filled waveguide (or empty waveguide for $\mathcal{E}_1 = \mathcal{E}_2 = 1$) which we must approach whenever $\delta \ll 1$. This suggests that the solutions may be linearized in the neighbourhood of the wavenumber of the homogeneously filled waveguide.

We regard the dielectric discontinuity as a small perturbation that causes small shifts of the radial wavenumbers in the two dielectrics K_1 and K_2 from their common value k_c which they had in the homogeneous case. We therefore introduce the small quantities

$$s = k_1/k_c - 1 = \bar{x}/k_c b - 1 \quad \text{and} \quad t = k_2/k_c - 1 = \bar{y}/k_c b - 1 \tag{23}$$

(see Eq. 11) as new variables. Later we shall work out approximate solutions by linearizing the equation in s and t. In doing so we are primarily interested to solve the problem which is outlined in the next chapter.

II. The Fresnel "dragging" effect in a waveguide partially filled with a moving dielectric.

Dragging coefficient. The dependence of electromagnetic propagation on the velocity v of the medium through which this propagation occurs is known as the Fresnel effect.The motion of the medium changes the wavelength λ for the medium at rest into a new value λ' which to first order in v/c is given by

$$\lambda'/\lambda = k/k' = n/n' = 1 \pm \varkappa vn/c \tag{24}$$

where the dragging coefficient $\varkappa$ is given by[6)]

$$\varkappa = 1 - \frac{1}{n^2} + \frac{\omega}{n}\frac{\partial n}{\partial \omega} = n^{-2}\left[n^2 - 1 + \omega^2 \frac{\partial (n^2)}{\partial(\omega^2)}\right] \tag{25}$$

The two signs in Eq. (24) refer to a wave travelling parallel or antiparallel to the velocity of the medium.

The derivation of Eq. (25) shows that its validity is not restricted to plane waves in an infinite medium but can be applied to any kind of "medium" which has the effect of changing the vacuum wavenumber in the direction of propagation in the ratio $k_z/k_o = n(\omega)$ which then is the generalized meaning of a refractive index.

This "refractive index" of a waveguide which is homogeneously filled with a dielectric is then given by

$$n^2 = k_z^2/k_o^2 = \varepsilon - k_c^2/k_o^2 \quad (k_c = \text{cut-off wavenumber in the empty waveguide})$$

$$\text{or} \quad n^2 - 1 = \varepsilon - 1 - k_c^2 c^2 \omega^{-2} \tag{26}$$

It is easy to show that whenever n^2-1 (which in the homogeneous medium is equal to 4π times the electric susceptibility χ_e) is of the form

$$n^2 - 1 = \text{const.} \cdot \bar{\omega}^2$$

the bracket of Eq. (25) vanishes, leaving $\varkappa = 0$. Therefore all contributions to n^2-1 which are proportional to ω^{-2} can be neglected in calculating $\varkappa$.

We therefore have a coefficient $\varkappa$ given by:

$$\varkappa = n^{-2}[n^2 - 1 + \omega^2 \partial(n^2)/\partial(\omega^2)] = n^{-2}[\varepsilon - 1 + \omega^2 \partial\varepsilon/\partial(\omega^2)] \tag{27}$$

This equation still refers to the homogeneous filling.

Effective dielectric constant. We think of a homogeneous medium which replaces the two dielectrics in the sense that it produces the same k_z. This medium is then said to have the effective dielectric constant.

$$k_z^2 = \varepsilon_{eff} k_o^2 - k_c^2 = k_o^2 \varepsilon_2 - k_2^2 = k_o^2 \varepsilon_1 - k_1^2 \tag{28}$$

Hence it is evident that for the inhomogeneous dielectric in Eq. (27) ε simply has to be replaced by ε_{eff}:

$$\varkappa = n^{-2}[\varepsilon_{eff} - 1 + \omega^2 \partial\varepsilon_{eff}/\partial(\omega^2)] \tag{29}$$

Dragging of guided waves. In the experiments[7] the change of phase angle along a plasma column of length L due to the dragging effect was measured. This increment is obtained from Eq. (24):

$$\begin{aligned}\Delta &= (k_z' - k_z)L = -k_z L \varkappa n v/c \\ &= -k_o L (v/c)[\varepsilon_{eff} - 1 + \omega^2 \partial\varepsilon_{eff}/\partial\omega^2]\end{aligned} \tag{30}$$

The same expression for Δ would be obtained for plane waves in a homogeneous medium with dielectric constant ε_{eff}.
This leads us to the formulation of the following theorem:

> The phase change (in absolute magnitude) exhibited by the guided wave through the motion of the polarizable matter within the waveguide at velocity v is equal to that which would be exhibited by a plane wave which propagates freely through a homogeneous medium with $\varepsilon = \varepsilon_{eff}$ moving at the same velocity v.

For further evaluation of the bracket in Eq. (30) we will always assume $\varepsilon_2 = 1$ and we express with help of Eq. (28) ε_{eff} in terms of t which has been defined by Eq. (23):

$$\varepsilon_{eff} - 1 = -2t k_c^2/k_o^2 = -2t c^2 k_c^2 \omega^{-2} \tag{31}$$

A final expression for Δ is then obtained:

$$\Delta = k_o L \frac{v}{c} \cdot 2c^2 k_c^2 \frac{dt}{d(\omega^2)} \tag{32}$$

The functional dependence of t on experimental parameters will be derived for a cold collisionless plasma during the following chapters for different waveguide modes.

III. Approximate solutions for the partly plasma filled waveguide for the lowest modes.

There exists already a considerable literature on totally or partly plasma filled waveguides and resonant cavities. A survey is given by Allis et al.[7]. In this literature however the interest has been mainly in the case $\omega \lesssim \omega_p$ where the signal frequency does not exceed the plasma frequency and phenomena like resonance, slow and backward waves occur. The case $\omega \gg \omega_p$ has - to our knowledge - not been treated in a way that would be directly applicable to our problem of the Fresnel dragging effect.

Symmetric (m = 0) modes of magnetic type. We consider first the modes which correspond to the H_{on} modes of the empty guide. The normalized radial wavenumbers x and y are determined as the n^{th} root of Eq. (22a) combined with Eq. (20) which we repeat:

$$\frac{K}{x} - \frac{M}{y} = 0 \qquad (22a) \qquad ; \qquad y^2 - x^2 = A^2 \qquad (20)$$

Here K and M are defined in Eq. (15). In a cold collisionless plasma δ is given by

$$\delta = \omega_p^{\,2} / \omega^2 .$$

In the last chapter we have already assumed $\mathcal{E}_2 = 1$. This makes A^2 independent of ω^2:

$$A^2 = a^2 k_0^2 \, \mathcal{E}_2 \, \delta = a^2 \omega_p^2 / c^2 \tag{33}$$

The solutions (x, y) of Eqs. (22a) and (20) will therefore not depend on ω^2. According to Eq. (23) this will be true also for s and t. Eq. (32) then tells us that there will be no dragging effect.

E_{01} mode. We consider now the first symmetric mode of electric type and therefore have to solve the combination of Eqs. (22b) and (20), i.e. (with $\mathcal{E}_2 = 1$, $\mathcal{E}_1 = 1 - \delta$)

$$(1-\delta)\frac{K}{x} - \frac{L}{y} = 0 \qquad (22b) \qquad ; \qquad y^2 - x^2 = A^2 = a^2 k_0^2 \delta \qquad (20)$$

K and L are again as defined in Eq. (15), whereas δ is supposed to be a perturbation small compared to unity. The configuration to be perturbed is characterized by:

$$\mathcal{E}_1 = 1 \; , \quad A^2 = 0. \quad \text{and} \quad x = y \quad \text{or } k_1 = k_2 = k_c .$$

Inserting this into the above equation and using the explicit expressions for K and L yields:

$$\frac{y J_0'(y)}{J_0(y)} = y \, \frac{J_0'(y) N_0(\beta y) - N_0'(y) J_0(\beta y)}{J_0(y) N_0(\beta y) - N_0(y) J_0(\beta y)} \tag{34}$$

The last equation is solved whenever βy equals one of the zeros of J_0. This clearly must be so because the modes of the empty (or homogeneously filled) guide must be reproduced. In this case we are interested only in the first zero which is given by

$$y = b k_c = r_0 = 2{,}4 \tag{35}$$

The first zero of $N_o'(y)$ is at $y = 2,2$ which is close to r_o. The first two zeros of $N_o(y)$ are at $y = .9$ and $y = 3.95$. We now approximate the functions J_o and N_o by parabolae J_o, N_o, which have the same zeros:

$$J_0^*(x) = c_1(x-r_0)(x+r_0) = c_1(x^2-r_0^2) \quad ; \quad J_0^{*\prime} = 2c_1 x$$
$$N_0^*(x) = c_2(x-r_0-3/2)(x-r_0+3/2) \quad ; \quad N_0^{*\prime} = 2c_2(x-r_0) \tag{36}$$

The zero of N_o' will then coincide with r_o which only slightly differs from the true zero of N_o' at $x = 2,2$. The constants c_1, c_2 need not to be specified numerically as they will cancel in the approximations K^*, M^* for the functions K_o and M_o which are given by:

$$K^* = \frac{J_0^{*\prime}(x)}{J_0^*(x)} = \frac{2x}{x^2-r_0^2}$$

$$L^* = \frac{J_0^{*\prime}\bar{N}_0^* - N_0^{*\prime}\bar{J}_0^*}{J_0^*\bar{N}_0^* - N_0^*\bar{J}_0^*} = \frac{y(\bar{y}-r_0-3/2)(\bar{y}-r_0+3/2)-(y-r_0)(\bar{y}-r_0)(\bar{y}+r_0)}{(y^2-r_0^2)(\bar{y}-r_0-3/2)(\bar{y}-r_0+3/2)-(\bar{y}^2-r_0^2)(y-r_0-3/2)(y-r_0+3/2)}$$

where we used the bars to denote arguments which are increased by the factor $\beta = b/a$, as we did earlier.

Taking now the quantities (see Eq. (23))

$$s = \bar{x}/r_0 - 1 \qquad \text{and} \qquad t = \bar{y}/r_0 - 1 \tag{37}$$

to be small compared to unity, we can neglect their powers > 1 and express K^* and M^* in terms of s and t. The result can be written in the form:

$$x K^{*-1} = -r_0^2\beta^2\,\Gamma\,(1 - s/\Gamma) \qquad \text{with} \quad \Gamma = \gamma(1+\tfrac{\gamma}{2}) \ \text{and} \ \gamma = \beta - 1 = \frac{b-a}{a} \tag{38}$$

$$x L^{*-1} = -r_0^2\beta^2\,\Gamma\,(1+\zeta t) \qquad \text{with} \quad \zeta = 2\,\frac{1-.46\gamma}{1+\gamma/2} + 5.13\gamma \tag{39}$$

Eq. (20) between x and y reads to first order in s and t:

$$t - s = B_0^2/2 \ ; \quad B_0^2 = b^2 r_0^{-2} k_0^2 \delta = b^2 r_0^{-2} c^{-2}\omega_p^2 \tag{40}$$

Using the linearized forms of Eqs. (38) and (39) in Eq. (22b) and eliminating s by means of Eq. (40) we can solve for t:

$$2t = \frac{2\Gamma\delta + B_0^2}{5\Gamma(1-\delta)+1} \tag{41}$$

Inserting this result into Eqs. (31) and (32) we obtain:

$$\varepsilon_{eff} - 1 = \frac{1 + 2\Gamma c^2 k_c^2 \omega^{-2}}{1 + 5\Gamma} \cdot \frac{\omega_p^2}{\omega^2 - \omega_{R0}^2} \tag{42}$$

with

$$\omega_{R0}^2 = \frac{5\Gamma}{1+5\Gamma}\,\omega_p^2 \tag{43}$$

and

$$\Delta = -k_0 L \frac{v}{c}(1-\varepsilon_{eff}) \cdot \frac{2\Gamma\left(\frac{r_0}{b}\right)^2\left(\frac{c}{\omega_p}\right)^2 + \frac{5\Gamma}{1+5\Gamma}}{2\Gamma\left(\frac{r_0}{b}\right)^2\left(\frac{c}{\omega}\right)^2 + 1} \tag{44}$$

For $\Gamma = 0$ which means vanishing thickness of the outer dielectric (Eq. (38)), Δ vanishes. This clearly must be so since a waveguide with homogeneous filling is approached. However, in the infinitesimally thin outer layer t will remain finite. On the other hand if $\Gamma \to \infty$ i.e. if the inner cylinder collapses to a very thin dielectric wire our approximation of N by a parabola becomes invalid and we cannot use the above formulae.

The curve Δ vs. ω_p^2 can be seen to start with finite slope from the origin.

<u>The H_{11}-like dipole mode.</u> The first among the higher dipole modes is the one that becomes the H_{11}-mode when dielectric homogeneity is approached. We have to use now the entire Eq. (22) and the mode becomes of the hybrid type. The procedure is again to use parabolic approximations for the J and N-functions:

$$J_1^* = c_3 x (x - 2r_1) \;; \quad J_1^{*\prime} = 2c_3(x - r_1)$$

$$N_1^* = c_4(x - 1.2r_1)(x - 2.8r_1) \;; \quad N_1^{*\prime} = 2c_4(x - 2r_1) \quad \text{with } J_1'(r_1) = 0 \tag{45}$$

Here we used the properties that the zeros of $J_1(x)$ and $N_1'(x)$ to a high degree of approximation are given by $x = 2r_1$. The constants c_3 and c_4 again are arbitrary. The development of $\bar{x}$ and $\bar{y}$ is now around $bk_c = r_1$ and given by

$$\bar{x} = \beta x = r_1(1+s) \quad \text{and} \quad \bar{y} = \beta y = r_1(1+t) \tag{46}$$

Eq. (20) becomes now to first order in s and t

$$t - s = B_1^2/2 \;; \quad B_1^2 = b^2 r_1^{-2} k_0^2 \delta = b^2 r_1^{-2} c^{-2} \omega_p^2 \tag{47}$$

For K, M and L we consequently find the approximations:

$$x K^* = \frac{x J_1^{*\prime}}{J_1^{*\prime}} = \frac{2\gamma}{1+2\gamma} \; \frac{1 - s/\gamma}{1 - s/(1+2\gamma)} \qquad \text{where } \gamma = \beta - 1$$

$$y M^* = y \frac{J_1^{*\prime}\overline{N_1^{*\prime}} - N_1^{*\prime}\overline{J_1^{*\prime}}}{J_1^{*}\overline{N_1^{*\prime}} - N_1^{*}\overline{J_1^{*\prime}}} = \frac{2\gamma}{1+2\gamma} \; \frac{1+2t}{1-\Gamma_1 t} \qquad \text{where } \Gamma_1 = 1.36 + 3.36 \frac{\gamma^2}{1+2\gamma} \tag{48}$$

$$y L^* = y \frac{J_1^{*\prime}\overline{N_1^{*}} - \overline{J_1^{*}} N_1^{*\prime}}{J_1^{*}\overline{N_1^{*}} - \overline{J_1^{*}} N_1^{*}} = -\frac{2}{\gamma} \; \frac{1+2.36\gamma}{2+3.36\gamma} \cdot \frac{1-\Gamma_2 t}{1-\Gamma_3 t} \qquad \text{where } \Gamma_2 = 1.36 \frac{1+1.47\gamma}{1+2.36\gamma} \text{ and } \Gamma_3 = 1.36 \frac{1+1.24\gamma}{1+1.68\gamma}$$

yL^* becomes nearly independent of t since $\Gamma_2 \approx \Gamma_3$. We may then consider this dependence to be of second order and write

$$y L^* = -\Gamma_4/\gamma \qquad \text{with} \qquad \Gamma_4 = \frac{1+2.36\gamma}{1+1.68\gamma} \xrightarrow{\gamma \ll 1} 1 + .7\gamma \tag{49}$$

With help of Eqs.(46) to(49) the dispersion equation (22) is written in terms of s and t. Setting m = 1 the left side of this equation becomes

$$[Z] = [1 - \delta B_1^{-2}(1+2t)]\, A^4 = r_1^4 \beta^{-4} B_1^2 \cdot [B_1^2 - \delta(1+B_1^2) - \delta\, 2s] \tag{50}$$

For the [M] -bracket on the right side we obtain:

$$[M] = [y^2 x K^* - x^2 y M^*] = \frac{2 r_1^2}{(1+\gamma\cdot 2)\beta^2}\left[(1+2t)\frac{\gamma - s}{1 - s/(1+2\gamma)} - (1+2s)\gamma \frac{1+2t}{1-\Gamma_1 t}\right] \tag{51}$$

Treating now as small quantities $s/(1 + 2\gamma)$ and Γ_1 t but not s/γ and making use of Eq. (47) we express M entirely in terms of s. The product $(1 + 2\,t)/\gamma$ appears twice with opposite sign and cancels. There is a product 2s · 2t which is of second order. We shall see within short that 2t roughly equals B_1^2 for $\gamma \ll 1$ (which is the case of main interest). This allows us to treat this product in an approximate fashion rather than drop it completeely. Evaluating the bracket we obtain:

$$[M] = -r_1^2 \beta^{-2} \gamma (1+2\gamma)^{-1} \left\{ \Gamma_1 B_1^2 + 2s\left[\gamma^{-1}(1+B_1^2) + \Gamma_1 + \Gamma_5\right]\right\} \qquad \text{with} \qquad \Gamma_5 = \frac{1+4\gamma}{1+2\gamma} \xrightarrow{\gamma \ll 1} 1 + 2\gamma \tag{52}$$

The last equation will enable us to obtain already a first order solution to our problem (if we call s = t = 0 the "zeroth" order solution). For a vanishingly small dielectric discontinuity we must approach the unperturbed magnetic H_{11}-mode which expresses itself by the fact that the [Z]-bracket due to the factor A^4 is of second order in δ. We shall see within short that on the right side [L] has a rather large value within the whole range of our approximation. The first order solution is therefore given by

$$[M] = 0 \quad \text{or} \quad 2s = -\frac{\Gamma_1}{\gamma^{-1}(1+B_1^2) + \Gamma_1 + \Gamma_5} B_1^2 \overset{\substack{\delta \ll 1 \\ \gamma \ll 1}}{\approx} -1.36\,\gamma\, B_1^2 \tag{53}$$

and confirms the above treatment of the $2s \cdot 2t$ -term. This first order solution does not contain the frequency ω (no dragging).

Finally, the [L]-bracket is given by

$$[L] = [(1-\delta)y^2 x K - x^2 y L] = \frac{r_1^2}{\beta^2}\left\{\frac{\Gamma_4}{\gamma} + \frac{2(1-\delta)\gamma(1+B_1^2)}{1+2\gamma} + \right.$$
$$\left. + 2s\left[\frac{\Gamma_4}{\gamma} + \frac{(1-\delta)\left(2\gamma + \frac{\gamma}{1+2\gamma} - 1 - B_1^2\right)}{1+2\gamma}\right]\right\} \tag{54}$$

From now we shall neglect terms $\propto \gamma$ as compared to terms $\propto \gamma^{-1}$. This amounts to a linearization in γ after splitting off γ^{-1}. It is not necessary to do this but it simplifies the formulas. In our later application to an inhomogeneous plasma the magnitude of γ which is representative is only vaguely determined and the errors introduced by the present procedure are unlikely to be larger than those arising from the imperfect knowledge of γ even for values of γ as large as .6.
We then obtain for [L]:

$$[L] = r_1^2 \beta^{-2} \Gamma_4 \left\{1 + (1-\Phi)2s\right\} \quad \text{with} \quad \Phi = (1-\delta)(1+B_1^2)\gamma(1+2\gamma)^{-1}\Gamma_4^{-1} \tag{55}$$

The Φ-term will be neglected later. We see that for small γ the [L] -bracket is large, a behaviour which is approximately inverse to that of the [M]-bracket.
For the product [L] [M] we obtain to first order in 2s:

$$-\Gamma_6 \beta^4 r_1^{-4}[L][M] = \Gamma_1 B_1^2 + \left[\gamma^{-1}(1+B_1^2) + \Gamma_1 + \Gamma_5 + \Gamma_1 B_1^2 (1-\Phi)\right] 2s \tag{56}$$
$$\text{where} \quad \Gamma_6 = (1+2\gamma)/\Gamma_4$$

We combine Eq. (56) with Eq. (50) for [Z] through Eq. (22) and solve for 2s:

$$2s = -\frac{[\Gamma_1 + \Gamma_6 B_1^2 - \delta\Gamma_6(1+B_1^2)]\, B_1^2}{\gamma^{-1}(1+B_1^2) + \Gamma_1 + \Gamma_5 + \Gamma_1(1-\Phi)B_1^2 - \Gamma_6 B_1^2 \delta} \approx -\frac{\Gamma_1 + \Gamma_6 B_1^2 - \delta\Gamma_6(1+B_1^2)}{\gamma^{-1}(1+B_1^2) + \Gamma_1 + \Gamma_5 + \Gamma_1(1-\Phi)B_1^2} B_1^2$$

The term $\delta \cdot \Gamma_6 \; B_1^2$ in the denominator (which through B_1^2 is quadratic in δ) has a much smaller influence than the term $\delta \cdot \Gamma_6(1 + B_1^2)$ in the numerator and is therefore neglected. As we did earlier we also have to neglect terms $\propto \gamma$ with respect to terms $\propto \gamma^{-1}$ in the denominator, that means we set $\Gamma_5 = 1$ and we drop the Φ-term:

$$2s = -\frac{\Gamma_1 + \Gamma_6 B_1^2 - \delta\Gamma_6(1+B_1^2)}{(1+B_1^2)(\gamma^{-1} + \Gamma_1) + 1} B_1^2 \tag{57}$$

Eq. (57) gives the second order solution for the shift of the radial wavenumber in dielectric "one" which for small δ tends towards the first order solution of Eq. (52) as $B_1^2 = r_1^{-2} b_1^2 k_0^2 \delta \rightarrow 0$
We also give the expression for 2t treating higher order terms in γ in the same way:

$$2t = B_1^2 + 2s = \frac{1 + \gamma(1+B_1^2)^{-1} + \gamma\Gamma_6\delta}{1 + \gamma(1+B_1^2)^{-1} + \gamma\Gamma_1} B_1^2 \tag{58}$$
$$\approx \frac{1 + \gamma\Gamma_6\delta}{1 + \gamma\Gamma_1} B_1^2$$

We calculate $\mathcal{E}_{eff}$ and Δ from Eq. (58) in analogy to what has been done earlier for the E_{01} mode and find

$$\mathcal{E}_{eff} - 1 = - \frac{1 + \gamma(1+B_1^2)^{-1}}{1 + \gamma\Gamma_1 + \gamma(1+B_1^2)^{-1}} \cdot \frac{\omega_p^2}{\omega^2 - \omega_{R1}^2} \tag{59}$$
$$\text{with} \quad \omega_{R1}^2 = \frac{\gamma\Gamma_6}{1+\gamma(1+B_1^2)^{-1}}\,\omega_p^2 < \omega_p^2$$

In order to arrive at Eq. (59) we have used the approximate equality: $1+\gamma\Gamma_6\delta \approx (1-\gamma\Gamma_6\delta)^{-1}$. This way of writing Eq. (59) shows the equivalency with a resonance occurring at $\omega_{R1} < \omega_p$ by the action of an electrostatic restoring force which reacts against any attempted displacement of the bulk of electrons from the ion background. However, at $\omega = \omega_{R1}$ Eq. (59) cannot be expected to be valid as in our deviation we have always assumed $\delta = \omega_p^2/\omega^2 \ll 1$.

Finally Δ is given by

$$\Delta = - k_0 L \frac{v}{c} \cdot \frac{\gamma\Gamma_6}{1+\gamma(1+B_1^2)^{-1} + \gamma\Gamma_1} \cdot \frac{\omega_p^4}{\omega^4} \tag{60}$$

The curve Δ vs. ω_p^2 now starts with zero slope from the origin.

<u>The E_{11}-like dipole mode.</u> We will in short consider this mode too as it shows an interesting phenomenon of sign reversal in the frequency dependence of $\mathcal{E}_{eff}$. We arrive at this mode if we develop our parabola approximation of the J and N function around the zero of J_1, which we call r_2 and which for our purpose can be taken to be equal $2\,r_1$. The approximations of Eq. (45) with $2r_1$ replaced by r_2 are:

$$J_1^* = x(x - r_2)\ ; \quad J_1^{*\prime} = 2x - r_2\ ; \quad N_1^* = (x - .6r_2)(x - 1.4r_2)\ ; \tag{61}$$
$$N_1^{*\prime} = 2(x - r_2)$$

whereas for the wavenumbers in analogy to Eqs. (46) we have now

$$\bar{x} = \beta x = r_2(1+s) \quad \text{and} \quad \bar{y} = \beta y = r_2(1+t) \tag{62}$$

and in analogy to Eqs. (48) we obtain

$$x K^* = (1-\gamma+2s)/(s-\gamma) \tag{63}$$
$$y M^* = \gamma/(.08 + .16\gamma - .42\gamma^2)$$
$$y L^* = -\frac{1}{\gamma}\,\frac{1-\gamma+(2-12.5\gamma)t}{1+(2-5.25\gamma)t}$$

The brackets of Eq. (22) are now given by

$$[z] = [1-\delta B_2(1+2t)]A^4 = r_2^2\beta^{-2}B_2^4[1-\delta B_2^{-2}(1+2t)] \tag{64}$$
$$[M] = r_2^2\beta^{-2}\left[\frac{(1+2t)(1-\gamma+2s)}{s-\gamma} - \frac{(1+2s)\gamma}{.08+.16\gamma+.42\gamma^2}\right]$$
$$[L] = r_2^2\beta^{-2}\left[\frac{(1-\delta)(1+2t)(1-\gamma+2s)}{s-\gamma} + \frac{1+2s}{\gamma}\cdot\frac{1-\gamma+(2-12.5\gamma)t}{1+(2-5.25\gamma)t}\right]$$
$$\text{where} \quad B_2 = \frac{b}{r_2}\cdot\frac{\omega_p}{c}$$

The situation regarding the relative magnitude of the [L] and [M] -bracket in the vicinity of $\delta = s = t = 0$ is now reversed. The [L] -bracket tends to zero and the [M] -bracket reaches a relatively large (negative) value for small γ.

The first order solution is now given by

$$[L] = 0 \quad \text{or} \quad (1-\delta)(1+2t)\,\frac{1-\gamma+2s}{s-\gamma} = -\frac{1+2s}{\gamma}\cdot\frac{1-\gamma+(2-12.5\gamma)t}{1+(2-5.25\gamma)t}$$

Eliminating s by means of

$$2s = 2t - B_2^{\,2}$$

and neglect of second order terms in s, t and δ finally leads to

$$t = \frac{1-\gamma}{1+\gamma+3.25\gamma^2+5.25\gamma^3}\,(B_2^2/2+\gamma\delta) \tag{65}$$

Eq. (65) predicts a change of sign of t from positive to negative values at $\gamma = 1$. This will cause $\mathcal{E}_{eff}$ -1 and Δ as well to change their sign at $\gamma = 1$. Our approximation, however, becomes of doubtful validity in this range as the N_1 function deviates more and more from the supposed parabola as γ is increased over unity. In the following we want to confirm the result with the proper use of the J_1- and N_1-functions.

We first express the derivatives of J_1 and N_1 occurring in K and L in terms of J_o and N_o according to the rule $Z_1' = Z_o - Z_1/x$ where Z_1 can be either J_1 or N_1:

$$K = \frac{J_1'(x)}{J_1(x)} = \frac{J_0(x)}{J_1(x)} - \frac{1}{x} \quad \text{and} \quad L = \frac{J_1'\bar{N}_1 - N_1'\bar{J}_1}{J_1\bar{N}_1 - N_1\bar{J}_1} = \frac{J_0\bar{N}_1 - N_0\bar{J}_1}{J_1\bar{N}_1 - N_1\bar{J}_1} - \frac{1}{y}$$
$$= \frac{J_0 - (N_0/\bar{N}_1)\bar{J}_1}{J_1 - (N_1/\bar{N}_1)\bar{J}_1} - \frac{1}{y} \tag{66}$$

Numerators and denominators are now again expanded around r_2. Setting again (Eq. 62):

$$x = \alpha\bar{x} = \alpha r_2(1+s) \quad \text{and} \quad y = \alpha\bar{y} = \alpha r_2(1+t) \quad \text{with} \quad \alpha = \beta^{-1} = a/b$$

and introducing the symbols

$$J_{oo} = J_0(r_2)\;;\quad J_{o\alpha} = J_0(\alpha r_2)\;;\quad J_{1o} = J_1(r_2) = 0\;;\quad J_{1\alpha} = J_1(\alpha r_2) \tag{67}$$

with analogous ones for the N-functions, we have the expansions

$$J_0(y) = J_{o\alpha} - J_{1\alpha}\,\alpha r_2 t\;;\quad J_1(\bar{y}) = J_{oo} r_2 t\;;\quad J_1(y) = J_{1\alpha} + (\alpha r_2 J_{o\alpha} - J_{1\alpha})\,t \tag{68}$$

$$L = \frac{J_{o\alpha} - (\alpha r_2 J_{1\alpha} + \frac{J_{oo}}{N_{1o}} r_2 N_{o\alpha})\,t}{J_{1\alpha} + (\alpha r_2 J_{o\alpha} - J_{1\alpha} - \frac{J_{oo}}{N_{1o}} r_2 N_{1\alpha})\,t} - \frac{1}{\alpha r_2(1+t)} \tag{69}$$

Replacing t by s and ignoring $N_{o\alpha}$, $N_{1\alpha}$ we obtain K:

$$K = \frac{J_{o\alpha} - \alpha r_2 J_{1\alpha} s}{J_{1\alpha} - (\alpha r_2 J_{o\alpha} - J_{1\alpha})\,s} - \frac{1}{\alpha r_2(1+s)} \tag{70}$$

With help of the ratios:

$$a_{01} = J_{0\alpha}/J_{1\alpha} \;; \qquad b_{00} = N_{0\alpha}/J_{0\alpha} \;; \qquad b_{11} = N_{1\alpha}/J_{1\alpha}$$

L takes the form

$$L = a_{01}\,\frac{1-(\alpha r_2 a_{01}^{-1} + J_{00} N_{10}^{-1} b_{00})t}{1+(\alpha r_2 a_{01} - 1 - J_{00} N_{10}^{-1} b_{11})t} - \frac{1}{\alpha r_2 (1+t)}$$

or, linearizing the denominators

$$\frac{M}{Y} = \alpha^{-1} r_2^{-1} a_{01}\left\{1+\left[J_{00}N_{10}^{-1}(b_{11}-b_{00}) - \alpha r_2 (a_{01}+a_{01}^{-1})\right]t\right\} - \alpha^{-2} r_2^{-2}(1-2t)$$

Replacing $t \rightarrow s$ and setting $J_{00}N_{10}^{-1} = 0$ yields

$$\frac{K}{X} = \alpha^{-1} r_2^{-1} a_{01}\left\{1 - \alpha r_2 (a_{01}+a_{01}^{-1})s\right\} - \alpha^{-2} r_2^{-2}(1-2s)$$

The first order solution can now be obtained from

$$[L] = 0 \qquad \text{or} \qquad (1-\delta)\,\frac{K}{X} - \frac{L}{Y} = 0$$

and is given by

$$t = \frac{(\alpha r_2 a_{01} - 1)\delta + \left[1 - \tfrac{1}{2}\alpha^2 r_2^2 (a_{01}+a_{01}^{-1})\right]B_2^2}{\alpha r_2 a_{01} r_2 J_{00} N_{10}^{-1} (b_{00}-b_{11})}$$

$$= \frac{\pi}{2}\,\frac{N_{10}}{r_2 J_{00}}\left\{J_{1\alpha}(\alpha r_2 J_{0\alpha} - J_{1\alpha})\delta + J_{1\alpha}^2\left[1 - \frac{\alpha^2 r_2^2}{2}\left(\frac{J_{0\alpha}}{J_{1\alpha}} + \frac{J_{1\alpha}}{J_{0\alpha}}\right)\right]B_2^2\right\} \qquad (71)$$

where the last form has been obtained by reintroducing the old symbols and observing the relation $N_{0\alpha}\, J_{1\alpha} - J_{0\alpha}\, N_{1\alpha} = 2/\pi r_2 \alpha$.

The coefficient of δ changes its sign where

$$J_{0\alpha} - J_{1\alpha}/\alpha r_2 = J_{1\alpha}' = 0 \qquad \text{or} \qquad \alpha r_2 = 1.84 \quad \alpha = \frac{1.84}{3.83} \qquad (72)$$

$$\gamma = 1.08$$

which is indeed close to the value $\gamma = 1$ obtained by the approximation used earlier.

The results for t, $\mathcal{E}_{eff}$ and Δ can now be given in the following form:

$$2t = Q_0(Q_1 B_2^2 + Q_2\delta) \qquad (71a)$$

with

$$Q_0 = \frac{\pi}{r_2}\frac{N_{10}}{J_{00}} = -.84\,; \quad Q_1 = J_{1\alpha}^2\left[1 - \frac{\alpha^2 r_2^2}{2}\left(\frac{J_{0\alpha}}{J_{1\alpha}} + \frac{J_{1\alpha}}{J_{0\alpha}}\right)\right]$$

$$Q_2 = J_{1\alpha}\left[\alpha r_2 J_{0\alpha} - J_{1\alpha}\right] \gtrless 0 \text{ for } \alpha \gtrless .48$$

$$\mathcal{E}_{eff} - 1 = -\,Q_0\left\{Q_1 \frac{r_2^2}{b^2}\frac{c^2}{\omega^2} + Q_2\right\}\frac{\omega_p^2}{\omega^2} \qquad (73)$$

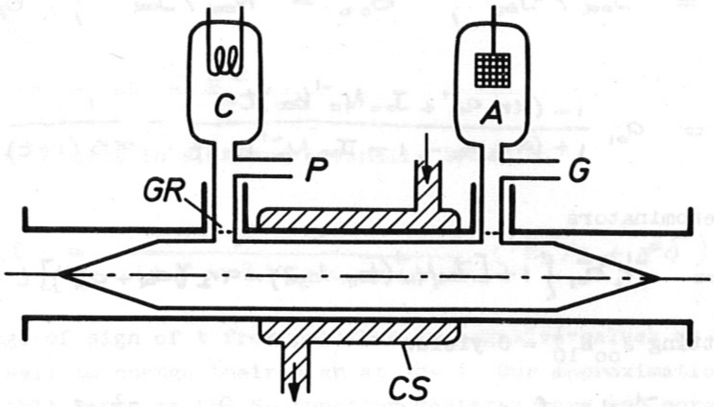


Fig. 1

C oxide coated cathode
A anode
P to pumping lead
G gas inlet
Cs cooling system
GR grids

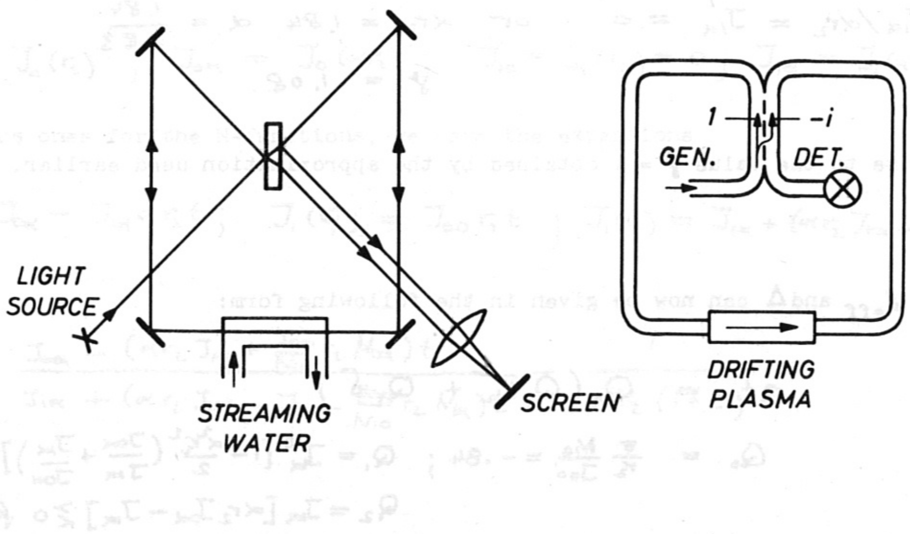
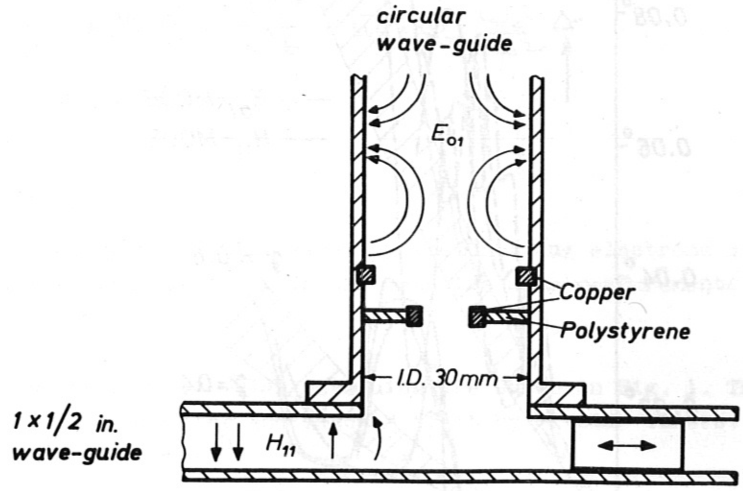
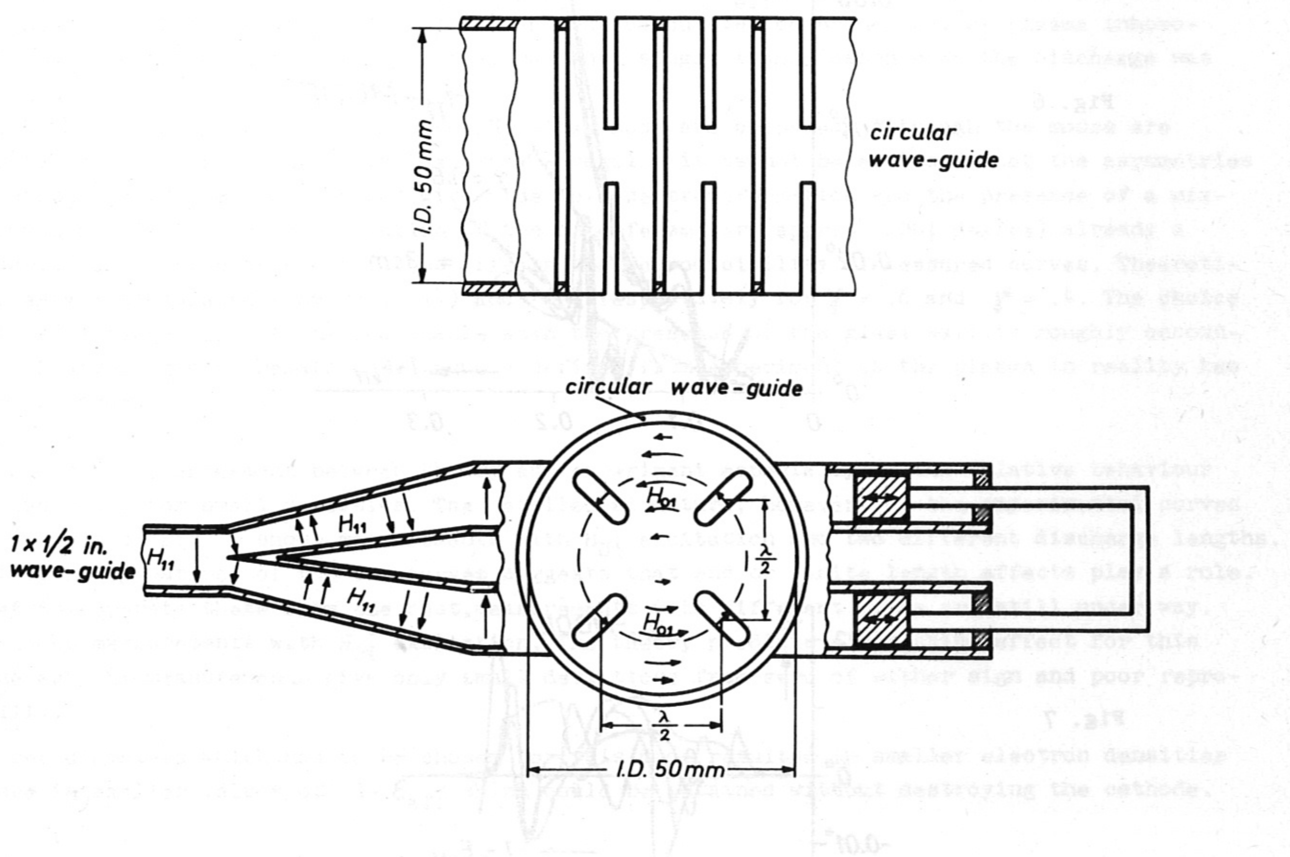


Fig. 2



circular wave-guide
E_{01}
Copper
Polystyrene
I.D. 30mm
1 x 1/2 in. wave-guide
H_{11}

Fig. 3



I.D. 50 mm
circular wave-guide
circular wave-guide
1 x 1/2 in. wave-guide
H_{11}
H_{11}
H_{11}
H_{01}
H_{01}
$\frac{\lambda}{2}$
$\frac{\lambda}{2}$
I.D. 50mm

Fig. 4

Fig. 5

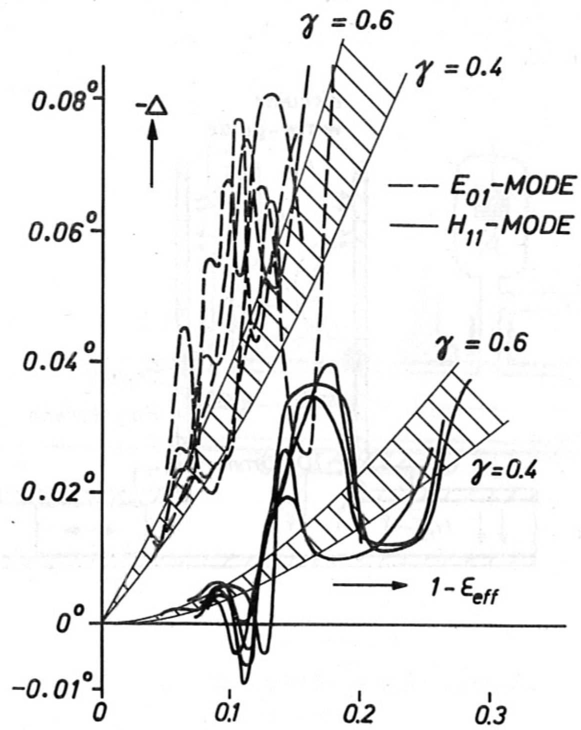


Fig. 6

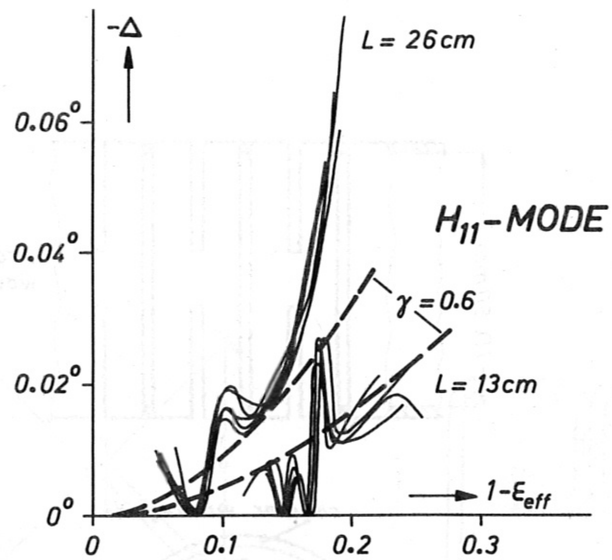
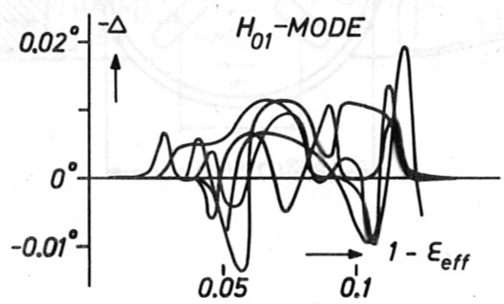


Fig. 7



$$\begin{aligned}\Delta &= k_o L \frac{v}{c} \cdot 2c^2 k_c^2 \cdot \frac{dt}{d(\omega^2)} \\ &= -k_o L \frac{v}{c} \cdot \frac{2r_2^2}{k_o^2 b^2} \cdot Q_o \cdot Q_2 \cdot \frac{\omega_p^2}{\omega^2}\end{aligned} \tag{74}$$

Comparison with experiment

Experiments on the dragging of microwaves by the drifting electrons of a low -pressure discharge have already been reported[8)9)]. In the meantime some measurements have also been made with the H_{01} mode.

The discharge and the surrounding circular waveguide are shown in Fig. 1. They are inserted into the path of a microwave interferometer and matched to it by outside tuners. This interferometer will be described in detail elsewhere[8)].

Briefly it may be quoted that the wave coming from the generator is split into two seperate trains travelling through the discharge in opposite directions. When the direction of the wave trains is reversed a small shift in the relative phase of the two trains is observed which is attributed to the dragging effect.

The principle of measurement is essentially the one used by Fizeau[11)] in his historical measurements with light propagating through streaming water (Fig. 2).
The E_{01} and H_{01} modes have been produced by means of couplers and filters on both sides of the discharge section. They are shown in Figs. 3 and 4. The E_{01}-coupler-and-filter has been described in [13)]. The H_{01} mode filter consisted of a system of azimuthal slots in the wall of the circular waveguide (Fig. 4b) which was surrounded by absorbing material.

In Figs. 5 to 7 as defined in Eq. (30) is plotted vs.1- ε_{eff}. In the two previous papers[8)9)] this quantity had been designated by ω_p^2/ω^2 and was calculated under neglect of plasma inhomogeneity from the phase shift which was measured in a single transmission when the discharge was switched on.
In Fig. 5 the dragging of the E_{01} and the H_{11}-like mode are compared. Although the modes are substantially pure when produced in the empty waveguide it cannot be excluded that the asymmetries of the discharge to a certain extent give rise to mode transformation and the presence of a mixture of modes. With the high resolution of the interferometer (approx. .001 degree) already a small variation of discharge conditions affects the reproducibility of measured curves. Theoretical curves are claculated from Eqs. (44) and (60) respectively for γ = .6 and γ = .4. The choice of γ in this range seems to be reasonable when the presence of the glass wall is roughly accounted for. However, γ must remain only vaguely defined from experiment as the plasma in reality has no sharp boundary.

Fig. 5 shows crude agreement between theory and experiment especially in the relative behaviour of the two modes for small densities. The detailed structure, however, of the experimental curves is not predicted. Fig. 6 shows measurements with H_{01} excitation and two different discharge lengths. The different appearance of the two curves suggests that end or finite length effects play a role. In order to seperate these from the rest,measurements with different sizes are still under way.
Fig. 7 shows measurements with H_{01} excitation. The theory predicts a vanishing effect for this mode whereas the measurements give only small deviations from zero of either sign and poor reproducibility.

The larger diameters which had to be chosen for this mode resulted in smaller electron densities and hence in smaller values of 1- ε_{eff} which could be attained without destroying the cathode.

Conclusions

The model of a circular waveguide filled partially with a rod of cold and collisionless plasma is shown to explain largely the observed dragging effect for various modes and frequencies considerably above ω_p, the electron plasma frequency. Physically this behaviour may be attributed to the possibility of resonance due to collective electron oscillations at frequencies of the order of ω_p which occurs whenever the mode has an electric field normal to the plasma density gradient.

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List of Symbols

c	velocity of light in vacuum
ω	angular frequency of fields
b	inner radius of a circular waveguide
a	radius of the dielectric discontinuity
α	a/b
β	$b/a = 1/\alpha$
γ	$(b-a)/a = \beta - 1$
μ	magnetic permeability
ε	dielectric constant (in general)
ε_1	dielectric constant for $o < r < a$, in particular $\varepsilon_1 = 1 - \omega_p^2/\omega^2$
ε_2	dielectric constant for $a < r < b$, in particular $\varepsilon_2 = 1$
ε_{eff}	effective dielectric constant (Eq. (28))
δ	$(\varepsilon_2 - \varepsilon_1)/\varepsilon_2$ (relative) magnitude of the dielectric discontinuity
k_o	ω/c, vacuum wavenumber
k, k_1, k_2	radial wavenumber, in general, in the inner, in the outer dielectric zone
k_z	z-component of wavenumber
k_p	$k_o\,(\varepsilon_2 - \varepsilon_1)^{1/2}$, in particular ω_p/c
k_c	r_n/b, where r_n is explained below. Cut-off wavenumber of the empty or homogeneously filled waveguide referring to a particular mode
x, y	$k_1 a$, $k_2 a$
$\bar{x}$, $\bar{y}$	$k_1 B' = \beta x$, $k_2 b = \beta y$
A^2	$y^2 - x^2 = a^2 k_p^2$
J, $\bar{J}$	$J(y)$, $J(\bar{y})$
N, $\bar{N}$	$N(y)$, $N(\bar{y})$
K, L, M	see Eq. (15)
[L], [M], [Z]	electric, magnetic factor and product of the two in the characteristic equation (Eq. (22))
s, t	relative deviations of the radial wavenumbers in the two dielectrics from k_c, quantities in which the characteristic equation is linearized (Eq. (23))
r_o, r_2	given by the first zero $\neq 0$ of $J_o(r_o) = 0$ respectively $J_1(r_2) = 0$
r_1	given by the first zero of $J_1'(r_1) = 0$
B_n	$b k_p r_n^{-1}$
v	velocity of refractive medium along z-axis
$\varkappa$	dragging coefficient
Δ	phase shift along a sample of length L due to the dragging effect
Γ	$\gamma(1 + \gamma/2)$
Γ_1	$1.36 + 3.36\,\gamma^2/(1+\gamma) \xrightarrow{\gamma \ll 1} 1.36$
Γ_2	$1.36\,(1 + 1.47\gamma)/(1 + 2.36\gamma) \xrightarrow{\gamma \ll 1} 1.36(1 - .89\gamma)$
Γ_3	$1.36\,(1 + 1.24\gamma)/(1 + 1.68\gamma) \xrightarrow{\gamma \ll 1} 1.36(1 - .44\gamma)$
Γ_4	$(1 + 2.36\gamma)/(1 + 1.68\gamma) \xrightarrow{\gamma \ll 1} 1 + .7\gamma$
Γ_5	$(1 + 4\gamma)/(1 + 2\gamma) \xrightarrow{\gamma \ll 1} 1 + 2\gamma$
Γ_6	$(1 + 2\gamma)/\Gamma_4 \xrightarrow{\gamma \ll 1} 1 + 1.3\gamma$

ζ	$2\,\frac{1-.46\gamma}{1+\gamma/2}+5.13\,\gamma$
J_{no}, $J_{n\alpha}$	$J_n(r_2)$; $J_n(\alpha r_2)$
N_{no}, $N_{n\alpha}$	$N_n(r_2)$; $N_n(\alpha r_2)$

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