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Abstract

Toroidal magnetic fields $\underset{\sim}{B} = \sum_{\nu \geq 0} \varepsilon^{\nu} \underset{\sim}{B}_{\nu}$ where the unperturbed field $\underset{\sim}{B}_0$ has closed lines of force, are considered. On rather weak assumptions it is shown that single-valued formal solutions $F = \sum_{\nu \geq 0} \varepsilon^{\nu} F_{\nu}$ of the equation $\underset{\sim}{B} \cdot \nabla F = 0$ exist, and that the asymptotic magnetic surfaces $F = const$ are uniquely determined up to any order. Recursion formulae are derived which allow calculation of any order of F from its lower orders, and explicite expressions are given for F_0 which depend on whether the rotational transform is of the same or of higher order than the perturbing field. For the special cases of stellarator-like vacuum fields a necessary condition is derived for the asymptotic magnetic surfaces to be toroidally closed. As another application the lowest order of an adiabatic invariant is constructed for the longitudinal guiding centre motion.

1. Introduction

A magnetic field $\underset{\sim}{B}$ is said to be "toroidal" if a "meridional surface" exists on which $\underset{\sim}{B}$ is nowhere tangential, and which is intersected infinitely many times by every line of force. Let $\underset{\sim}{x}$ be a point of a meridional surface, and let $\underset{\sim}{x}'(\underset{\sim}{x})$ be that point where the line of force, intersecting this surface at $\underset{\sim}{x}$, the first time intersects it again. The vector $\delta\underset{\sim}{x} = \underset{\sim}{x}' - \underset{\sim}{x}$ is then called the "displacement vector", and the mapping $\underset{\sim}{x} \rightarrow \underset{\sim}{x}'$ is called "magnetic mapping". Its deviation from the identity is called "rotational transform". Magnetic surfaces may be defined as being generated by carrying invariant curves of the magnetic mapping along the lines of force. They may be defined alternatively as the surfaces $F = const$ of any single-valued solution of the equation

$$\underset{\sim}{B} \cdot \nabla F = 0 . \qquad (1.1)$$

In the present paper this second definition will be used as a starting point. The properties of magnetic surfaces were discussed, for example, by Kruskal and Kulsrud (1958), and Hamada (1962).

Magnetic surfaces play an important role in many branches of plasma physics. In ideal MHD theory, for example, their existence is necessary for static equilibria. In the theory of charged particle motion as well as in the related microscopic theory of collisionless plasmas, they provide, in the limit of small gyroradii, a useful tool for constructing constants of the guiding centre motion.

A given magnetic field does not, in general, form surfaces. On the other hand, there are two classes of toroidal fields for which such surfaces trivially exist, viz. symmetric fields and closed-line fields. The latter are degenerate in the sense that the magnetic surfaces are not unique. One should expect that approximate magnetic surfaces exist in an asymptotic sense for fields which deviate only a little from a "fundamental" field forming exact surfaces. Such fields may be represented as

$$\underset{\sim}{B} = \sum_{\nu \geq 0} \varepsilon^{\nu} \underset{\sim}{B}_{\nu}(\underset{\sim}{x}) , \qquad (1.2)$$

where the equation

$$\underset{\sim}{B}_{0} \cdot \nabla F_{0} = 0 \qquad (1.3)$$

possesses single-valued solutions. One then writes

$$F = \sum_{\nu \geq 0} \varepsilon^{\nu} F_{\nu}(\underset{\sim}{x}) \qquad (1.4)$$

and looks for single-valued solutions F_{ν} of the formal expansion of eq. (1.1)

$$\sum_{\mu=0}^{\nu} \underset{\sim}{B}_{\nu-\mu} \cdot \nabla F_{\mu} = 0 \quad (\nu = 1, 2, 3, \ldots) . \qquad (1.5)$$

Clearly the series (1.4) will in general diverge for any $\varepsilon \neq 0$ even if its coefficients exist to all orders. The N th order asymptotic magnetic surfaces

$$\sum_{\nu=0}^{N} \varepsilon^{\nu} F_{\nu} = const \tag{1.6}$$

will nevertheless be useful if ε is sufficiently small.

The functions F_{μ} being known for $\mu < \nu$ eq. (1.5) is of the form

$$\underset{\sim}{B}_{0} \cdot \nabla F_{\nu} = S_{\nu} \quad (\nu = 1, 2, 3, \dots), \tag{1.7}$$

where S_{ν} is a known function. Such an equation is called a "magnetic differential equation". Newcomb (1959) proved that it possesses single-valued solutions if, and only if, the condition

$$\oint dl \, \frac{S_{\nu}}{|\underset{\sim}{B}_{0}|} = 0 \tag{1.8}$$

is satisfied on every closed line of force of the fundamental field $\underset{\sim}{B}_{0}$. The integral is once around these lines, l being the arclength. If $\underset{\sim}{B}_{0}$ has a rotational transform, then the unperturbed magnetic surfaces $F_{0} = const$ are uniquely determined by eq. (1.3). Hence the function S_{1} depends essentially on $\underset{\sim}{B}_{1}$ only, and the lowest order Newcomb condition is fulfilled only for special perturbing fields. This difficulty is connected with so-called "resonances" and with the question of the "stability of magnetic surfaces", c.f. Solov'ev (1966), Solov'ev and Shafranov (1966a), Rosenbluth et al. (1966), Filonenko et al. (1967), Vuillemin and Gourdon (1967).

If on the other hand the fundamental field has no rotational transform, then the situation is entirely different, because the surfaces $F_0 = const$ are not determined uniquely by eq. (1.3). In this case one can choose the function F_0 such that the Newcomb condition for the next nontrivial order is satisfied. It has been shown for many more or less special cases that asymptotic magnetic surfaces are then uniquely determined to the lowest order or to the lowest few orders, c.f. Greene and Johnson (1961), Korablev et al. (1961), Mercier (1964), McNamara and Whiteman (1966), Solov'ev (1966), Solov'ev and Shafranov (1966b), Lindner and Lortz (1967), Vuillemin and Gourdon (1967).

The present paper confirms this statement for the most general magnetic fields with small rotational transform and extends it to all orders. The existence and the uniqueness of asymptotic magnetic surfaces is thus proved in all orders. This result can also be concluded from the theory of nearly recurrent Hamiltonian systems of Kruskal (1962), which applies to the present problem because the equations for the lines of force can be put into Hamiltonian form, as shown by McNamara and Whiteman (1966). The approach used in the present paper has, however, the advantage of yielding relatively simple recursion formulae for the calculation of any order of the function $\bar{F}$.

Given a real magnetic field with small rotational transform, the choice of the expansion (1.2), particularly the choice of the corresponding fundamental field, is arbitrary, and the asymptotic magnetic surfaces (1.6) will depend on this choice. Only in special cases is a natural expansion parameter, and hence also a natural fundamental field, given. In vacuum fields, such as stellarators, which are produced by two systems of current carrying wires, the ratio of the currents is a natural expansion parameter because it is the only relevant dimensionless quantity which can easily be altered experimentally. Hence the magnetic field of the larger current is a natural fundamental field, provided its rotational transform is zero. In stability theory of closed-line plasma equilibria the appropriate fundamental field is, of course, the equilibrium field. Though it would thus suffice for many applications to represent the perturbing field by only one term, we nevertheless consider the most general infinite power series (1.2). Besides the endeavour for generality, we have also a practical reason for doing this: The rotational transform can be of the same order as the perturbing field or of any higher order. If it is of higher order, then great simplification is achieved by a transformation which yields the field as an infinite power series even if it consisted only of a finite number of terms originally. Hence we do not introduce any additional difficulties when considering the most general problem from the beginning.

2. Displacements

We represent the magnetic field as

$$\underset{\sim}{B} = \nabla \mu \times \nabla v . \tag{2.1}$$

Since $\underset{\sim}{B}$ is a series, the flux functions μ and v are series too,

$$\mu = \sum_{\nu \geq 0} \varepsilon^{\nu} \mu_{\nu}(\underset{\sim}{x}) \quad , \quad v = \sum_{\nu \geq 0} \varepsilon^{\nu} v_{\nu}(\underset{\sim}{x}) , \tag{2.2}$$

and the various orders of $\underset{\sim}{B}$ are given by

$$\underset{\sim}{B}_{\nu} = \sum_{\mu=0}^{\nu} \nabla \mu_{\mu} \times \nabla v_{\nu-\mu} \quad (\nu = 0, 1, 2, \ldots) . \tag{2.3}$$

The flux functions are multi-valued because they are constant on the lines of force, which are assumed not to be closed. In general only the lowest orders μ_0 and v_0 , being flux functions of the fundamental field, are single-valued. Hence they are well suited to serve as coordinates. We choose an additional arbitrary function $w_0(\underset{\sim}{x})$ which increases monotonically along the lines of force of $\underset{\sim}{B}_0$. For convenience we assume it to increase by a constant amount σ_0 once around these closed lines. If $\underset{\sim}{B}_o$ is, for instance, a vacuum field, then w_o can thus be chosen as its potential. The transformation

$$\psi = u_o(\underset{\sim}{x}) \quad , \quad \chi = v_o(\underset{\sim}{x}) \quad , \quad \sigma = w_o(\underset{\sim}{x}) \tag{2.4}$$

possesses an inverse if $\underset{\sim}{B}_o \neq 0$ because the determinant

$$D = (\nabla u_o \times \nabla v_o) \cdot \nabla w_o \tag{2.5}$$

is then positive. Hence the quantities ψ, χ, and σ may serve as coordinates. Writing the inverse of the transformation (2.4) as

$$\underset{\sim}{x} = \underset{\sim}{x}(\psi, \chi, \sigma) , \tag{2.6}$$

we have the periodicity property

$$\underset{\sim}{x}(\psi, \chi, \sigma + \sigma_o) = \underset{\sim}{x}(\psi, \chi, \sigma) . \tag{2.7}$$

Any derivatives are transformed by using the relations

$$\frac{\partial \underset{\sim}{x}}{\partial \psi} = \frac{\nabla v_o \times \nabla w_o}{D} \; , \quad \frac{\partial \underset{\sim}{x}}{\partial \chi} = \frac{\nabla w_o \times \nabla u_o}{D} \; , \quad \frac{\partial \underset{\sim}{x}}{\partial \sigma} = \frac{\nabla u_o \times \nabla v_o}{D} . \tag{2.8}$$

In the coordinate system (ψ, χ, σ) the contravariant components of the fields $\underset{\sim}{B}_\nu$ are, according to eqs. (2.3) and (2.8) ,

$$\left.\begin{aligned}
\underset{\sim}{B}_\nu \cdot \nabla\psi &= D\left[-\frac{\partial u_\nu}{\partial\sigma} + \sum_{\mu=1}^{\nu-1} \frac{\partial(u_\mu, v_{\nu-\mu})}{\partial(\chi,\sigma)}\right],\\
\underset{\sim}{B}_\nu \cdot \nabla\chi &= D\left[-\frac{\partial v_\nu}{\partial\sigma} + \sum_{\mu=1}^{\nu-1} \frac{\partial(u_\mu, v_{\nu-\mu})}{\partial(\sigma,\psi)}\right],\\
\underset{\sim}{B}_\nu \cdot \nabla\sigma &= D\left[\frac{\partial u_\nu}{\partial\psi} + \frac{\partial v_\nu}{\partial\chi} + \sum_{\mu=1}^{\nu-1} \frac{\partial(u_\mu, v_{\nu-\mu})}{\partial(\psi,\chi)}\right],\\
&(\nu = 1, 2, 3, \dots).
\end{aligned}\right\} \tag{2.9}$$

Here and henceforth we adopt the convention that a sum is to be omitted whenever the upper summation limit is smaller than the lower one. If the solenoidal field $\underset{\sim}{B}_\nu$ and functions u_μ, v_μ $(1 \leq \mu < \nu)$ are given, then the eqs. (2.9) represent a system of inhomogeneous differential equations, the solvability condition of which is automatically satisfied. Hence the functions u_ν, v_ν can be calculated for any possible choice of the lower order flux functions.

The displacements $\delta\psi$ and $\delta\chi$ in the meridional surfaces $w_0 = \sigma$ are defined by

$$\left.\begin{aligned}
u(\psi+\delta\psi, \chi+\delta\chi, \sigma+\sigma_0, \varepsilon) &= u(\psi,\chi,\sigma,\varepsilon),\\
v(\psi+\delta\psi, \chi+\delta\chi, \sigma+\sigma_0, \varepsilon) &= v(\psi,\chi,\sigma,\varepsilon).
\end{aligned}\right\} \tag{2.10}$$

The solutions $\delta\psi(\psi,\chi,\sigma,\varepsilon)$ and $\delta\chi(\psi,\chi,\sigma,\varepsilon)$ of these equations are series in powers of ε, from which the displacement vectors are calculated according to

$$\delta\underset{\sim}{X} = \underset{\sim}{X}(\psi+\delta\psi, \chi+\delta\chi, \sigma) - \underset{\sim}{X}(\psi,\chi,\sigma). \tag{2.11}$$

If we denote by f^* the increase of a function f once around the closed lines of force of $\underset{\sim}{B}_0$,

$$f^*(\psi,\chi,\sigma) = f(\psi,\chi,\sigma+\sigma_0) - f(\psi,\chi,\sigma), \tag{2.12}$$

then the solutions of eqs. (2.10) are

$$\delta\psi = -\varepsilon^n u_n^* + O(\varepsilon^{n+1}), \quad \delta\chi = -\varepsilon^n v_n^* + O(\varepsilon^{n+1}), \tag{2.13}$$

where n is the smallest integer for which the increases u_n^* and v_n^* are not both identically zero. Clearly the rotational transform is $O(\varepsilon^n)$. We will henceforth use the symbol "n" exclusively for the order of the rotational transform.

The single-valuedness in space of the fields $\underset{\sim}{B}_\nu$ implies, in view of eq. (2.7), that their components are periodic in σ. Hence eqs. (2.9) imply

$$\frac{\partial u_n^*}{\partial\sigma} = \frac{\partial v_n^*}{\partial\sigma} = \frac{\partial u_n^*}{\partial\psi} + \frac{\partial v_n^*}{\partial\chi} = 0 \tag{2.14}$$

for $\nu = n$. We conclude that a function $\phi(\psi,\chi)$ exists such that

$$u_n^* = -\frac{\partial\phi}{\partial\chi}, \quad v_n^* = \frac{\partial\phi}{\partial\psi}. \tag{2.15}$$

The eqs. (2.13) for the displacements have thus to lowest order the canonical form

$$\delta\psi = \varepsilon^n\frac{\partial\phi}{\partial\chi} + O(\varepsilon^{n+1}), \quad \delta\chi = -\varepsilon^n\frac{\partial\phi}{\partial\psi} + O(\varepsilon^{n+1}). \tag{2.16}$$

Substituting this into eq. (2.11) for the displacement vectors and using eqs. (2.8) now yields

$$\delta \underset{\sim}{x} = \varepsilon^{n} D^{-1} \nabla\phi \times \nabla\sigma + O(\varepsilon^{n+1}), \tag{2.17}$$

from which it is immediately concluded that

$$\underset{\sim}{B} \times \delta \underset{\sim}{x} = \varepsilon^{n} \nabla\phi + O(\varepsilon^{n+1}). \tag{2.18}$$

Hence the magnetic field as well as the displacement vectors are to lowest order tangential to the surfaces $\phi = const$ From this property it may be anticipated that these surfaces coincide with the zero-order magnetic surfaces. In Sec. 4 this will be formally proven to be the case.

The function ϕ is single-valued in space, because the flux functions can be chosen such that they and their increases in σ are single-valued on any contour in a meridional surface. Furthermore, eq. (2.18) implies that ϕ does not depend on the special choice of the flux functions.

In order to calculate the function ϕ let us first consider its differential. Using eqs. (2.15) we have

$$d\phi = v_n^* \, d\psi - u_n^* \, d\chi = \int_0^{\sigma_0} d\sigma \left(\frac{\partial v_n}{\partial \sigma} d\psi - \frac{\partial u_n}{\partial \sigma} d\chi \right), \tag{2.19}$$

where the integral may simply be evaluated between the limits 0 and σ_0 because the result does not depend on σ.

Introducing the infinitesimal vector $\underset{\sim}{d}$ from the point (ψ, χ, σ) to the point ($\psi + d\psi, \chi + d\chi, \sigma$), which satisfies

$$\underset{\sim}{d} \cdot \nabla\psi = d\psi, \quad \underset{\sim}{d} \cdot \nabla\chi = d\chi, \quad \underset{\sim}{d} \cdot \nabla\sigma = 0, \tag{2.20}$$

we compute

$$\begin{aligned} d\phi &= -\int_0^{\sigma_0} d\sigma\, D^{-1}\left[(\underset{\sim}{B_0} \cdot \nabla u_n)(\underset{\sim}{d} \cdot \nabla\chi) - (\underset{\sim}{B_0} \cdot \nabla v_n)(\underset{\sim}{d} \cdot \nabla\psi)\right] \\ &= -\int_0^{\sigma_0} d\sigma\, D^{-1}(\underset{\sim}{B_0} \times \underset{\sim}{d}) \cdot (\nabla\psi \times \nabla v_n + \nabla u_n \times \nabla\chi) \\ &= -\oint \underset{\sim}{dx} \cdot (\underset{\sim}{d} \times \underset{\sim}{\hat{B}}), \end{aligned} \tag{2.21}$$

where

$$\underset{\sim}{\hat{B}} = \nabla\psi \times \nabla v_n + \nabla u_n \times \nabla\chi . \tag{2.22}$$

The resulting integral is a closed-contour integral along the lines of force of $\underset{\sim}{B_0}$. Since $\underset{\sim}{dx} \times \underset{\sim}{d}$ is a surface element of the infinitesimal annular surface between the two neighbouring lines of force (ψ, χ) and ($\psi + \delta\psi, \chi + \delta\chi$), $d\phi$ is simply the flux of the field $\underset{\sim}{\hat{B}}$ through this surface. We conclude

$$\phi = \iint \underset{\sim}{df} \cdot \underset{\sim}{\hat{B}}, \tag{2.23}$$

where the integral is over any annular surface limited by the field line (ψ, χ) and another arbitrary fixed reference field line (ψ_0, χ_0).

According to eq. (2.3) an equivalent expression for $\hat{\underset{\sim}{B}}$ is

$$\hat{\underset{\sim}{B}} = \underset{\sim}{B}_n - \sum_{\nu=1}^{n-1} \nabla u_\nu \times \nabla v_{n-\nu} . \qquad (2.24)$$

Let ℓ denote the order of the perturbing field, i.e. $\underset{\sim}{B} - \underset{\sim}{B}_0 = O(\varepsilon^\ell)$. The functions u_ν, v_ν can be chosen to be zero for $1 \leq \nu < \ell$. Hence in the special case $n = \ell$ the function ϕ takes the simple form

$$\phi = \iint \underset{\sim}{df} \cdot \underset{\sim}{B}_\ell . \qquad (2.25)$$

If, on the other hand, $n > \ell$ (the case $n < \ell$ is excluded by the present expansion), then $u_\ell^* = v_\ell^* = 0$, and

$$\underset{\sim}{A}_\ell = \psi \nabla v_\ell - \chi \nabla u_\ell \qquad (2.26)$$

is a single-valued vector potential for $\underset{\sim}{B}_\ell$. Hence

$$\iint \underset{\sim}{df} \cdot \underset{\sim}{B}_\ell = \oint_{\psi,\chi = \mathrm{const}} \underset{\sim}{dx} \cdot \underset{\sim}{A}_\ell - \oint_{\psi_0,\chi_0 = \mathrm{const}} \underset{\sim}{dx} \cdot \underset{\sim}{A}_\ell$$

$$= \Big[\psi v_\ell^* - \chi u_\ell^* \Big]_{\psi_0,\chi_0}^{\psi,\chi} = 0 \qquad (2.27)$$

in this case. We conclude that the rotational transform is of higher order than the perturbing field if, and only if, the flux of the lowest order perturbing field $\underset{\sim}{B}_\ell$ through the closed lines of force of the fundamental field $\underset{\sim}{B}_0$ is constant. For the special case $\ell = 1$ this was previously shown by Lindner and Lortz (1967).

In order to eliminate the flux functions $\mathcal{U}_\nu$ and v_ν we first apply Stokes' theorem to the integral over the second term of the expression (2.24) and obtain

$$\phi = \iint d\underset{\sim}{f} \cdot \underset{\sim}{B}_n - \int_0^{\sigma_0} d\sigma \sum_{\nu=1}^{n-1} u_\nu \frac{\partial v_{n-\nu}}{\partial \sigma} + C , \qquad (2.28)$$

where the constant C depends on the above mentioned reference field line. The use of eqs. (2.9), which in general is a rather involved procedure, yields for instance

$$\left.\begin{aligned}
n=1: \quad \phi &= \iint d\underset{\sim}{f} \cdot \underset{\sim}{B}_1 + C , \\
n=2: \quad \phi &= \iint d\underset{\sim}{f} \cdot \underset{\sim}{B}_2 - \int_0^{\sigma_0} d\sigma \frac{\underset{\sim}{B}_1 \cdot \nabla \chi}{D} \int_0^{\sigma} d\sigma \frac{\underset{\sim}{B}_1 \cdot \nabla \psi}{D} + C , \\
n=3: \quad \phi &= \iint d\underset{\sim}{f} \cdot \underset{\sim}{B}_3 + \int_0^{\sigma_0} d\sigma \Big[\frac{\underset{\sim}{B}_2 \cdot \nabla \psi}{D} \int_0^{\sigma} d\sigma \frac{\underset{\sim}{B}_1 \cdot \nabla \chi}{D} - \\
&- \frac{\underset{\sim}{B}_2 \cdot \nabla \chi}{D} \int_0^{\sigma} d\sigma \frac{\underset{\sim}{B}_1 \cdot \nabla \psi}{D} \Big] + \frac{1}{2} \int_0^{\sigma_0} d\sigma \Big[\frac{\partial}{\partial \chi} \frac{\underset{\sim}{B}_1 \cdot \nabla \psi}{D} \Big(\int_0^{\sigma} d\sigma \frac{\underset{\sim}{B}_1 \cdot \nabla \chi}{D} \Big)^2 - \\
&- \frac{\partial}{\partial \psi} \frac{\underset{\sim}{B}_1 \cdot \nabla \chi}{D} \Big(\int_0^{\sigma} d\sigma \frac{\underset{\sim}{B}_1 \cdot \nabla \psi}{D} \Big)^2 \Big] + C .
\end{aligned}\right\} \qquad (2.29)$$

The function ϕ was previously derived by Lindner and Lortz (1967) for the case $n=2$, $\underset{\sim}{B}_1 = 0$. Its extremely simple representation for $n=1$ has, to our knowledge, not yet been appreciated.

3. Guiding centre motion

It is a well-known fact that in the limit $\varepsilon = 0$, in which the lines of force are closed, an adiabatic invariant series

$$\mathcal{J} = \sum_{\nu \geq 0} \varepsilon_L^{\nu} \, \mathcal{J}_{\nu} \tag{3.1}$$

can be constructed for the longitudinal guiding centre motion, where

$$\varepsilon_L \sim m v / (e B_0 L) \tag{3.2}$$

is the ratio of the particle Larmor radius and the scale length L of the field $\underset{\sim}{B}_0$. The zero-order term of the series (3.1) is

$$\mathcal{J}_0 (K, \mu, \Psi, \chi) = \oint dl \, v_{\parallel} = \int d\sigma \, D^{-1} B_0 \sqrt{2(K - \mu B_0)} \, , \tag{3.3}$$

where the integration is either back and forth between two zeros of the integrand, or once around a line of force, depending on whether the particle is trapped or not. The quantities K and μ are the particle energy and magnetic moment,

$$K = \tfrac{1}{2} v^2 \quad , \quad \mu = \tfrac{1}{2} v_{\perp}^2 / B_0 \, . \tag{3.4}$$

The displacement due to the drifts a guiding centre suffers during one circuit around its closed zero-order orbit is

$$\delta\psi = \frac{m}{e}\,\frac{\partial \mathcal{J}_0}{\partial \chi} + O\left(\varepsilon_L^2\right), \quad \delta\chi = -\frac{m}{e}\,\frac{\partial \mathcal{J}_0}{\partial \psi} + O\left(\varepsilon_L^2\right), \tag{3.5}$$

while the corresponding circulation period is

$$\tau = \frac{\partial \mathcal{J}_0}{\partial K} + O\left(\varepsilon_L\right). \tag{3.6}$$

Turning now to the case $\varepsilon \neq 0$ we observe that the displacement of an untrapped particle due to the rotational transform is given by eqs. (2.16). Hence the total displacement is

$$\delta\psi = \frac{m}{e}\,\frac{\partial \tilde{\mathcal{J}}_0}{\partial \chi} + O\left(\varepsilon_L^2 + \varepsilon^{n+1}\right), \quad \delta\chi = -\frac{m}{e}\,\frac{\partial \tilde{\mathcal{J}}_0}{\partial \psi} + O\left(\varepsilon_L^2 + \varepsilon^{n+1}\right), \tag{3.7}$$

where

$$\tilde{\mathcal{J}}_0 = \mathcal{J}_0 \pm \varepsilon^n \frac{e}{m}\,\phi. \tag{3.8}$$

The relative sign depends on the sense in which the particle circulates. The circulation period may now be written as

$$\tau = \frac{\partial \tilde{\mathcal{J}}_0}{\partial K} + O\left(\varepsilon_L + \varepsilon\right) \tag{3.9}$$

because ϕ does not depend on the variable K. Hence the function $\tilde{\mathcal{J}}_0$ has for any ε precisely the same properties as the function $\mathcal{J}_0$ for $\varepsilon = 0$ and is thus the zero-order term of an adiabatic invariant series for untrapped particles. This invariant is relavant if the quantities ε_L and ε^n are both small and have the same order of magnitude. If $\varepsilon^n \ll \varepsilon_L \ll 1$, then the term arising from the rotational transform is negligible. If, on the other hand, $\varepsilon_L \ll \varepsilon^n \ll 1$, then the function ϕ itself is an approximate constant of the guiding centre motion.

Substitution of the expressions (3.3) and (2.28) into eq. (3.8) yields

$$\tilde{J}_0 = \oint d\sigma \left[\frac{B_0}{D} \sqrt{2(K-\mu B_0)} \pm \varepsilon^n \frac{e}{m} \left(\frac{\underset{\sim}{A}_n \cdot \underset{\sim}{B}_0}{D} - \sum_{\nu=1}^{n-1} u_\nu \frac{\partial v_{n-\nu}}{\partial \sigma} \right) \right] , \quad (3.10)$$

where $\underset{\sim}{A}_n$ is any single-valued vector potential for the field $\underset{\sim}{B}_n$. The expression (3.10) holds also for trapped particles because the second term integrates to zero for these. It may be generalized in the obvious way so as to include a small electric field as well as a weak time dependence of the fields. In the special cases $\underset{\sim}{B} - \underset{\sim}{B}_0 = O(\varepsilon^n)$ the adiabatic invariant may then simply be written as

$$\tilde{J} = \oint d\underset{\sim}{x} \cdot (\underset{\sim}{v} \pm e\underset{\sim}{A}/m) + O(\varepsilon^{n+1} + \varepsilon_L + \varepsilon_E + \varepsilon_t). \quad (3.11)$$

Here $\underset{\sim}{A}$ is any single-valued vector potential for the total magnetic field, $\underset{\sim}{v}$ is the guiding centre velocity, whose relevant component is

$$v_\parallel = \sqrt{2(K-\mu B - e\varphi/m)} , \quad (3.12)$$

where φ is a scalar potential of the parallel component of the electric field $\underset{\sim}{E}$. The additional parameters

$$\varepsilon_E \sim E/(vB) \quad , \quad \varepsilon_t \sim L v^{-1} \partial/\partial t \qquad (3.13)$$

are assumed to have the same order of magnitude as the parameter ε_L . The expression (3.11) was previously derived by Hastie et al. (1967) for the more special cases of static magnetic fields forming exact magnetic surfaces.

4. Zero-order magnetic surfaces

We try to determine the function F_o from the requirement that the eqs. (1.5) have single-valued solutions, i.e. that

$$F_\nu^{\,*} = 0 \qquad (\nu = 1, 2, 3, \ldots). \qquad (4.1)$$

It turns out that one has to consider eq. (1.5) up to $\nu = n$ for this purpose. Since this would become rather involved for $n > 1$, we first introduce some new coordinates $\psi', \chi',$ and σ', in terms of which the determination of F_o will be much simpler.

Since the functions u_ν and v_ν are single-valued for $\nu < n$, the field

$$\underset{\sim}{B}_o' = \nabla\left(\sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu\right) \times \nabla\left(\sum_{\nu=0}^{n-1} \varepsilon^\nu v_\nu\right) \qquad (42)$$

has closed lines of force and may therefore be taken as a new fundamental field. The flux functions of this field are thus, together with σ, suitable coordinates. They are calculated from the original coordinates according to

$$\left.\begin{aligned} \psi' &= \psi + \sum_{\nu=1}^{n-1} \varepsilon^{\nu} u_{\nu}(\psi, \chi, \sigma) \\ \chi' &= \chi + \sum_{\nu=1}^{n-1} \varepsilon^{\nu} v_{\nu}(\psi, \chi, \sigma) \\ \sigma' &= \sigma \end{aligned}\right\} \tag{4.3}$$

If we define functions $u_{\nu}(\psi', \chi', \sigma')$, $v_{\nu}(\psi', \chi', \sigma')$ by

$$\left.\begin{aligned} u(\psi, \chi, \sigma) &= \psi' + \sum_{\nu \geq 1} \varepsilon^{\nu} u_{\nu}'(\psi', \chi', \sigma'), \\ v(\psi, \chi, \sigma) &= \chi' + \sum_{\nu \geq 1} \varepsilon^{\nu} v_{\nu}'(\psi', \chi', \sigma'), \end{aligned}\right\} \tag{4.4}$$

then the contravariant components of the various terms in the corresponding expansion

$$\underset{\sim}{B} = \sum_{\nu \geq 0} \varepsilon^{\nu} \underset{\sim}{B}_{\nu}' \tag{4.5}$$

are given by

$$\left.\begin{aligned} \underset{\sim}{B}_{\nu}' \cdot \nabla \psi' &= D' \Big[- \frac{\partial u_{\nu}'}{\partial \sigma'} + \sum_{\mu=1}^{\nu-1} \frac{\partial (u_{\mu}', v_{\nu-\mu}')}{\partial (\chi', \sigma')} \Big], \\ \underset{\sim}{B}_{\nu}' \cdot \nabla \chi' &= D' \Big[- \frac{\partial v_{\nu}'}{\partial \sigma'} + \sum_{\mu=1}^{\nu-1} \frac{\partial (u_{\mu}', v_{\nu-\mu}')}{\partial (\sigma', \psi')} \Big], \\ \underset{\sim}{B}_{\nu}' \cdot \nabla \sigma' &= D' \Big[\frac{\partial u_{\nu}'}{\partial \psi'} + \frac{\partial v_{\nu}'}{\partial \chi'} + \sum_{\mu=1}^{\nu-1} \frac{\partial (u_{\mu}', v_{\nu-\mu}')}{\partial (\psi', \chi')} \Big], \\ &(\nu = 1, 2, 3, \dots) \end{aligned}\right\} \tag{4.6}$$

where

$$\mathcal{D}' = (\nabla\psi' \times \nabla\chi')\cdot\nabla\sigma'. \tag{4.7}$$

Equations (4.6) are analogs of eqs. (2.9). Any other relations of Sec. 2 hold as well if any quantities are replaced by the corresponding primed quantities. Solving the eqs.(4.4) for the functions u_ν' and v_ν' up to the n-th order, we find

$$u_\nu' = v_\nu' = 0\ , \quad (1 \leq \nu < n) \tag{4.8}$$

and

$$\left.\begin{aligned} u_n'(\psi,\chi,\sigma) &= u_n(\psi,\chi,\sigma) \\ v_n'(\psi,\chi,\sigma) &= v_n(\psi,\chi,\sigma) \end{aligned}\right\} \tag{4.9}$$

Hence

$$\underset{\sim}{B} - \underset{\sim}{B}_0' = O(\varepsilon^n) \tag{4.10}$$

and $n' = n$. Writing now

$$F = \sum_{\nu \geq 0} \varepsilon^\nu F_\nu'(\psi', \chi', \sigma'), \tag{4.11}$$

the equations for the asymptotic magnetic surfaces are

$$\sum_{\mu=0}^{\nu} \underset{\sim}{B}_{\nu-\mu}' \cdot \nabla F_\mu' = 0\ , \quad (\nu = 0,1,2,3,\dots), \tag{4.12}$$

$$F_\nu'^{\,*} = 0. \tag{4.13}$$

We have thus reduced our general problem to the relatively simple special cases of perturbing fields being of the same order as the rotational transform. Note that if $\underset{\sim}{B} - \underset{\sim}{B}_o = O(\varepsilon^n)$ originally, the functions μ_ν and v_ν can be chosen to be zero for $1 \leq \nu < n$, and the transformation (4.3) then reduces to the identity.

In order to determine the function F_o' we need the operators $\underset{\sim}{B}_o' \cdot \nabla$ and $\underset{\sim}{B}_n' \cdot \nabla$, which in terms of the primed coordinates are

$$\underset{\sim}{B}_o' \cdot \nabla = D' \frac{\partial}{\partial \sigma'} \tag{4.14}$$

$$\underset{\sim}{B}_n' \cdot \nabla = D' \left[\left(\frac{\partial \mu_n'}{\partial \psi'} + \frac{\partial v_n'}{\partial \chi'} \right) \frac{\partial}{\partial \sigma'} - \frac{\partial \mu_n'}{\partial \sigma'} \frac{\partial}{\partial \psi'} - \frac{\partial v_n'}{\partial \sigma'} \frac{\partial}{\partial \chi'} \right] . \tag{4.15}$$

For $\nu = 0$ eq. (4.12) implies that F_o' is an arbitrary function of ψ' and χ' only, which thus automatically satisfies eq. (4.13). The next $n-1$ orders merely yield the same statements for the functions $F_1', \dots, F_{n-1}'$. In the n-th order the function F_o' reappears the first time. This order is

$$\frac{\partial F_n'}{\partial \sigma'} = \frac{\partial \mu_n'}{\partial \sigma'} \frac{\partial F_o'}{\partial \psi'} + \frac{\partial v_n'}{\partial \sigma'} \frac{\partial F_o'}{\partial \chi'} . \tag{4.16}$$

Integration with respect to σ' yields

$$F_n'^{*} = \mu_n'^{*} \frac{\partial F_o'}{\partial \psi'} + v_n'^{*} \frac{\partial F_o'}{\partial \chi'} . \tag{4.17}$$

According to eqs. (2.15) this is zero if, and only if,

$$\frac{\partial(\phi', F_o')}{\partial(\psi', \chi')} = 0. \tag{4.18}$$

We conclude that the surfaces $F_o' = const$ coincide with the surfaces $\phi' = const$, and that the function F_o' has the form

$$F_o'(\psi', \chi') = H(\phi'(\psi', \chi')) \tag{4.21}$$

where H is an arbitrary function of one variable. Recalling that, in view of eq. (4.10),

$$\phi' = \iint d\underset{\sim}{f} \cdot \underset{\sim}{B}_n', \tag{4.22}$$

and applying the transformation (4.2-7) to lowest order, we finally obtain

$$F_o(\psi, \chi) = H(\phi(\psi, \chi)). \tag{4.23}$$

Zero-order magnetic surfaces are thus uniquely determined in any case.

5. Higher orders

We now proceed to any orders of eqs. (4.12-13). We omit the primes, but bear in mind that in order to obtain F as a function of the position vector we have to transform the results back to the original coordinates.

We have seen that the σ-dependence of F_0 is given by the zeroth order of the eq. $\underset{\sim}{B} \cdot \nabla F = 0$, while its dependence on the other two coordinates is restricted by the σ-integral over the n-th order equation, i.e. by the requirement $F_n^* = 0$ We now show that the analog holds for any order.

We make our notation uniform by defining F_μ to be zero for $\mu < 0$. We may then write

$$D \frac{\partial F_\nu}{\partial \sigma} + \sum_{\mu \geq n} \underset{\sim}{B}_\mu \cdot \nabla F_{\nu-\mu} \tag{5.1}$$

for any ν. Equation (5.1) implies

$$F_\nu(\psi, \chi, \sigma) = f_\nu(\psi, \chi, \sigma) + g_\nu(\psi, \chi), \tag{5.2}$$

where

$$f_\nu = - \int_0^\sigma d\sigma \frac{1}{D} \sum_{\mu \geq n} \underset{\sim}{B}_\mu \cdot \nabla F_{\nu-\mu}, \tag{5.3}$$

and g_ν is arbitrary. Now F_ν does not appear in the next $n-1$ orders. When eq. (5.2) is substituted for F_ν and eq. (4.15) is used for the term $\underset{\sim}{B}_n \cdot \nabla g_\nu$, the $(\nu + n)$-th order, integrated with respect to σ, yields

$$\mu_n^* \frac{\partial g_\nu}{\partial \psi} + v_n^* \frac{\partial g_\nu}{\partial \chi} = \int_0^{\sigma_0} d\sigma \frac{1}{D} \Big(\underset{\sim}{B}_n \cdot \nabla f_\nu + \sum_{\mu \geq n+1} \underset{\sim}{B}_\mu \cdot \nabla F_{\nu+n-\mu} \Big). \quad (5.4)$$

Let us henceforth assume that the coordinates ψ and χ have from the beginning been chosen such that ϕ is a function of ψ only, i.e. such that the surfaces $\psi = const$ coincide with the zero-order magnetic surfaces. This implies

$$\mu_n^* = 0 \, , \quad \frac{\partial v_n^*}{\partial \chi} = 0. \quad (5.5)$$

Note that according to eq. (4.9) these relations hold for the primed quantities (if, and only if, they hold for the unprimed ones.). Equation (5.4) is now readily integrated to yield

$$g_\nu(\psi, \chi) = h_\nu(\psi) + $$
$$+ \frac{1}{v_n^*} \int_0^{\chi} d\chi \int_0^{\sigma_0} d\sigma \frac{1}{D} \Big(\underset{\sim}{B}_n \cdot \nabla f_\nu + \sum_{\mu \geq n+1} \underset{\sim}{B}_\mu \cdot \nabla F_{\nu+n-\mu} \Big), \quad (5.6)$$

where h_ν is arbitrary.

The function F can be constructed order by order from eqs. (5.2-3) and (5.6). If the perturbing fields are analytical, then it obviously exists to all orders. It depends however on the choice of the functions h_ν. If these functions are gathered in a single function

$$h(\psi, \varepsilon) = \sum_{\nu \geq 0} \varepsilon^\nu h_\nu(\psi), \quad (5.7)$$

then there is a one-to-one correspondence between the function h and the function F.

In order to show that the surfaces $F = const$ are to all orders independent of h, we prove the formal relation

$$F(\psi, x, \sigma, \varepsilon) = h(\tilde{F}(\psi, x, \sigma, \varepsilon), \varepsilon), \qquad (5.8)$$

where $\tilde{F}$ corresponds to the special choice $h = \psi$, i.e. to $h_0 = \psi$, and $h_\nu = 0$ for $\nu > 0$. A straightforward expansion in powers of ε of eq. (5.8) yields

$$F_k = \sum_{i \geq 0} \sum_{j \geq 0} \frac{h_i^{(j)}}{j!} \tilde{S}_j^{(k-i)}. \qquad (5.9)$$

Here $h_i^{(j)}$ is the j-th derivative of h_i at $\psi = \tilde{F}_0$, and the functions $\tilde{S}_r^{(s)}$ are defined to be zero ~~for~~ if $s < 0$ or if $r < 0$, while for $s \geq 0$ they are defined by

$$\left(\sum_{\mu \geq 1} \varepsilon^\mu \tilde{F}_\mu\right)^r = \sum_{s \geq 0} \varepsilon^s \tilde{S}_r^{(s)}. \qquad (5.10)$$

Hence

$$\tilde{S}_r^{(s)} = \left\{ \begin{array}{lll} \sum\limits_{\nu_1, \ldots, \nu_r} \prod\limits_{i=1}^{r} \tilde{F}_{\nu_i} & \text{for} & 1 \leq r \leq s \\ 1 & \text{for} & r = s = 0 \\ 0 & \text{otherwise,} & \end{array} \right\} \qquad (5.11)$$

where the summation is over those r-tuples $(\nu_1, \cdots, \nu_r)$ of positive integers ν_i which satisfy $\nu_1 + \nu_2 + \cdots + \nu_r = s$.

We prove eq. (5.9) by induction to be a consequence of eqs. (5.2-3) and (5.6). For $k = 0$ it reads

$$F_o = h_o(\tilde{F}_o), \tag{5.12}$$

while eqs. (5.2-3) and (5.6) yield

$$\tilde{F}_o = \Psi, \tag{5.13}$$

$$F_o = h_o(\Psi), \tag{5.14}$$

which implies eq. (5.12). Next, we assume that eq. (5.9) holds for any $k < \nu$. Substitution into eq. (5.1) then yields

$$\frac{\partial F_\nu}{\partial \sigma} = -\frac{1}{D} \sum_{i \geq o} \sum_{j \geq o} \frac{1}{j!} \sum_{\mu \geq 1} \underset{\sim}{B}_\mu \cdot \nabla \left(h_i^{(j)} \tilde{S}_j^{(\nu-\mu-i)} \right) =$$

$$= -\frac{1}{D} \sum_{i \geq o} \sum_{j \geq o} \frac{1}{j!} \sum_{\mu \geq 1} \left(h_i^{(j)} \underset{\sim}{B}_\mu \cdot \nabla \tilde{S}_j^{(\nu-\mu-i)} + h_i^{(j+1)} \tilde{S}_j^{(\nu-\mu-i)} \underset{\sim}{B}_\mu \cdot \nabla \tilde{F}_o \right), \tag{5.15}$$

where eq. (5.13) has been used. Since the second term vanishes for $j = -1$, we may replace j by $j-1$ in it. Replacing μ by $\mu - 1$ in the first term simultaniously, and resubtracting the additionally arising $(\mu = 1)$ -term, we have

$$\frac{\partial F_\nu}{\partial \sigma} = \sum_{i \geq o} \sum_{j \geq o} \frac{h_i^{(j)}}{j!} \frac{\partial \tilde{S}_j^{(\nu-i)}}{\partial \sigma} - \tag{5.16}$$

$$-\frac{1}{D} \sum_{i \geq o} \sum_{j \geq o} \frac{h_i^{(j)}}{j!} \sum_{\mu \geq 1} \left(\underset{\sim}{B}_{\mu-1} \cdot \nabla \tilde{S}_j^{(\nu-\mu-i+1)} + j\, \tilde{S}_{j-1}^{(\nu-\mu-i)} \underset{\sim}{B}_\mu \cdot \nabla \tilde{F}_o \right).$$

Here the second sum vanishes because of the identity

$$\sum_{\mu\ge 1}\left(\underset{\sim}{B}_{\mu-1}\cdot\nabla\tilde{S}_r^{(s+1-\mu)} + r\,\tilde{S}_{r-1}^{(s-\mu)}\,\underset{\sim}{B}_{\mu}\cdot\nabla\tilde{F}_0\right) = 0. \tag{5.17}$$

This identity is readily verified for $2\le r\le s$ by evaluating

$$\begin{aligned}
&\sum_{\mu\ge 1}\underset{\sim}{B}_{\mu-1}\cdot\nabla\tilde{S}_r^{(s+1-\mu)} = \\
&= r\sum_{\mu\ge 1}\sum_{\nu_r\ge 1}\left(\underset{\sim}{B}_{\mu-1}\cdot\nabla\tilde{F}_{\nu_r}\right)\tilde{S}_{r-1}^{(s+1-\mu-\nu_r)} = \\
&= r\sum_{\mu\ge 1}\sum_{\ell\ge\mu}\left(\underset{\sim}{B}_{\mu-1}\cdot\nabla\tilde{F}_{\ell+1-\mu}\right)\tilde{S}_{r-1}^{(s-\ell)} = \\
&= r\sum_{\ell\ge 1}\tilde{S}_{r-1}^{(s-\ell)}\sum_{1\le\mu\le\ell}\underset{\sim}{B}_{\mu-1}\cdot\nabla\tilde{F}_{\ell+1-\mu} = \\
&= -\,r\sum_{\ell\ge 1}\tilde{S}_{r-1}^{(s-\ell)}\,\underset{\sim}{B}_{\ell}\cdot\nabla\tilde{F}_0\,,
\end{aligned} \tag{5.18}$$

while for other values of r and s it is obvious.*) Equations (5.3) and (5.16) now yield

$$f_\nu = \sum_{i\ge 0}\sum_{j\ge 0}\frac{h_i^{(j)}}{j!}\left(\tilde{S}_j^{(\nu-i)} - \left[\tilde{S}_j^{(\nu-i)}\right]_{\sigma=0}\right). \tag{5.19}$$

*) For $r=1$ eq. (5.17) reduces to eq. (1.5)

Substitution of eqs. (5.9) and (5.19) into eq. (5.6) yields

$$g_\nu = h_\nu + \frac{1}{v_n^*}\int_0^x dx \int_0^{\sigma_0} d\sigma \frac{1}{D} \sum_{\mu \geq 1} \underset{\sim}{B}_\mu \cdot \nabla \sum_{i\geq 0} \sum_{j \geq 0} \frac{h_i^{(j)}}{j!} \tilde{S}_j^{(n+\nu-\mu-i)} -$$
$$- \frac{1}{v_n^*}\int_0^x dx \int_0^{\sigma_0} d\sigma \frac{1}{D} \underset{\sim}{B}_n \cdot \nabla \sum_{i\geq 0} \sum_{j\geq 0} \frac{h_i^{(j)}}{j!} \left[\tilde{S}_j^{(\nu-i)}\right]_{\sigma=0} . \qquad (5.21)$$

Here the second term vanishes according to eqs(5.15-17) because $\tilde{S}_r^{i(s)}$ is periodic in σ. Using eqs. (4.15) and (5.5) in the last term, we find

$$g_\nu = h_\nu + \int_0^x dx \frac{\partial}{\partial x} \sum_{i\geq 0} \sum_{j \geq 0} \frac{h_i^{(j)}}{j!} \left[\tilde{S}_j^{i(\nu-i)}\right]_{\sigma=0} . \qquad (5.22)$$

Observing that $\tilde{F}_\mu$, and hence also $\tilde{S}_r^s$, is zero for $x = \sigma = 0$ and that the x-derivative of the ($i = \nu, j = 0$)-term is zero, we finally obtain

$$g_\nu = \sum_{i \geq 0} \sum_{j\geq 0} \frac{h_i^{(j)}}{j!} \left[\tilde{S}_j^{(\nu-i)}\right]_{\sigma=0} . \qquad (5.23)$$

Equations (5.19) and (5.23) now yield eq. (5.9) for $k=\nu$, q.e.d.

A problem remains if the zero-order magnetic surfaces $\psi = const$ are toroidally closed. In this case x is an angle-like variable, and hence the components of the fields $\underset{\sim}{B}_\nu$ are periodic in x with some period $x_0(\psi)$. We have to show that the functions F_ν are then also periodic in x because otherwise they would not be single-valued in space.

We again proceed by induction. The function F_0 is trivially periodic in χ because it depends only on Ψ. Assuming that F_μ is periodic for $\mu < \nu$, we immediately conclude from eqs. (5.3) and (5.6) that f_μ is periodic for $\mu \leq \nu + n - 1$ while g_ν is periodic if, and only if,

$$\int_0^{\chi_0} d\chi \int_0^{\sigma_0} d\sigma \frac{1}{D} \left(\underset{\sim}{B}_n \cdot \nabla f_\nu + \sum_{\mu \geq n+1} \underset{\sim}{B}_\mu \cdot \nabla F_{n+\nu-\mu} \right) = 0. \qquad (5.24)$$

We rewrite this as

$$\int_0^{\chi_0} d\chi \int_0^{\sigma_0} d\sigma \frac{1}{D} \underset{\sim}{B}_{n+\nu} \cdot \nabla F_0 + \int_0^{\chi_0} d\chi \int_0^{\sigma_0} d\sigma \frac{1}{D} \nabla \cdot \underset{\sim}{A} = 0, \qquad (5.25)$$

where

$$\underset{\sim}{A} = f_\nu \underset{\sim}{B}_n + \sum_{\mu=n+1}^{n+\nu-1} F_{n+\nu-\mu} \underset{\sim}{B}_\mu , \qquad (5.26)$$

and show that the two terms vanish separately. As to the first term, we observe that the elements of the surfaces $\Psi = const$ are

$$\underset{\sim}{df} = D^{-1} d\chi \, d\sigma \, \nabla \Psi . \qquad (5.27)$$

Hence

$$\int_0^{\chi_0} d\chi \int_0^{\sigma_0} d\sigma \frac{1}{D} \underset{\sim}{B}_{n+\nu} \cdot \nabla F_0 = \frac{dF_0}{d\Psi} \oiint \underset{\sim}{df} \cdot \underset{\sim}{B}_{n+\nu} . \qquad (5.28)$$

Since this integral is over a closed surface, it vanishes in accordance with the Gauss theorem. In the second term of eq. (5.25) we write the integrand in terms of the coordinates ψ, χ, and σ by means of

$$\frac{1}{D}\nabla\cdot\underset{\sim}{A} = \frac{\partial}{\partial\psi}\frac{1}{D}\underset{\sim}{A}\cdot\nabla\psi + \frac{\partial}{\partial\chi}\frac{1}{D}\underset{\sim}{A}\cdot\nabla\chi + \frac{\partial}{\partial\sigma}\frac{1}{D}\underset{\sim}{A}\cdot\nabla\sigma . \quad (5.29)$$

Since all quantities are periodic in σ and, by assumption, in χ as well, we thus obtain

$$\int_0^{\chi_0} d\chi \int_0^{\sigma_0} d\sigma \frac{1}{D}\nabla\cdot A = \frac{d}{d\psi}\int_0^{\chi_0} d\chi \int_0^{\sigma_0} d\sigma \frac{1}{D}\underset{\sim}{A}\cdot\nabla\psi = \frac{d}{d\psi}\oint\!\!\oint d\underset{\sim}{f}\cdot\underset{\sim}{A}, \quad (5.30)$$

where eq. (5.27) has again been used. In order to show that the r.h.s. of eq. (5.30) is zero, we need a modified version of the identity (5.17). We define

$$P_\ell = \begin{cases} F_\ell & \text{for } 1 \le \ell \le \nu - 1 \\ f_\ell & \text{for } \nu \le \ell \le \nu + n - 1 \\ 0 & \text{otherwise.} \end{cases} \quad (5.31)$$

The quantities P_ℓ are periodic in σ and in χ, and satisfy

$$\sum_{\mu\ge 0} \underset{\sim}{B}_\mu\cdot\nabla P_{\ell-\mu} = -\underset{\sim}{B}_\ell\cdot\nabla F_0 \quad (5.32)$$

for $0 \le \ell \le \nu + n - 1$. If we further define some quantities $S_r^{(s)}$ by means of eq. (5.11), where $\tilde{F}_{\nu_i}$ is replaced by P_{ν_i}, then the identity

$$\sum_{\mu\ge 1}\left(\underset{\sim}{B}_{\mu-1}\cdot\nabla S_r^{(s+1-\mu)} + r\, S_{r-1}^{(s-\mu)}\underset{\sim}{B}_\mu\cdot\nabla F_0\right) = 0 \quad (5.33)$$

is verified in the same way as the identity (5.17) to hold for $s \leq r+\nu+n-2$. It does not hold for larger values of s because it is based on eq. (5.32), which is valid only for $\ell \leq \nu+n-1$. The reason is that the quantities P_ℓ appear up to $\ell = s-r+1$ in $S_r^{(s)}$. Hence $P_{\nu+n}$ would appear in eq. (5.33) if $s > r+\nu+n-2$. Since the identity (5.33) involves only periodic functions, we obtain, when integrating with respect to σ and χ,

$$\frac{dR_r^{(s)}}{d\Psi} + r\,\frac{dF_0}{d\Psi}\,R_{r-1}^{(s)} = 0 \qquad (s \leq r+\nu+n-2), \qquad (5.34)$$

where

$$R_r^{(s)} = \sum_{\mu \geq 1} \oint\!\!\oint d\underset{\sim}{f} \cdot \underset{\sim}{B}_\mu \, S_r^{(s-\mu)}. \qquad (5.35)$$

Iteration of eq. (5.34) yields

$$R_1^{(\nu+n)} = \frac{1}{m!}\left[-\left(\frac{dF_0}{d\Psi}\right)^{-1}\frac{d}{d\Psi}\right]^{m-1} R_m^{(\nu+n)} \qquad (5.36)$$

for any $m \geq 1$. Since $R_m^{(\nu+n)}$ is zero for $m \geq \nu+n$, $R_1^{(\nu+n)}$ is also zero. The proof of periodicity of F_ν is now completed by observing that the expression (5.30) equals $dR_1^{(\nu+n)}/d\Psi$.

We have thus proved that for analytical magnetic fields asymptotic magnetic surfaces exist uniquely to all orders. These surfaces can be constructed by the following steps:

1) Calculate the function ϕ according to eqs. (2.29) with the help of any preliminary flux functions of the fundamental field.

2) Construct flux functions of the whole field such that ϕ is a function of ψ only.

3) Introduce the coordinates ψ', χ', and σ' according to eqs. (4.3) and compute the functions u'_ν and v'_ν from eqs. (4.4) and then the fields B'_ν from eqs. (4.6).

4) Calculate the functions $F'_\nu(\psi', \chi', \sigma')$ from eqs. (5.2-3) and (5.6).

5) Determine the functions $F_\nu(\psi, \chi, \sigma)$ by substituting eqs. (4.3) and expanding in powers of ε. The N -th order magnetic surfaces are then given by eq. (1.6).

6. The angle of the rotational transform

In this section we consider toroidally closed surfaces. As in Sec. 5, let χ be an angle-like variable, F_0 depending on ψ only. If the symbol * denotes, as above, the increase in σ of a function, and ** the increase in χ, then, for exact surfaces $F = const$ the angle ι of the rotational transform is

$$\iota = 2\pi\, G^* / G^{**}, \qquad (6.1)$$

where F and G are flux functions, i.e.

$$\underset{\sim}{B} = \nabla F \times \nabla G \tag{6.2}$$

Turning now to the case of asymptotic surfaces, we take eq. (6.1) as a definition as well. Since the displacements are $O(\varepsilon^n)$, the thus defined rotational transform angle will come out as a series of the form

$$\iota = \sum_{\nu \geq n} \varepsilon^\nu \iota_\nu(F). \tag{6.3}$$

This series is conveniently obtained via the coordinates ψ', χ', and σ'. Again dropping the primes, we choose

$$F_0 = \psi \ , \quad G_0 = \chi . \tag{6.4}$$

The higher orders of G are determined by

$$\left.\begin{aligned}
\underset{\sim}{B}_\nu \cdot \nabla\psi &= D\left[-\frac{\partial F_\nu}{\partial\sigma} + \sum_{\mu=1}^{\nu-1} \frac{\partial(F_{\nu-\mu}, G_\mu)}{\partial(\chi,\sigma)}\right] \\
\underset{\sim}{B}_\nu \cdot \nabla\chi &= D\left[-\frac{\partial G_\nu}{\partial\sigma} + \sum_{\mu=1}^{\nu-1} \frac{\partial(F_{\nu-\mu}, G_\mu)}{\partial(\sigma,\psi)}\right] \\
\underset{\sim}{B}_\nu \cdot \nabla\sigma &= D\left[\frac{\partial F_\nu}{\partial\psi} + \frac{\partial G_\nu}{\partial\chi} + \sum_{\mu=1}^{\nu-1} \frac{\partial(F_{\nu-\mu}, G_\mu)}{\partial(\psi,\chi)}\right] \\
&(\nu = 1, 2, 3, \dots).
\end{aligned}\right\} \tag{6.5}$$

These equations are analogs of eqs. (2.9) or (4.6) for the flux functions u_ν and v_ν. Since the fields $\underset{\sim}{B}_\nu$ as well as the σ-derivatives of the functions F_ν are zero for $1 \leq \nu \leq n-1$, the second of eqs. (6.5) implies that the σ-derivatives of the functions G_ν are also zero for $\nu \leq n-1$. For $\nu = n$ it implies

$$\frac{\partial G_n}{\partial \sigma} = \frac{\partial v_n}{\partial \sigma}, \tag{6.6}$$

where eq. (4.6) has been used. Hence

$$G^* = \varepsilon^n v_n^* + O(\varepsilon^{n+1}), \tag{6.7}$$

$$G^{**} = \chi_o + O(\varepsilon), \tag{6.8}$$

and

$$\iota/2\pi = \varepsilon^n v_n^* / \chi_o + O(\varepsilon^{n+1}). \tag{6.9}$$

This result could already have been anticipated from eqs. (2.16) for the lowest order displacements. The present method enables us, however, to construct the higher orders of ι in a systematic way.

It turns out that the increases of the functions G_ν can be obtained direct from eqs. (6.5), without solving them explicitly. We restrict ourselves to illustrating this by calculating the second non-vanishing order of ι for the special case $n = 1$, $\underset{\sim}{B} = \underset{\sim}{B}_o + \varepsilon \underset{\sim}{B}_1$. Here a convenient choice is

$$F = \Psi + \varepsilon\left(\mu_1 - \omega/v_1^*\right) + O(\varepsilon^2) \quad , \tag{6.10}$$

where

$$\omega(\Psi,\chi) = \int_0^{\sigma_0} d\sigma\, \mu_1 \frac{\partial v_1}{\partial \sigma} \quad . \tag{6.11}$$

Using this in eqs. (6.5) and then employing eqs. (2.9), we compute

$$G_1^{**} = \int_{\chi}^{\chi+\chi_0} d\chi \frac{\partial G_1}{\partial \chi} = \int_0^{\chi_0} d\chi \left(\frac{1}{D} \underset{\sim}{B}_1 \cdot \nabla\sigma - \frac{\partial F}{\partial \Psi}\right) =$$

$$= \int_0^{\chi_0} d\chi \left(\frac{\partial v_1}{\partial \chi} + \frac{\partial}{\partial \Psi} \frac{\omega}{v_1^*}\right) = \frac{d}{d\Psi} \frac{1}{v_1^*} \int_0^{\chi_0} d\chi\, \omega + v_1^{**}\Big|_{\chi=0} . \tag{6.12}$$

Note that v_1^{**} does not depend on σ . Similarly

$$G_2^* = \int_{\sigma}^{\sigma+\sigma_0} d\sigma \frac{\partial G_2}{\partial \sigma} = \int_{\sigma}^{\sigma+\sigma_0} d\sigma \frac{\partial(F_1, G_1)}{\partial(\sigma,\Psi)} =$$

$$= \int_{\sigma}^{\sigma+\sigma_0} d\sigma \left(\frac{\partial}{\partial\sigma} F_1 \frac{\partial G_1}{\partial \Psi} - \frac{\partial}{\partial \Psi} F_1 \frac{\partial G_1}{\partial \sigma}\right) = \tag{6.13}$$

$$= F_1 \frac{\partial G_1^*}{\partial \Psi} - \frac{\partial}{\partial \Psi} \int_0^{\sigma_0} d\sigma \left(\mu_1 - \frac{\omega}{v_1^*}\right) \frac{\partial v_1}{\partial \sigma} =$$

$$= \left(\mu_1 - \frac{\omega}{v_1^*}\right) \frac{\partial v_1^*}{\partial \Psi} \quad .$$

In terms of the coordinates Ψ, χ , and σ the rotational transform angle is thus

$$\frac{\iota}{2\pi} = \frac{\varepsilon G_1^* + \varepsilon^2 G_2^*}{G_0^{**} + \varepsilon G_1^{**}} + O(\varepsilon^3) = \varepsilon \frac{G_1^*}{G_0^{**}} + \varepsilon^2\left(\frac{G_2^*}{G_0^{**}} - \frac{G_1^* G_1^{**}}{G_0^{**2}}\right) + O(\varepsilon^3) \tag{6.14}$$

$$= \varepsilon \frac{v_1^*}{\chi_0} + \varepsilon^2\left[\frac{1}{\chi_0} \frac{\partial v_1^*}{\partial \Psi}\left(\mu_1 - \frac{\omega}{v_1^*}\right) - \frac{1}{\chi_0^2} v_1^* \left(\frac{d}{d\Psi} \frac{1}{v_1^*} \int_0^{\chi_0} d\chi\, \omega + v_1^{**}\Big|_{\chi=0}\right)\right] .$$

Substituting

$$\psi = F - \varepsilon \left[\mu_1(F,\chi,\sigma) - \frac{\omega(F,\chi)}{v_1^*(F)} \right] + O(\varepsilon^2), \tag{6.15}$$

we finally obtain

$$\frac{\iota}{2\pi} = \varepsilon \frac{v_1^*}{\chi_0} \left[1 - \varepsilon \frac{1}{\chi_0} \frac{d}{d\psi} \frac{1}{v_1^*} \int_0^{\chi_0} d\chi\, \omega + v_1^{**} \Big|_{\chi=0} \right]_{\psi=F} + O(\varepsilon^3). \tag{6.16}$$

It should be mentioned that this result corresponds only to the special choice (6.10) for F . The F -dependence of ι corresponding to any other choice can easily be obtained from eq. (6.16) because ι is a scalar, when F is transformed according to eq. (5.8).

7. Stellarator-like configurations

In many experimental devices one produces an axisymmetric toroidal vacuum field and then tries to superimpose on it another small vacuum field such that the whole field forms toroidally closed asymptotic surfaces, in the interior of which it has no singularities. We show that this is not possible, if the rotational transform is of the order of the perturbing field.

Since for regular fields the function ϕ , when written as a function of any two regular coordinates, is also regular, the asymptotic magnetic surfaces are closed for sufficiently small ε if, and only if, this function has a local extremum. Let $\underset{\sim}{B} - \underset{\sim}{B}_0 = O(\varepsilon^n)$, and $s = const, z = const$ be the lines of force of $\underset{\sim}{B}_0$, and let $\varphi(s, \vartheta, z)$ be a potential of $\underset{\sim}{B}_n$, where s, ϑ, and z are cylindrical coordinates. The function ϕ , being the flux of $\underset{\sim}{B}_n$ through any surface limited by the field line (s, z) and a reference field line (s_0, z_0), is then

$$\phi(s,z) = \int_{s_0}^{s} ds\, s \frac{\partial}{\partial z} \bar{\varphi}(s, z_0) - s \int_{z_0}^{z} dz \frac{\partial}{\partial s} \bar{\varphi}(s,z) , \tag{7.1}$$

where

$$\bar{\varphi} = \int_0^{2\pi} \varphi\, d\vartheta . \tag{7.2}$$

Using

$$\frac{1}{s} \frac{\partial}{\partial s} s \frac{\partial}{\partial s} \bar{\varphi} + \frac{\partial^2}{\partial z^2} \bar{\varphi} = 0, \tag{7.3}$$

which follows from $\Delta \varphi = 0$ by integration with respect to ϑ, we find

$$\frac{\partial \phi}{\partial z} = - s \frac{\partial \bar{\varphi}}{\partial s} , \quad \frac{\partial \phi}{\partial s} = s \frac{\partial \bar{\varphi}}{\partial z} . \tag{7.4}$$

Hence

$$\frac{\partial^2 \phi}{\partial s^2} - \frac{1}{s}\,\frac{\partial \phi}{\partial s} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (7.5)$$

and the function ϕ cannot have a local extremum according to the maximum principle for elliptic differential equations.

The asymptotic magnetic surfaces of such vacuum fields can thus only be toroidally closed if either the lowest order perturbing field has singularities in the interior or the rotational transform is of higher order. Examples of the first possibility are multipoles, where conductors represent singularities, while the second possibility is realized by those stellarators in which the flux of the perturbing field is zero owing to the symmetry of the helical windings. Examples of configurations in which none of these two conditions is met are torsatrons. Hence their asymptotic surfaces either enclose conductors or run to infinity in the limit of vanishing helical currents.

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