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An absolutely MHD-stable axisymmetric closed field line equilibrium +)

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ABSTRACT

The MHD-stability of special axisymmetric toroidal equilibria with a hardcore and closed field lines is demonstrated.

1. Introduction

Although abundant literature on the stability of scalar pressure MHD equilibria exists, only very few absolutely MHD-stable equilibria are known so far. Most of the literature deals with special perturbations and the stability criteria associated with them. In many cases the importance of the derived criteria is not clear. On the one hand other more dangerous perturbations may exist and on the other hand, some criteria may impose too severe restrictions when the wave length of the perturbations considered is very small and hence the application of macroscopic equations becomes doubtful. Many considerations are restricted to low β and may become invalid in high β systems.

Although absolute MHD stability may be too strong a requirement - it includes stability with respect to small wavelength perturbations and ignores stabilizing effects such as finite gyroradii etc. - it would be desirable to have a more complete list of absolutely stable equilibria. These may serve as the starting point for a more refined stability analysis. And, as in the following paper, some of the most dangerous perturbations may impose just or almost as stringent restrictions as the requirement for absolute stability.

The equilibria considered here have been extensively studied in ref. [1] . They are axisymmetric and have closed meridional magnetic field lines produced by toroidal currents. Ref. [1] presents a very useful form of the energy integral in which two components of the perturbations are eliminated by minimisation after Fourier analysis in the ignorable coordinate. Moreover, a necessary stability criterion is derived in general, and necessary and sufficient criteria are derived in two limiting cases (low β, almost circular fieldlines).

[1] Bernstein I.B., Frieman E.A., Kruskal M.D. and Kulsrud R.M. (1958) Proc.R.Soc. A 244, 17

We shall show in this paper, assuming "reasonably" shaped flux surfaces, that these equilibria can only be stable if a hard core field is superposed on the plasma currents. In addition we give some more stability criteria in the general finite β case and show that stable equilibria actually exist, i.e. that the criteria derived can be satisfied.

2. Notation, equilibrium equations and energy variation

In this section we give a brief review of section V in ref.1 as far as is needed for our purposes. For the reader's convenience we use the same notation as used in ref.1.

If a stream function $\psi = \psi(r, z)$ is introduced the (meridional) magnetic field $\underline{B}$ can be written

$$\underline{B} = -\underline{e}_\theta \times \nabla\psi / r \tag{1}$$

in cylindrical coordinates (r, θ, z). ($\underline{e}_\theta$ is the unit vector in the θ-direction). If χ = const are orthogonal trajectories to the flux surfaces ψ = const, the set (ψ, θ, χ) forms a right-handed orthogonal system which will be employed alternatively with the cylindrical coordinates. From $j \times B = \nabla p$ we obtain

$$\frac{\partial}{\partial\psi}(J B^2) = -J p'(\psi) \tag{2}$$

$$J = 1 / B \cdot |\nabla\chi| \tag{3}$$

p is a function of ψ only and is related to the currents $\underline{j} = j \cdot \underline{e}_\theta$ by

$$p'(\psi) = j / r \tag{4}$$

The energy variation of the system

$$\delta W = \int d\tau \left\{ \mathrm{curl}^2(\underline{\xi} \times \underline{B}) + \underline{\xi} \cdot [\mathrm{curl}(\underline{\xi} \times \underline{B})] \times \underline{j} + (\underline{\xi} \cdot \nabla p)\,\mathrm{div}\,\underline{\xi} + \gamma p\, \mathrm{div}^2 \underline{\xi} \right\}$$

can be decomposed into

$$\delta W = \delta W_0 + 2 \sum_{m=1}^{\infty} \delta W_m$$

by making use of the periodicity in Θ:

$$\underline{\xi} = \sum_m \underline{\xi}_m \, e^{im\Theta}$$

δW_m contains only $\underline{\xi}_m$ and is given by

$$\delta W_m = \frac{\pi}{2} \int d\psi\, d\chi\, J \left\{ (B^2 + \gamma p) \left[Y_m + \frac{\partial X_m}{\partial \psi} + \frac{X_m \left(p' + \gamma p \frac{\partial \ln J}{\partial \psi}\right) + \frac{\gamma p}{J} \frac{\partial Z_m}{\partial \chi}}{} \right]^2 \right.$$

$$+ \frac{r^2}{m^2 J} \left(\frac{\partial Y_m}{\partial \chi} \right)^2 + \frac{1}{r^2 B^2 J^2} \left(\frac{\partial X_m}{\partial \chi} \right)^2 + p' X_m^2 \frac{\partial \ln J}{\partial \psi} - \frac{p'^2 X_m^2}{B^2}$$

$$+ \frac{1}{\frac{1}{B^2} + \frac{1}{\gamma p}} \left[X_m \frac{\partial \ln J}{\partial \psi} + \frac{1}{J} \frac{\partial Z_m}{\partial \chi} - \frac{p' X_m}{B^2} \right] \qquad (5)$$

where $\quad X_m = r B \xi_m^{\psi}, \qquad Y_m = \frac{m}{r} \xi_m^{\Theta}, \qquad Z_m = \xi_m^{\chi} / B$

For absolute stability it suffices that δW_∞ be positive. Supressing the subscript ∞ and minimising δW_m in the limiting case $m = \infty$ yields

$$\delta W = \frac{\pi}{2} \int d\psi\, \delta W(\psi)$$

where

$$\delta W(\psi) = \frac{1}{2\pi} \left(L' + \frac{V'}{\gamma p} \right) f^2 + \int d\chi \left[\frac{1}{r^2 B^2 J} \left(\frac{\partial X}{\partial \chi} \right)^2 + p' J D X^2 \right] \qquad (6)$$

and

$$L' = 2\pi \int d\chi \; J / B^2 \qquad (7)$$

$$V' = 2\pi \int d\chi \, J \tag{8}$$

$$D = -\frac{2}{rB^3} \, \underline{e}_\psi \cdot (\underline{B} \cdot \underline{\nabla} \, \underline{B}) = -\frac{2}{rB} (\underline{e}_\psi \cdot \underline{\kappa}) \tag{9}$$

$$f = \frac{2\pi \int d\chi \, J D X}{L' + V'/\gamma p} \tag{10}$$

$\underline{\kappa}$ is the vector of curvature of the $\underline{B}$-lines.

3. Some necessary stability conditions

First we shall show that in practical situations we must place a current carrying hard core inside the plasma in order to obtain stability. For this purpose we assume that there is no hard core but only plasma currents inside the plasma. We show that $\delta W'$ then becomes negative in the neighbourhood of the magnetic axis, provided that the surfaces are "reasonably" shaped there. Setting

$$d\sigma = J D \, d\chi \tag{11}$$

we introduce a new variable σ with the period

$$\Sigma = \int d\chi \, J D \tag{12}$$

If we choose

$$X = \sin(2\pi\sigma / \Sigma') \tag{13}$$

then

$$f \sim \int d\chi \, J D = \int d\sigma \, X = 0$$

and

$$\delta W(\psi) = \int d\sigma \left[\frac{D}{r^2 B^2} \left(\frac{\partial X}{\partial \sigma} \right)^2 + p' X^2 \right] \tag{14}$$

Obviously

$$\delta W(\psi) \leq \left(\frac{D}{r^2 B^2}\right)_{max} \cdot \int \left(\frac{\partial \chi}{\partial \sigma}\right)^2 d\sigma + p' \int \chi^2 d\sigma$$

and with (13) this becomes

$$\delta W \leq \left[\left(\frac{D}{r^2 B^2}\right)_{max} \frac{4\pi^2}{\Sigma^2} + p' \right] \cdot \frac{\Sigma}{2} \qquad (15)$$

We now introduce polar coordinates ρ, φ around the magnetic axis r=R, z=o (Fig.1) and calculate p', rB, D and Σ by expanding with respect to ρ. Assuming analyticity of the flux surfaces (cf. [2]) we have to lowest order in

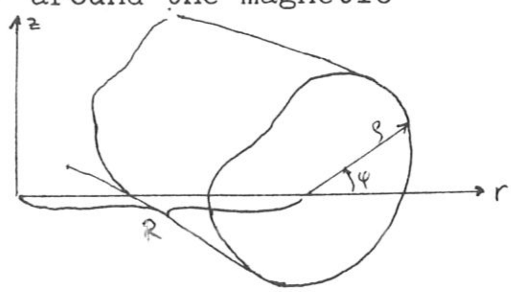

Fig.1

$$\psi = (1 - \varepsilon^2 \cos 2u)\, \rho^2 \qquad (16)$$

where $u = \varphi - \delta$ and $\varepsilon < 1$.

δ is a constant which gives the inclination of the ellipses (16) towards the z-axis. Differentiation of (16) yields

$$\underline{\nabla} \psi = 2\rho \{ 1 - \varepsilon^2 \cos 2u \, , \, \varepsilon^2 \sin 2u \}$$

From this and (1) we get

$$rB = 2\rho \left(1 - 2\varepsilon^2 \cos 2u + \varepsilon^4\right)^{1/2} \qquad (17)$$

Using rot $\underline{B} = \underline{j}$ it follows from (1) and (3) that

$$p' = -4/R^2 \qquad (18)$$

Since the curvature of the ellipses has the opposite direction to $\underline{e}_\psi$, we get

[2] Mercier C. (1963) Nuclear Fusion 3, 89

$$D = \frac{2\kappa}{rB} \qquad (19)$$

from eq.(9). κ is given by

$$\kappa = \frac{(1-\varepsilon^4)(1-\varepsilon^4\cos 2u)}{\varrho\,(1-2\varepsilon^2\cos 2u+\varepsilon^4)^{3/2}} \quad . \qquad (20)$$

Finally, we have to calculate Σ, eq.(12). Because of eq.(3)

$$J\,d\chi = \frac{d\chi}{B\,|\nabla\chi|} = \frac{ds_\chi}{B} \qquad (21)$$

where

$$ds_\chi = \frac{\varrho\,(1-2\varepsilon^2\cos 2u+\varepsilon^4)^{1/2}}{1-\varepsilon^2\cos 2u}\,du \qquad (22)$$

is the element of arc along ψ = const. Therefore, from (17) (19), (20) and (22) we have

$$\Sigma = \frac{(1-\varepsilon^4)R}{2\psi}\int_0^{2\pi}\frac{1-\varepsilon^2\cos 2u}{(1-\varepsilon^2\cos 2u+\varepsilon^4)^2}\,du \quad . \qquad (23)$$

Note that in expanding with respect to ϱ we have used $r = R+\varrho\cos u$. The integral in eq.(23) can be calculated [+)] to give

$$\Sigma = \frac{R\pi}{\psi(1-\varepsilon^4)} \quad .$$

In evaluating $\delta W(\psi)$ we have to find the maximum of D/r^2B^2. From (19), (20) and (17) we have

$$\frac{D}{r^2B^2} = \frac{(1-\varepsilon^4)}{4R^2\psi^2}\left(\frac{1-\varepsilon^2\cos 2u}{1-2\varepsilon^2\cos 2u+\varepsilon^4}\right)^3$$

This expression assumes its maximum at $\cos 2u = 1$, hence

$$\left(\frac{D}{r^2B^2}\right)_{max} = \frac{1+\varepsilon^2}{4R^2\psi^2(1-\varepsilon^2)^2}$$

+) $$\int_0^{2\pi}\frac{\cos x}{(a+\cos x)^2}\,dx = -\left[\frac{d}{dq}\int_0^{2\pi}\frac{dx}{a+q\cos x}\right]_{q=1} = -2\pi\left(a^2-1\right)^{-3}$$

and together with (18) and (23) we finally obtain from (15)

$$\delta W \leq \left[(1+\varepsilon^2)^3 - 4\right] \cdot \frac{\Sigma}{2R^2}$$

It follows that $\delta W < 0$ if $\varepsilon \lesssim 0.77$. By a more refined but somewhat lengthy calculation it could be shown that $\delta W < 0$ at least up to $\varepsilon = 0.9$. An exact minimisation of δW in the neighbourhood of the magnetic axis is sure to give a negative δW for still higher values of ε.

We conclude that the plasma is unstable in the neighbourhood of the magnetic axis for all "reasonably" shaped ψ surfaces (not too large ε and hence not too slender ellipses). As an expedient we place a current carrying hard core at $\rho = 0$. Since the plasma should be separated from this we assume that the hard core is surrounded by a vacuum region and that the plasma starts at some surface ψ_0 with p=0. It can easily be seen from the equilibrium equations that the plasma currents must be opposite to the hard core currents where the pressure increases with ρ. This means that the hard core field will be weakened by the plasma currents and may reverse its direction.

We shall show next that the magnetic field must reverse wherever it becomes zero and that the plasma becomes unstable wherever the field reverses. Equation (2) can be integrated over χ to give

$$\frac{\partial}{\partial\psi} \oint B\, ds_\chi = - p' q$$

where the notation $q = \oint J\, d\chi = \oint \frac{ds_\chi}{B}$ has been introduced. Note that q is a positive quantity and that B in $\oint B\, ds_\chi$ is the absolute value. After integration over ψ we obtain

$$\oint_\psi B\, ds_\chi = \int_{\psi_0} B\, ds_\chi - \int q\, dp$$

Since the left-hand side is positive and hence increases towards both sides from a $B=0$ line and since q is positive, dp must change sign when a B=0 line is crossed. If we exclude the singular case in which j becomes zero together with B , we can conclude from $\nabla p = \underline{j} \times \underline{B}$ that $\underline{B}$ must reverse.

In order to prove instability, we consider a special perturbation X_I with $\partial X_I/\partial\chi = 0$. Then from (6), (10) and (21) we get

$$\delta W = X_I^2 \int ds_\chi \frac{D}{B}\left(\frac{2\pi\gamma p \int ds_\chi D/B}{V' + \gamma p L'} + p'\right) .$$

Now as $B \to 0$ we have $D \sim \frac{1}{B}$ from (9), $V' \sim \frac{1}{B}$ from (8), $L' \sim 1/B^3$ from (7) and therefore the first term in parenthesis is O(B). On the other hand, $p' = j/r = O(1)$ according to eq.(4). Hence the p'-term dominates in the neighbourhood of a $B=0$ line, and we have

$$\delta W = X_I^2 \int ds_\chi \frac{Dp'}{B} . \qquad (24)$$

Together with $\underline{B}$, $\nabla\psi$ and $\underline{e}_\psi$ also reverse and according to eq.(9) D must change its sign. Now if $\int ds_\chi Dp'/B$ is positive on one side of the $B=0$ contour, it will be negative on the other. (We assumed that $j \neq 0$ and therefore $p' = j/r$ does not change sign). Since X_I can be localised to the ψ surfaces where the integral in (24) is negative we have proven instability.
The stability condition thus derived is that the magnetic field should not reverse.

We shall conclude this section by proving that a certain average of the field line curvature (signed) has to be positive:

$$\oint JD\, d\chi = \oint \frac{u}{rB} \frac{ds_\chi}{B} > 0 \qquad (25)$$

is a necessary criterion for absolute stability. We mention that a very similar criterion has been derived for stability with respect to interchanges in the case of an axially symmetric equilibrium at low β [3]

$$\int_{s_1}^{s_2} \frac{\kappa}{rB} \frac{ds}{B} > 0 .$$

However, in this case the criterion is neither necessary nor sufficient for absolute stability [4] .

We prove the criterion by decomposing a general perturbation into a χ -independent component X_I and a remainder $X_{II} = X - X_I$ and by assuming that

$$\int JD\,d\chi = 2\int \frac{\kappa}{rB} \frac{ds_\chi}{B} = 0$$

on some surface $\psi = const$.

Note that $\kappa = -\underline{e}_\psi \cdot \underline{\kappa}$ is signed. Then

$$\delta W(\psi) = \frac{2\pi}{L' + V'/\gamma p}\left(\int d\chi\, JD X_I\right)^2 + \int d\chi \left[\frac{1}{r^2B^2J}\left(\frac{\partial X_{II}}{\partial \chi}\right)^2 + p' JD X_I^2\right]$$
$$+ 2p' X_I \int JD X_{II}\, d\chi \qquad (16)$$

X can be chosen such that $\int JD X_I d\chi \neq 0$ for fixed X_{II}, X_I can be made so large that the last term in (26) dominates. In addition, X_I can be given such a sign that $\delta W(\psi)$ becomes negative and we have proven instability if the integral in eq.(25) becomes zero. On the other hand, since it is positive in the vacuum field which surrounds the hard core (κ is positive for all χ since the vacuum field lines are everywhere curved towards the hard core), it must also be positive in the whole plasma domain.

[3] Rosenbluth M.N. and Langmuir C.L. (1957) Ann. Phys. 1, 120

[4] Grad H. (1964) Phys. Fl. 7, 1283

Physically, condition (25) means that the fieldlines should not contain too large sections where the curvature is directed away from the hardcore.

An equivalent form of (25) is

$$V'' > p'L' \qquad (27)$$

This form is obtained from grad $(p + B^2/2) = (\underline{B}\cdot\nabla)\underline{B}$ by insertion into (9) and using the definitions (7) and (8) in $\int JD\,d\chi > 0$.

4. Absolutely stable solutions

Thus far we have obtained a number of necessary conditions by considering the energy variation at certain singular field lines.

We now turn to a more general consideration of the energy variation and show how equilibria can eventually be obtained for which δW is always positive.

Let us again decompose X into a χ-independent component X_I and a remainder $X_{II} = X - X_I$. If we choose

$$X_I = \int d\chi\, JDX \Big/ \int d\chi\, JD \qquad (28)$$

then

$$\int d\chi\, JD X_{II} = 0 \qquad (29)$$

Since we have to satisfy inequality (25), X_I does always exist, and this decomposition is always possible. This enables us to split $\delta W(\psi)$ into two independent contributions:

$$\delta W(\psi) = \delta W_I(\psi) + \delta W_{II}(\psi)$$

where

$$\delta W_I(\psi) = X_I^2 \left[\frac{2\pi \gamma p \left(\int d\chi \, JD \right)^2}{V' + \gamma p L'} + p' \int JD \, d\chi \right] \tag{30}$$

$$\delta W_{II}(\psi) = \int d\chi \left[\frac{1}{r^2 B^2 J} \left(\frac{\partial X_{II}}{\partial \chi} \right)^2 + p' J D X_{II}^2 \right] \tag{31}$$

If $\delta W_I(\psi) > 0$ for all X_I with $\partial X_I / \partial \chi = 0$ and if $\delta W_{II}(\psi) > 0$ for all X_{II} with $\int d\chi \, JD X_{II} = 0$, then $\delta W(\psi) > 0$ for all X and we obtain absolute stability.

Provided inequality (25) is fulfilled then $\delta W_I(\psi)$ is positive if $p' > 0$. Therefore, $\delta W_I(\psi) > 0$ imposes a stability criterion only in the region where p decreases towards the wall. Using

$$\int JD \, d\chi = V'' - L' p'$$

this criterion is

$$-p' \leq \gamma p \frac{(V'' - p' L')}{V' + \gamma p L'} \tag{32}$$

which imposes the very strong condition that the plasma pressure may not decay to zero within a finite distance from the hard core. On the contrary it has to decay more slowly than the curves

$$p = \text{const} \cdot \exp\left[- \gamma \int \frac{V'' - p' L'}{V' + \gamma p L'} \, d\psi \right] \tag{33}$$

If we want to achieve low pressure at a wall which is located at some distance from the hardcore, then we may get a limitation on the maximum plasma pressure by this condition. (Of course also the equilibrium equations give an upper limit of the pressure maximum)

In the low pressure boundary close to the wall we have $p'L' \ll V''$ and $\gamma p L' \ll V'$. Neglecting small terms in the integrand eq.(33) can be integrated to give

$$p V'^{\gamma} = \text{const.}$$

In a straight cylinder these curves can easily be shown to give

$$p / \varrho^{2\gamma} = \text{const.}$$

which shows that p will decay reasonably fast in the boundary layer.
Since $\delta W_{II}(\psi)$, eq.(31), is positive for small p (according to eq.(32) p' is small if p is small) we have the result that

$$p'/p + \gamma V''/V' > 0 \qquad (32a)$$

together with

$$V'' > 0 \qquad (27a)$$

are necessary and sufficient criteria for absolute stability in the boundary layer.

We add two remarks about the boundary layer.

First, the condition of finite pressure up to the wall was derived from perturbations with infinitely small wavelength in the Θ-direction, $m = \infty$. We can show that this necessity arises from long-wave perturbations as well.

If we assume that our perturbations are constant on the B-lines, $\partial/\partial\chi = 0$, then we obtain from (5) after minimisation with respect to Y_m

$$\delta W_m(\psi) = X_m^2 \int d\chi \left[\frac{\gamma p B^2 J D^2}{\gamma p + B^2} + p' J D \right]$$

From this we obtain in the boundary layer the stability condition

$$-p' \leq \gamma p \frac{\int J D^2 d\chi}{\int J D d\chi}$$

which again requires finite pressure throughout up to the wall and allows only for a somewhat faster decay.

Our second remark concerns the practical significance of this condition. As we have seen in the case of a straight cylinder the pressure can decay rather rapidly towards the wall ($p \sim \varrho^{-10/3}$ for $\gamma = 5/3$). On the other hand, finite resistivity and other diffusion effects will enforce a low pressure layer up to the wall even in cases where an ideal MHD equilibrium would be separated from this by a vacuum region. Therefore, in practice the low pressure layer of our equilibria will be no disatvantage.

We now have to consider $\delta W_{II}(\psi)$, eq. (31). In the appendix we shall prove the inequality

$$\delta W_{II} \geq \int d\chi \; J X_{II}^2 \left[\frac{4\pi^2 \left(\int J D d\chi\right)^2 r^2 B^2}{\left(\int r^2 B^2 J d\chi\right)^3 \int \frac{J D^2}{r^2 B^2} d\chi} + p' D \right] \qquad (34)$$

which is valid for all X_{II} with

$$\int J D X_{II} \, d\chi = 0 \quad . \qquad (35)$$

Therefore, if

$$-p' D \leq \frac{4\pi^2 \left(\int J D d\chi\right)^2 r^2 B^2}{\left(\int r^2 B^2 J d\chi\right)^3 \int \frac{J D^2}{r^2 B^2} d\chi} = A \qquad (36)$$

then δW_{II} will be positive.

First we consider the region where $p'(\psi) > 0$. If also $D > 0$ then absolute stability is warranted provided that our previous condition (27) is not violated. If we let p increase slowly enough on surfaces where D takes on negative values, stability is still achieved.

In the region of decaying pressure $p'(\psi) < 0$ we can achieve stability by choosing $|p'|$ smaller than A/D_{max} (inequality (36)) and according to inequality (32). Both inequalities require that the pressure does not decay too rapidly towards the wall. Since A stays finite, condition (32) will be dominant at lower pressures, i.e. close to the wall.

Thus it is shown that by choosing $p(\psi)$ appropriately we can actually obtain solutions for which δW is positive for all possible perturbations. The most stringent restrictions which are imposed by stability requirements are the need for a hard core, the existence of a low pressure boundary layer up to the wall and the exclusion of field reversal.

Conditions (32) and (36) - the latter condition being sufficient in conjunction with the others but not necessary - may impose a limit on the pressure maximum. More precisely, they may render possible only a lower pressure maximum than the equilibrium theory without stability considerations.

For completeness we give an upper bound of the pressure which is implied by the pure equilibrium theory.

$\underline{j} \times \underline{B} = \nabla p$ can be written as $\nabla(p + B^2/2) = (\underline{B} \cdot \nabla)\underline{B}$. Using div $\underline{B} = 0$, we get from the Gauss theorem

$$\oint_{\psi} (p + B^2/2)\, df = \int_{\psi_0} \frac{B^2}{2}\, df$$

ψ_0 is the inner flux surface where $p = 0$. $(\underline{B}\cdot\nabla)\underline{B} = \nabla\cdot(\underline{B}\,\underline{B})$ does not give a contribution since the normal component B_n

vanishes on the flux surfaces. From this we get

$$p \leq \frac{\frac{1}{2}\int\limits_{\psi_0} B^2 df}{\int\limits_{\psi} df} < \frac{\frac{1}{2}\int\limits_{\psi_0} B^2 df}{\int\limits_{\psi_0} df}$$

Hence the maximum fluid pressure must be lower than the average magnetic pressure on the plasma boundary with the interior vacuum region.

Remark

We have seen that $p(\psi)$ can be chosen such that stability is achieved. The equilibrium problem posed by a stable pressure distribution is the following:
around the hard core we have the well known vacuum magnetic field of a ring conductor up to a certain flux surface ψ_0. For $\psi > \psi_0$ we have to solve the elliptic equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = -r^2 p'(\psi)$$

with the boundary condition $\psi = \psi_0$ on the given flux surface ψ_0 = const. (Dirichlets Problem). Solutions to this problems are known to exist [5].

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[5] H.Grad and H.Rubin, in Proceedings of the Second Conference on the Peaceful Uses of Atomic Energy (United Nations, Geneva, 1958) Vol 31.

Appendix. Derivation of inequality (34)

We consider the quantity $\int_0^L y'^2 dx$ and derive a lower bound for it under the constraining side condition $\int_0^L ay\, dx = 0$. a and y are both assumed to be periodic functions in x of period L. Then

$$y = \sum_k y_k e^{i2\pi kx/L} \tag{A 1}$$

$$y_{-k} = y_k^*$$

and

$$\int_0^L y'^2 dx = \sum_k \left(\frac{2\pi k}{L}\right)^2 |y_k|^2 L \geqslant \left(\frac{2\pi}{L}\right)^2 \sum_{k\neq 0} |y_k|^2 L \tag{A 2}$$

The constraint $\int_0^L ay\, dx = 0$ becomes

$$\sum_k a_k y_k^* = 0 \tag{A 3}$$

where $a = \sum_k a_k e^{i2\pi kx/L}$ has been used.

From (A 3)

$$a_0 y_0 = - \sum_{k\neq 0} a_k y_k^*$$

Applying Schwartz's inequality we get

$$\sum_{k\neq 0} |y_k|^2 \geqslant \frac{(a_0 y_0)^2}{\sum_{k\neq 0} |a_k|^2}$$

and hence

$$\frac{\sum_k |a_k|^2 \sum_{k\neq 0} |y_k|^2}{\sum_{k\neq 0} |a_k|^2} = \left(1 + \frac{a_0^2}{\sum_{k\neq 0} |a_k|^2}\right) \sum_{k\neq 0} |y_k|^2 \geqslant \frac{a_0^2}{\sum_{k\neq 0} |a_k|^2} \left(y_0^2 + \sum_{k\neq 0} |y_k|^2 \right)$$

or

$$\sum_{k\neq 0} |y_k|^2 \geqslant \frac{a_0^2}{\sum_k |a_k|^2} \sum_k |y_k|^2$$

From this and (A 2) we get, using

$$a_0 = \int_0^L a\,dx / L , \qquad \sum_k |y_k|^2 = \int_0^L y^2 dx / L , \qquad \sum_k |a_k|^2 = \int_0^L a^2 dx / L$$

$$\int_0^L y'^2 dx \geq \frac{4\pi^2}{L^3} \frac{\left(\int_0^L a\,dx\right)^2}{\int_0^L a^2 dx} \int_0^L y^2 dx \,. \tag{A 5}$$

From this result we can easily derive (34). Put

$$dx = r^2 B^2 J\, d\chi$$

then

$$\int d\chi \frac{1}{r^2 B^2 J} \left(\frac{\partial X_{II}}{\partial \chi}\right)^2 = \int \left(\frac{\partial X_{II}}{\partial x}\right)^2 dx$$

and our side condition (35) writes

$$\int \frac{D}{r^2 B^2} X_{II}\, dx = 0$$

which means $a = D / r^2 B^2$. If (A 5) is now applied, we obtain by going back from dx to $d\chi$ and using

$$L = \int dx = \int r^2 B^2 J\, d\chi$$

$$\int d\chi \frac{1}{r^2 B^2 J} \left(\frac{\partial X_{II}}{\partial \chi}\right)^2 \geq \frac{4\pi^2 \left(\int J D\, d\chi\right)^2 \int X_{II}^2\, r^2 B^2 J\, d\chi}{\left(\int r^2 B^2 J\, d\chi\right)^3 \int \frac{D^2 J}{r^2 B^2}\, d\chi} \tag{A 6}$$

If this is inserted in (31), we obtain (34).