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Dispersion Relation of an Inhomogeneous Vlasov Plasma

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ABSTRACT

An integral equation for the electric potential in Laplace-Fourier space is derived for the class of distribution functions:

$$F_{oj} = g_j(v_\perp^2; v_\parallel^2)\exp\left\{-a\left|\int_0^x \frac{B_z(x')}{B_o}dx' + \frac{m_j v_y}{e_j B_o}\right|\right\}; j = e, i$$

(which lead to density profiles everywhere infinitely differentiable). The problem of the stability of the solutions reduces to determining the singularities, in the ω-plane, of the solutions. A necessary and sufficient condition for the existence of singularities not independent of K_x is obtained. The problem is greatly simplified in the case of quasi-neutrality.

I. Introduction

The linear stability of a Vlasov plasma whose equilibrium distribution function depends on space coordinates has been treated hitherto by approximate methods: local approximation (Galeev, Oraevskii and Sagdeev, 1963), W.K.B. method (Krall and Rosenbluth 1965; Davidson and Kammash 1968). Such methods are applicable only when some severe limitations are fulfilled. When considering only wavelengths K_x^{-1} much larger that the mean Larmor radius of the particles, but still small with respect to the inhomogeneity length ($R_{i,e} \ll K_x^{-1} \ll \left(\frac{1}{n}\frac{dn}{dx}\right)^{-1}$), the integro-differential equation for the electric potential degenerates into a differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + D(\omega, x; K_y, K_z)\,\varphi = 0$$

and the W.K.B. method is applicable. If, moreover, the perturbation grows sufficiently in time before it propagates over an inhomogeneity length ($\mathrm{Im}(\omega) \gg \frac{1}{n}\frac{dn}{dx}\frac{\partial \mathrm{Re}(\omega)}{\partial K_x}$), it is not necessary to treat the boundary value problem, and we can apply the so-called "local approximation" by solving the equation

$$D\varphi = 0$$

Here we shall consider an initial value problem and use the well known property that a necessary and sufficient condition for a function f ($\underline{r}$,t) (or its Fourier transform f ($\underline{k}$,t)) to grow indefinitely in time is that its Laplace transform

$$\hat{f}(\underline{r},\omega) = \int_0^{+\infty} dt\, e^{i\omega t} f(\underline{r},t)$$

have a singularity (not necessarily a pole) in the half-plane $\mathrm{Im}(\omega) \geqslant 0$.

We show in Sect.2 that $\varphi(\omega, \underline{k})$ always has singularities for finite ω. They may or may not depend on the component of $\underline{k}$ in the direction where the equilibrium plasma is inhomogeneous (in our case k_x), the difference between the two kinds of singularities being that the former correspond to instabilities which may be absolute or convective in the x-direction, while the latter correspond only to instabilities absolute in the x-direction. In a warm homogeneous plasma there are no singularities which do not depend on $\underline{k}$.

In Sect.2 we deduce an integral equation for φ, where k_x is the independent variable, and k_y, k_z and ω are parameters; the eigenvalues of this equation (if they exist) are singularities independent of k_x.

There is an obvious connection between eigenvalues and the so-called "eigenmodes".
An eigenmode is a solution of the integral equation, of the form $\tilde{\varphi}(x,t;k_y,k_z)=e^{i\omega_0(k_y,k_z)t}\psi(x,k_y,k_z)$. Of course, this is possible only for particular, probably singular, initial conditions. The corresponding Laplace and Fourier transform has the form

$$\tilde{\varphi}(\omega,\underline{k}) = \frac{\psi(\underline{k})}{\omega - \omega_0(k_y,k_z)}$$

and $\omega_0(k_y,k_z)$ is therefore an eigenvalue of the integral equation for φ. If one neglects the possibly existing singularities in the ω-plane which depend on k_x, $\tilde{\varphi}$ is the asymptotic form (in time) of the exact solution φ for an arbitrary, sufficiently well behaved initial condition.

If we write the integral equation in the usual form

$$\varphi(K_x) = \varphi^\circ(\lambda, K_x) + \lambda \int \underline{\mathbb{K}}\,(K_x, K'_x; \lambda)\, \varphi(K'_x)\, dK'_x$$

where λ represents ω (or some function of ω), the corresponding (non-symmetric) kernel is a meromorphic function of λ : the solutions might have either "dispersive" singularities $\omega(K_x)$ or singularities ω independent of K_x (not necessarily poles) or both. Existing theories do not apply to this type of equation (Goodwin 1966).

We shall not try to determine the eigenvalues or give conditions for their existence. Our aim is to find what conclusions can be drawn from the asumption that singularities which do depend on k_x exist.

On this assumption we derive a necessary and sufficient condition in order that $\varphi(\omega, \underline{K})$ be singular at $\omega = \omega_0(K_x)$ or correspondingly at $K_x = K_0(\omega ..)$, in the form

(1) $$\mathbb{D}(\omega, \underline{K}) = 0$$

where D is a functional of the equilibrium distribution function and of the particle trajectories (Sect.3).

To obtain this result one must introduce in the integral equation (as we shall see in Sect.2) a function of the form $A_0(\omega)\, \delta(K_x - K_0(\omega))$. Difficulty in determining $A_0(\omega)$ (see Sect.4) makes it impossible to draw a definite conclusion on the existence of such singularities, also when $D(\omega, \underline{K}) = 0$.

This difficulty does not appear in the approximation of "quasi-neutrality" (see Sect.5). In that case, the dispersion relation has a particularly simple form, and because it is no longer necessary to introduce the function $A_0(\omega)\,\delta(K_x - K_0(\omega))$ it is legitimate to assert that in the approximation of quasi-neutrality

$$D(\omega, \underline{K}) = 0$$

is the necessary and sufficient condition for the existence of singularities in the ω-plane which depend on k_x.

II. Integral Equation

Let us consider an infinite equilibrium plasma described by the following equilibrium distribution function:

$$F_{oj} = g_j(v_\perp^2, v_{//}^2)\, \exp\left\{-a \left| \int_0^x \frac{B_z(x')}{B_o} dx' + \frac{m_j v_y}{e_j B_o} \right| \right\} ; \; j = e, i$$

$$B_z(x) = B_o - b(x) \; ; \; b(x) \geq 0 \; ; \; \int_{-\infty}^{+\infty} dx \; b(x) < \infty$$

$$\lim_{v_\perp \to \infty} g_j \, e^{\gamma v_\perp} = 0 \quad \text{for all } \gamma > 0 .$$

which leads to density profiles everywhere infinitely differentiable. We shall study the stability of F_o with respect to electrostatic perturbations. When considering two particle-species, the chosen equilibrium distribution functions lead to non neutral equilibrium: we will neglect the corresponding electric field in what follows.
The perturbation f can be obtained from the linearised Vlasov equation by integrating along the unperturbed trajectories of the particles (characteristics):

(2) $$f_j(\underline{r}, \underline{v}, t) = f_j(\underline{R}, \underline{V}, t')_{t'=0} + \frac{e_j}{m_j} \int_0^t dt' \; \underline{\nabla \varphi(\underline{R}, t')} \cdot \underline{\nabla_v F_{oj}(X, \underline{V})}$$

where

$$\underline{R} = (X, Y, Z) = \underline{R}(\underline{r}, \underline{v}, t'-t) = \underline{R}(\underline{r}, \underline{v}, u)$$

$$u = t' - t$$

$$\underline{V} = (V_x, V_y, V_z) = \underline{V}(\underline{r}, \underline{v}, t'-t) = \underline{V}(\underline{r}, \underline{v}, u)$$

and

$$\underline{R}_{t'=t} = \underline{r}$$
$$\underline{V}_{t'=t} = \underline{v}$$

Then the Poisson equation can be written:

$$(3) \qquad -\Delta\varphi(\underline{r},t) = \sum_{e,i} 4\pi e_j \iiint d\underline{v} \left[f_j\Big|_{t'=0} + \frac{e_j}{m_j}\int_0^t dt' \; \underline{\nabla\varphi(\underline{R},t')}\cdot\underline{\nabla_v F_{oj}(X,\underline{V})} \right]$$

The only condition we impose on the initial perturbation is that it be limited everywhere and its corresponding charge, and hence its total energy as well be finite:

$$(4) \qquad \iiint d\underline{r} \iiint d\underline{v} \; f_j\Big|_{t'=t=0} < \infty \quad ; \; f_j\Big|_{t'=t=0} \equiv f_j^{\circ}$$

It is easy to see that if this condition is fulfilled for $t = 0$, it will be verified for all t. This means that the Laplace transform of f_j° is L^2 in x, and applying the Fourier transform to it is justified.

We shall now proceed ad absurdum to show that the Laplace transforms of f and φ exist. The property that they are L^2 in x under condition (4) will then be obvious. We shall show first of all that

$$\lim_{t\to\infty} \frac{\underline{\nabla}\cdot\underline{E}}{\underline{E}} = 0 \quad ; \; \underline{E} = -\underline{\nabla}\varphi$$

is not possible in a region α. This will be proved for $\underline{E}$ vectors which only have the x-component; generalisation is easy. From the Vlasov equation it follows that

$$E = \frac{Df}{Dt}\left(\frac{\partial F_0}{\partial v_x}\right)^{-1}$$

If $\lim\limits_{t\to\infty} \frac{\underline{\nabla}\cdot\underline{E}}{\underline{E}} = 0$ in a region α, it must hold asymptotically in t that

(5) $$E = \frac{Df}{Dt}\left(\frac{\partial F_0}{\partial v_x}\right)^{-1} = H(t) + \mathcal{E}(x,t)$$

where $\lim\limits_{t\to\infty} \frac{\underline{\nabla}\cdot\underline{E}}{\underline{E}} = 0$ for $x \in \alpha$.

From (5) it follows that, for $t \to \infty$, E does not depend on x in α. On the other hand, (5) is an equation for f; together with Poisson's equation it gives an E which depends on x in α because Fo depends on x: which is a contradiction. Then $\lim\limits_{t\to\infty} \frac{\underline{\nabla}\cdot\underline{E}}{\underline{E}} \neq 0$ almost everywhere.

Now from Poisson's equation it follows that $\lim\limits_{t\to\infty} \frac{f}{\underline{\nabla}\cdot\underline{E}} \neq 0$

almost everywhere, at least in a finite region β of v. Then:

$$\lim_{t\to\infty} \frac{f\,/|E|}{\underline{\nabla}\cdot E\,/|E|} \neq 0$$

and therefore:

(6) $$\lim_{t\to\infty} \frac{f}{|E|} \neq 0$$

From the Vlasov equation it follows that

$$\left|\frac{Df}{Dt}\right| \leqslant |E|\,|\nabla_v F_0|$$

that is:

(7) $$\lim_{t\to\infty}\left|\frac{Df}{Dt}\right| \leqslant |f|\,|\nabla_v F_0|\,h$$

where $h(t\to\infty)$ is limited.

If we now assume that $\lim_{t\to\infty} E\,e^{-\gamma t} \to \infty$ for all finite $\gamma > 0$, in a region α of x, it follows from (6) that there would be a region β of v where

$$\lim_{t\to\infty} f\,e^{-\gamma t} \to \infty$$

for all finite $\gamma > 0$; but this property contradicts (7). The use of the Laplace transform is therefore legitimate. In an analogous way one deduces that the electric field cannot decay faster than $e^{-\delta t}$ ($0 < \delta$ finite) everywhere in x , hence $E(\omega, \underline{k})$ cannot be an entire function of for real $\underline{k}$.

We take now the Laplace-Fourier transform of (3):

(8) $$\varphi(\omega,\underline{k}) = G(\omega,\underline{k}) + \int_{-\infty}^{+\infty} dk'_x\,\mathbb{K}(\underline{k},\underline{k}';\omega)\,\varphi(\omega,\underline{k}')$$

where $\underline{K} = (K_x, K_y, K_z)$, $\underline{K}' = (K'_x, K_y, K_z)$

$$G(\omega,\underline{k}) = \sum_{e,i} 4\pi e_j \int_0^{+\infty} dt\, e^{i\omega t} \iiint d\underline{r}\, e^{i\underline{K}\cdot\underline{r}} \iiint d\underline{v}\, \frac{f_j(\underline{R},\underline{V},t')_{t'=0}}{K^2}$$

$$\mathbb{K}(\underline{K},\underline{K}';\omega) = \frac{1}{2\pi}\left[\frac{c^+(\underline{K}',\underline{K};\omega)}{K'_x - K_x + i a} - \frac{c^-(\underline{K}',\underline{K};\omega)}{K'_x - K_x - i a}\right]$$

$$c^{\pm}(\underline{K}',\underline{K};\omega) = \sum_{e,i} \frac{4\pi e_j^2}{m_j} \iiint d\underline{v} \int_0^{+\infty} du\, e^{i\omega u}\, \frac{e^{i(K'_x-K_x)\frac{v_y}{\Omega}}\, e^{i\underline{K}'\cdot(\underline{r}-\underline{R})}}{K^2} \left[\pm\frac{K_y a}{\Omega} + \underline{K}'\cdot\underline{\nabla}_{\underline{v}}\right] g$$

The existence of the Fourier transform can be shown directly by proving that the kernel of (8) is L^2 in K'_x and K_x , for all ω in the half-plane $Im(\omega) \geqslant \gamma$. This property allows us to use Fredholm's theorem; (8) has therefore one and only one quadratically integrable solution φ.

From now we shall consider K_y and K_z as parameters and write as arguments of the functions only ω and K_x (written shortly k). As long as ω lies in its convergence half-plane ($\omega_I \geqslant \gamma$), the products $c \mp$ are integrable and therefore the first two integrals in (8) are piecewise regular functions of k_x in the half-planes $Im(k_x) < a$ and $Im(k_x) > -a$ respectively. Their sum is regular in $-a < Im(k_x) < a$, and the same is therefore true of $\varphi(K)$.

It follows that a solution of (8) cannot have a singularity in the strip $-a < \mathrm{Im}(k_x) < a$ if it is not already singular on $\mathrm{Im}(k_x) \neq 0$.. Because φ is regular for real k, an $\mathrm{Im}(\omega) \geq \gamma$, $\varphi(\omega, K)$ (k real) would have no singularities for any value of ω. Note, however, that eq.(8) is not the most general Fourier-Laplace transform of (3). One can also add any function whose "image" in (x,t) space is identical to zero, i.e. any linear combination of functions of the form $A_s(\omega)\,\delta(K - K_s(\omega))$, where $K_s(\omega)$ are analytic functions with $\mathrm{Im}(k_s(\omega)) \neq 0$ for $\mathrm{Im}(\omega) > \gamma$ and $A_s(\omega)$ are analytic functions.

In other words, before transforming one can add to (3) the functions (identical to zero):

$$\lim_{\epsilon = 0} \left[e^{i(K_s(\omega)+i\epsilon)x} - e^{i(K_s(\omega)-i\epsilon)x} \right] \text{ if } x>0 \text{ and } 0 \text{ if } x<0$$

$$\text{where } \quad K_{sI}(\omega) \geq 0 \quad \text{for } \omega_I \geq \gamma$$

and

$$\lim_{\epsilon = 0} \left[e^{i(K_s(\omega)+i\epsilon)x} - e^{i(K_s(\omega)-i\epsilon)x} \right] \text{ if } x<0 \text{ and } 0 \text{ if } x>0$$

$$\text{when } \quad K_{sI}(\omega) \leq 0 \quad \text{for } \omega_I \geq \gamma$$

The introduction of the δ-functions can be seen as the analytical continuation of (8) in the ω-plane, when ω leaves its convergence half-plane $\mathrm{Im}(\omega) \geqslant \gamma$, where eq.(8) is originally defined.

In the next section we show that the solution of the equation one obtains in this way has singularities depending on k. It follows that a proof that the functions $A_s(\omega)$ and $k_s(\omega)$ can be uniquely determined is at the same time a proof of the existence of singularities depending on k. The functions $k_s(\omega)$ will in fact be uniquely determined in the next section; for the determination of the $A_s(\omega)$'s, only a plausible argument will be given in Sect. 4 .

III. Singularities of the solution

The integral equation for φ therefore has the form:

$$(9)\quad \varphi(\omega,K) = G(\omega,K) + \sum_s A_s(\omega)\,\delta(K-K_s(\omega)) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} dK' \left[\frac{c^+(K',K;\omega)}{K'-K+ia} - \frac{c^-(K',K;\omega)}{K'-K-ia}\right]\varphi(\omega,K')$$

It follows that φ has the form $\varphi = \sum_s A_s\,\delta(K-K_s(\omega)) + \tilde{\varphi}$

Let us now consider that value of φ where one of the k_s, say k_o, becomes real. Then (9) becomes (changing for simplicity A_o in $2\pi i A_o$):

$$(10)\quad 2\pi i A_0(\omega)\,\delta(k-k_0(\omega)) + \tilde{\varphi}(\omega,k) = G(\omega,k) + 2\pi i A_0(\omega)\,\delta(k-k_0(\omega)) +$$

$$+ i A_0(\omega)\left[\frac{c^+(k_0,k;\omega)}{k_0-k+ia} - \frac{c^-(k_0,k;\omega)}{k_0-k-ia}\right] + \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk'\left[\frac{c^+(k',k;\omega)}{k'-k+ia} - \frac{c^-(k',k;\omega)}{k'-k-ia}\right]\tilde{\varphi}(\omega,k')$$

Eq.(10) implies that $\tilde{\varphi}$ has singularities at the points $k_{\pm n} \equiv k_0 \pm ina$ $(n=1,2,\ldots)$ which are poles when the $c^{\pm}$ are entire functions, which is the case when $B_z = B_0$(low β). By analytical continuation of the integrals in (10), one obtains for the residues $\tilde{\varphi}_R$ of $i\tilde{\varphi}$ at theese points:

$$(11)\quad \begin{array}{ll} k = k_n\;,\; n>1 & \tilde{\varphi}_R(k_n) = c^+_{n-1}\,\tilde{\varphi}_R(k_{n-1}) \\ k = k_1 & \tilde{\varphi}_R(k_1) = c^+_0\;A_0 \\ k = k_{-1} & \tilde{\varphi}_R(k_{-1}) = -c^-_0\;A_0 \\ k = k_n\;,\; n<1 & \tilde{\varphi}_R(k_n) = c^-_{n+1}\,\tilde{\varphi}_R(k_{n+1}) \end{array}$$

$$c^+_n \equiv c^+(k'=k-ia=k_n)\;;\; c^-_n \equiv c^-(k'=k+ia=k_n)$$

From now on we shall consider only the case $B_z=B_0$; it is, however, possible to apply an analogous procedure to $B_z(x) = B_0 - b(x)$. When $B_z = B_0$, $\tilde{\varphi}$ is therefore the sum of a meromorphic function, $A_0\tilde{\varphi}_M$, and an entire function

$\widetilde{\varphi}_g$. From the Mittag-Leffler's theorem, (Sansone et al. 1960), it follows that $A_o \widetilde{\varphi}_M$ can be written:

$$A_o \widetilde{\varphi}_M = \sum_{n=1}^{\infty} \left\{ \left(\frac{K}{K_n}\right)^{\ell_n} \frac{\widetilde{A}_n}{K - K_n} - \left(\frac{K}{K_{-n}}\right)^{\ell_{-n}} \frac{\widetilde{A}_{-n}}{K - K_{-n}} \right\}$$

If we further impose that $A_o \widetilde{\varphi}_M$ be quadratic integrable, the exponents $\ell_{\pm n}$ must be zero. Then:

$$A_o \widetilde{\varphi}_M = \sum_{n=1}^{\infty} \left\{ \frac{\widetilde{A}_n}{K - K_n} - \frac{\widetilde{A}_{-n}}{K - K_{-n}} \right\} \tag{12}$$

From (11) it follows:

$$\begin{aligned} \widetilde{A}_{+n} &= A_o \prod_{n'=0}^{n'=n-1} C^{+}_{n'} \\ \widetilde{A}_{-n} &= A_o \prod_{n'=0}^{n'=n-1} C^{-}_{-n'} \end{aligned} \qquad n > 0 \tag{12a}$$

The allowed values for k_o are those which let (12) converge; i.e., the values k_o which let:

$$\lim_{n \to \infty} \left\{ \widetilde{A}_n + \widetilde{A}_{-n+q} \right\} = 0 \tag{13}$$

(q finite). This condition is of course necessary. We show later on that it is also sufficient.

If we define the quantities $(+A_o)$ and $\tilde{A}_n$ (n= ,1,...) as the values of a function $\tilde{A}^+(k)$ at the points $k=k_n=k_o+$ ina(n=0,1,-) and the quantities $(-A_o)$ and $\tilde{A}_{-n}$(n=1,2,..) as the values of a function $\tilde{A}^-(k)$ at the points $k=k_{-n}=k_o-$ ina(n=0,1-), eq. (13) can be written:

$$\lim_{n\to\infty} \left\{ \tilde{A}^+(K_n) + \tilde{A}^-(K_{-n+q}) \right\} = 0 \tag{14}$$

From (12a) it follows that $\tilde{A}^{\pm}(k)$ satisfy the difference equations:

$$\tilde{A}^+(K_o+ia) = \tilde{A}^+(K_o)\, c_o^+ = \tilde{A}^+(K_o)\, c^+(K_o, K_o+ia; \omega) \tag{15}$$

$$\tilde{A}^-(K_o-ia) = \tilde{A}^-(K_o)\, c_o^- = \tilde{A}^-(K_o)\, c^-(K_o, K_o-ia; \omega) \tag{16}$$

The solutions of (15) and (16) are given by Titchmarsh (1959); the solution of, say, (15) is the product of a particular solution $\overset{+}{A}(k)$ and of a periodic function P^+ with period ia.

The convergence condition (14) now has the form:

$$\lim_{n\to\infty} \left\{ A^+(K_n)\, P^+(K_n) + A^-(K_{-n+q})\, P^-(K_{-n+q}) \right\} = 0 \tag{17}$$

From the definition of $\tilde{A}^{+}(k)$ and $\tilde{A}^{-}(k)$ at $k=k_o$ it follows that

(18) $$A^{+}(K_o)\,P^{+}(K_o) + A^{-}(K_o)\,P^{-}(K_o) = 0$$

must be valid.

From this equation and (17) it follows that

(19) $$\lim_{n\to\infty}\left\{\frac{A^{+}(K_n)}{A^{-}(K_{-n+q})} - \frac{A^{+}(K_o)}{A^{-}(K_o)}\right\} = 0$$

This equation has a meaning only if

$$\lim_{n\to\infty}\frac{A^{+}(K_n)}{A^{-}(K_{-n+q})}$$

exists. In that case, we can define:

(20) $$D(\omega,K) \equiv \lim_{n\to\infty}\left\{\frac{A^{+}(K+ina)}{A^{-}(K-ina)} - \frac{A^{+}(K)}{A^{-}(K)}\right\}$$

The equation $D(\omega,k) = 0$ is a necessary condition for the existence of a singularity of $\tilde{\varphi}$ at the points $K_o(\omega) \pm ina$ $(n = 1,2,..)$ where $k_o(\omega)$ is a solution of $D = 0$. It is also sufficient if the following property of the $A^{\pm}$ is satisfied. Let $A^{\pm}$ have the asymtotic form:

$$A^{\pm}(K_{\pm n}) = \alpha^{\pm}(\pm n)\left[1 + \epsilon^{\pm}(\pm n)\right]$$

($\lim\limits_{n\to\infty} \epsilon^{\pm}(\pm n) = 0$). Then the series $\sum\limits_{n=1}^{\infty} \frac{\alpha^{-}(-n)}{n}\left[\epsilon^{+}(n) - \epsilon^{-}(-n)\right]$

must converge (it is for example sufficient that

$$\left|\alpha^{-}(-n)\left(\epsilon^{+}(n) - \epsilon^{-}(-n)\right)\right| < n^{-\delta}, \ \delta > 0 \ )$$

From (Titchmarsh,1959) it follows that $A^{\pm}(k)$ is defined for $\mathrm{Re}(k) = 0$ by:

$$\log A^{\pm}(K) = \int_{i\epsilon-\infty}^{i\epsilon+\infty} d\sigma \, \frac{e^{\pm\frac{\sigma K}{a}}}{e^{-i\sigma}-1} \int_{0}^{+i\infty} \frac{dK'}{ia} \, e^{\frac{\sigma K'}{a}} \log C^{\pm}(\pm K', \pm K' \pm ia; \omega)$$

(20a)

$$+ \int_{-i\epsilon-\infty}^{-i\epsilon+\infty} d\sigma \, \frac{e^{\mp\frac{\sigma K}{a}}}{e^{-i\sigma}-1} \int_{-i\infty}^{0} \frac{dK'}{ia} \, e^{\frac{\sigma K'}{a}} \log C^{\pm}(\pm K', \pm K' \pm ia; \omega)$$

$A^{\pm}(k)$, $\mathrm{Im}(k) = 0$, is defined as the analytic continuation of (20a).

It may be interesting to see how the homogeneous case can be obtained at the limit $a = 0$.

As first condition we must have

$$\lim_{n\to\infty} a \cdot n = 0$$

Then the dispersion relation (20) is an identity. The functions c^+ and c^- become equal. The difference equations (16) are now homogeneous algebraic equations for $A^+(k) \equiv A^-(k)$. The solubility condition is $c^+ = C = c^- = 1$, which is the well-known dispersion relation.

Eq.(9) takes the form:

$$A_0 \delta(k-k_0) + \tilde{\varphi}(k) = G(k) + B\,\delta(k-k_0) + A_0\, C(k)\,\delta(k-k_0) + C(k)\,\tilde{\varphi}(k)$$

It follows that:

$$B = (1-c)\,A_0$$

$$\tilde{\varphi} = \frac{G}{1-c}$$

Therefore there is no condition on k_0 and A_0.

It is thus seen that the δ-functions which were introduced in Sect. 2 are not necessary in the homogeneous case because the Fourier-Laplace transform of $A_0 \times (k-k_0)$ is identical to zero and A_0, B and k_0 do not appear in $\tilde{\varphi}$.

We have shown that if the solution to (10) has singularities in the ω-plane depending on k_x, they are given by (20). This means that if the A_s's are zero, the singularities in the ω-plane do not depend on k_x and the (possible) instabilities are absolute in the x-direction.

IV Entire part of the solution

We have calculated $A_0\widetilde{\varphi}_M$, the singular part of $\widetilde{\varphi}$, by introducing the functions $A_s(\omega)\,\delta(K - K_s(\omega))$. Because the inverse Fourier-Laplace of these functions is zero, only the fact that the A_s's are different from zero or not can prove that the singularities we have found are singularities of the solution to (10) or not.

We now give a plausible argument which determines the $A_s(\omega)$. The conclusions will be that the A_s are different from zero.

We have written $\varphi = A_0\,\delta + A_0\,\widetilde{\varphi}_M + \widetilde{\varphi}_g$.
If this "ansatz" is true, A_0 is a given functional of the kernel and G and could also be identical to zero.
Had we introduced a different value for A_0, let us call it A_0' , eq.(10) would now have the form

$$\varphi_g(\omega,K) = G(\omega,K) + \frac{A_0'}{2\pi}\int_{-\infty}^{+\infty} dK'\left[\frac{C^+(K';K;\omega) - C^+(K-ia;K;\omega)}{K'-K+ia} - \frac{C^-(K';K;\omega) - C^-(K+ia,K;\omega)}{K'-K-ia}\right]\widetilde{\varphi}_M(K')$$
$$+ \int_{-\infty}^{+\infty} dK'\,\mathbb{K}(K',K;\omega)\,\widetilde{\varphi}_g(K') + 2\pi i\,(A_0 - A_0')\,\delta(K-K_0)$$

Its solution has poles at $k = k_0 \pm ina$, with residues proportional to $(A_0 - A_0')$.

Let us now suppose that $G = (K - K_1) f$, where k_1 is, of course, a functional of A_o'. Division by $(k - k_1)$ gives:

$$\Psi(\omega,K) = f(\omega,K) + \frac{A_0'}{2\pi(K-K_1)} \int_{-\infty}^{+\infty} dK' \left[\frac{c^+(K',K;\omega) - c^+(K-ia,K;\omega)}{K'-K+ia} - \frac{c^-(K',K;\omega) - c^-(K+ia,K;\omega)}{K'-K-ia} \right] \tilde{\varphi}_M(K')$$

$$+ B\,\delta(K-K_1) + \int_{-\infty}^{+\infty} dK'\, \mathbb{K}(K',K;\omega) \frac{K'-K_1}{K-K_1} \Psi(K') + \frac{2\pi i (A_0 - A_0')}{K-K_1} \delta(K-K_0)$$

where

$$\Psi(\omega,K) = \frac{\varphi_g(\omega,K)}{K-K_1}$$

and B is a free constant.

If we choose A' in such a way that $k_1 = k_o$, we must have $(A_o - A_o') = 0$ in order that the coefficient of $\delta(k - k_o)$ remain finite at $k_1 = k_o$.

From this argument it follows that if A_o is determined in such a way that:

$$(22) \qquad \left\{ G(\omega,K) + A_0(\omega) \int_{-\infty}^{+\infty} dK' \left[\frac{c^+(K',K;\omega) - c^+(K-ia,K;\omega)}{K'-K+ia} - \frac{c^-(K',K;\omega) - c^-(K+ia,K;\omega)}{K'-K-ia} \right] \tilde{\varphi}_M(K') \right\}_{K=K_0} = 0$$

$\tilde{\varphi}_g$ is an entire function. From the quadratic integrability of the kernel it follows that $\tilde{\varphi}_g$ is L^2 in k. Since we know that there is only one L^2 solution to (3), $A_0\tilde{\varphi}_M + \tilde{\varphi}_g$ is the required solution.

V. Quasi-neutrality

All these difficulties disappear in the approximation of "quasi-neutrality". One then obtains singularities depending on k_x without introducing the δ-functions.

In this approximation the term φ in the integral equation is dropped and (9) becomes:

$$(23)\qquad 0 = G(\omega,K) + 2\pi i \sum_S A_S(\omega)\,\delta(K - K_S(\omega)) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} dK' \left[\frac{c^+(K',K;\omega)}{K'-K+ia} - \frac{c^-(K',K;\omega)}{K'-K-ia}\right]\varphi(\omega,K')$$

By letting one K_S, say K_0, become real, one sees that $A_0 \equiv 0$ must be valid. The analytic continuation of (23) for $\mathrm{Im}(K) > 0$ has the form:

$$-ic^+(K-ia,K;\omega)\,\varphi(\omega,K-ia) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} dK' \frac{c^+(K',K;\omega)}{K'-K+ia}\varphi(\omega,K') - \frac{1}{2\pi}\int_{-\infty}^{+\infty} dK' \frac{c^-(K',K;\omega)}{K'-K-ia}\varphi(\omega,K') + G(\omega,K) = 0$$

or $\quad -ic^+\varphi + \int dK'\, \mathbb{K}'\varphi + G = 0$

A necessary condition for the existence of a singularity with $\mathrm{Im}(K) > a$ is then:

$$(24)\qquad c^+(K-ia, K; \omega) = 0$$

For a singularity moving to the real k axis from below the necessary condition would be:

$$c^{-}(k+ia, k; \omega) = 0 \tag{25}$$

From (24) and (25) it follows that $\varphi(\omega, k)$ only has poles when $c^{\pm}$ are entire functions.

Let us show that (24) and (25) are also sufficient for the existence of singularities.

In general the equation $c^{+}(k-ia, k; \omega) = 0$ will have many solutions $k_s(\omega), s=0,1,..$ with Im $(k_s) > 0$ as Im $(\omega) \to +\infty$ too.

Let us assume that the solution of the integral equation remains finite at $k = k_0(\omega)$, for example. Then:

$$I_0 \equiv \int_{-\infty}^{+\infty} dk' \, \mathbb{K}'(k', k_0(\omega); \omega)\, \varphi(\omega, k') + G(\omega, k_0(\omega)) = 0 \tag{26}$$

Let us now consider:

$$I_s \equiv \int_{-\infty}^{+\infty} dk' \, \mathbb{K}'(k', k_s(\omega); \omega)\, \varphi(\omega, k') + G(\omega, k_s(\omega)) \,, \quad s \neq 0 \tag{27}$$

for the same arbitrary initial condition.

In (26) and (27), the integration is always performed along the real k axis so that the change of that integral when one chooses k_s instead of k_o depends only on the equilibrium through the kernel (since the k_s are the roots of $D=0$, which depends only on the equilibrium).

On the other hand, the change of G when k_o is replaced by k_s depends on the initial value. Therefore, in general, we have:

$$I_s \neq 0$$

for all $s \neq 0$, which means that $\varphi(\omega, k_s(\omega))$ has a pole for all $s \neq 0$. In this way, we have shown that, for one solution $k_o(\omega)$ at most, $\varphi(\omega, k)$ might have no pole.

Note now that the solutions $k_o(\omega)$ and $k_s(\omega)$ (with $s \neq 0$) will intersect at points which are independent of the initial condition. At such a point: $k^* = k_o(\omega^*) = k_s(\omega^*)$ one would get:

$$\varphi(\omega^*, k^*) = \varphi(\omega^*, k_o(\omega^*)) = \varphi(\omega^*, k_s(\omega^*)) \tag{28}$$

Eq.(28) is a contradiction since $\varphi(\omega, k_o(\omega))$ would remain finite and $\varphi(\omega, k_s(\omega))$ would diverge (except for some value ω which depends on the initial condition). We then conclude that conditions (24) and (25) are also sufficient.

The solution $\varphi(\omega, \kappa)$ is unique because the homogeneous equation corresponding to (23) has obviously only the trivial solution $\varphi = 0$.

Conclusions

Because of the particular class of unperturbed distribution functions which were chosen, it was possible to derive an exact dispersion relation for the whole plasma.
In the usual treatments (local approximation, W.K.B.method, ..) one considers in an approximate way only single modes. The difficulties which occur in generalising are of different types:

1) Considering inhomogeneous magnetic fields may add a new set of singularities of φ to the already existing set: When $\lim_{|x| \to \infty} b(x) \sim \exp(-\beta |x|)$, $\beta > 0$.

2) When densities decaying in x faster than exponentially are considered, the singularities $K_0 \pm i n a$ are shifted to infinity.

In the above treatment the question of the validity of linearisation is obviously not considered; the approximation $|f_j| \ll |F_{0j}|$ might break down locally (in x) even for short times, depending on F_{0j} and the initial perturbation. Furthermore, except for very small (or equal) ion and electron gyroradii ($R_i = R_e$) we do not have an exactly neutral equilibrium.

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