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Energy Balance Equation and Enhanced Collisional Plasma Diffusion

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ABSTRACT

An exact energy balance equation is derived from the first-order moments of the ion and electron transport equations. Thereby the particle loss of a stationary plasma through an isobaric surface is expressed as the sum of several terms each of which contains a specific force ($\underline{E}$-field, collisional friction, anisotropic pressure including inertial terms). The equation provides a general frame of interpretation for any stationary diffusion process or loss mechanism of a plasma. In particular it is shown that GALEEV-SAGDEEV diffusion in an axisymmetric torus cannot be explained by the friction term $\eta_{\perp} j_{\perp}^2$ or the Joule term $\underline{E}_{\perp} \cdot \underline{j}_{\perp}$, but comes about by pressure anisotropy. This result rests on the assumption that the perpendicular resistivity $\eta_{\perp}$ is not altered in order of magnitude by the presence of trapped particles, an assumption that is plausible on account of recent work by HINTON and OBERMAN. For TOKAMAK-like parameters the effective pressure anisotropies required for explaining GALEEV-SAGDEEV diffusion are very small, of the order of 10^{-5} to 10^{-6}.

1. INTRODUCTION

It is well known that the diffusion of a dense plasma across the magnetic field is linked to the frictional work ηj^2 by a socalled energy balance equation [1], [2], viz.

$$\oint n\underline{v}\cdot d\underline{S} = \oint \frac{dS}{|\mathrm{grad}\, p|}\cdot n\eta j^2 . \tag{1.1}$$

Here n = particle density, $\underline{v}$ = mass velocity, p = pressure, $\underline{j}$ = electric current density, and η = resistivity. The integrals extend over closed p = const surfaces. Equation (1.1) is derived from MHD equations or two-fluid equations with several assumptions, viz. isotropic pressure, p = p(n), stationarity, and others concerning the collision term.

Recent theories of enhanced collisional diffusion (ECD) in toroidal plasmas apply to the regime of long mean free paths where the assumption of isotropic pressure breaks down ([3] - [10]). The question arises whether this enhancement has to do with enhanced collisional friction, enhanced $\underline{E}\times\underline{B}$ drifts, or with other effects. We shall derive a more general form of eq. (1.1) that holds for anisotropic pressure and general collision term and that allows to discuss this question. This generalized energy balance equation will be exact and, hence, will offer an independent means of checking the consistency of results on ECD. The new energy balance equation is derived from the first-order moments of the plasma transport equations. A second expression for the particle loss could be gained from the second-order moments of the transport equations; but we shall not make use of this second possibility.

The generalized energy balance equation can be written in two equivalent forms. We shall interpret the various terms occurring therein. In particular, a certain term that dominates PFIRSCH-SCHLÜTER diffusion can be shown to arise from $\underline{E} \times \underline{B}$ drifts rather than from random walk under rather general conditions.

Throughout the paper a stationary toroidal plasma configuration with closed isobaric surfaces and confined by a magnetic field is considered. This implies the existence of stationary plasma sources. We shall consider a model in which the sources are located on some material surface, e. g. of a toroidal ring, while the plasma volume itself is assumed sourcefree. The plasma is also assumed to be fully ionized and quasineutral.

2. ENERGY BALANCE EQUATION

In this section the generalized balance equation will be derived which replaces eq. (1.1) in the case of anisotropic pressure, nonzero inertia, and a general collision term.

One starts from the transport equations for ions and electrons

(2.1) $$\frac{\partial f_j}{\partial t} + \underline{u} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{q_j}{m_j}\left(\underline{E} + \frac{1}{c}\,\underline{u} \times \underline{B}\right) \cdot \frac{\partial f_j}{\partial \underline{u}} = \left(\frac{\delta f_j}{\delta t}\right)_c ,$$

where the notation is standard, and $j = i, e$. We shall not need to specify the collision term. Defining velocity moments of f_j in the usual way one derives in particular the momentum equations for ions and electrons, which we write in the following form, valid for quasineutral plasma:

(2.2) $$m_j n \frac{\partial \underline{v}_j}{\partial t} = -\underline{\nabla} \cdot \underline{\underline{P}}_j + q_j n \underline{E} + \frac{q_j n}{c} \underline{v}_j \times \underline{B} + \underline{C}_j .$$

Here $q_i = +e$; $q_e = -e$; $\underline{C}_i$, $\underline{C}_e$ are the collision terms, with $\underline{C}_i + \underline{C}_e = 0$; and $\underline{\underline{P}}_i$, $\underline{\underline{P}}_e$ are the total kinetic pressure tensors of ions and electrons, viz.

(2.3) $$\underline{\underline{P}}_j = \underline{\underline{p}}_j + m_j n \underline{v}_j \underline{v}_j ,$$

where $\underline{\underline{p}}_i$ and $\underline{\underline{p}}_e$ are the usual pressure tensors of ions and electrons in their respective average rest frames.

Equations (2.2) may be combined to give the equation of motion and the generalized Ohm's law, which we write in their time-independent forms:

(2.4) $$\frac{1}{c}\,\underline{j}\times\underline{B} = \underline{\nabla}\cdot\underline{\underline{P}} = \mathrm{grad}\,P + \underline{\nabla}\cdot\Delta\underline{\underline{P}}$$

(2.5) $$\underline{E} + \frac{1}{c}\,\underline{v}\times\underline{B} = \underline{C} + \frac{1}{ne}\,\underline{\nabla}\cdot\underline{\underline{Q}}$$

We have introduced the mass velocity $\underline{v}$ the electric current density $\underline{j}$ the collision term $\underline{C}$ and the kinetic pressure tensors $\underline{\underline{P}}$ and $\underline{\underline{Q}}$. They are defined by

(2.6) $$m\,\underline{v} = m_i\,\underline{v}_i + m_e\,\underline{v}_e \qquad (m = m_i + m_e),$$

(2.7) $$\underline{j} = ne\,(\underline{v}_i - \underline{v}_e),$$

(2.8) $$\underline{\underline{P}} = \underline{\underline{P}}_i + \underline{\underline{P}}_e ,$$

(2.9) $$\underline{\underline{Q}} = \frac{1}{m}\left(m_i\,\underline{\underline{P}}_i - m_e\,\underline{\underline{P}}_e\right) \approx \underline{\underline{P}}_i ,$$

(2.10) $$\underline{C} = \underline{C}_e / ne .$$

The scalar pressure P, introduced in eq. (2.4), is defined by the trace of $\underline{\underline{P}}$ in the usual way. It is assumed that conservation of momentum holds for collisions separately; i.e. one has $\underline{C}_i = \underline{C}_{ie}$; $\underline{C}_e = \underline{C}_{ei}$; $\underline{C}_{ii} = \underline{C}_{ee} = 0$; $\underline{C}_{ie} + \underline{C}_{ei} = 0$. Even though collisions between like particles do not contribute to the collision terms $\underline{C}_j$, they can affect the pressure tensors $\underline{\underline{P}}_j$ and thereby contribute to the momentum balance of eqs. (2.2), (2.4), (2.5).

Scalar multiplication of eq. (2.4) with $n\underline{v}$ and of eq. (2.5) with $n\underline{j}$ leads to the following relation:

(2.11) $$n\underline{v}\cdot \mathrm{grad}\, P = n\underline{j}\cdot(\underline{E}-\underline{C}) - \frac{1}{e}\,\underline{j}\cdot(\underline{\nabla}\cdot\underline{\underline{Q}}) - n\underline{v}\cdot(\underline{\nabla}\cdot\Delta\underline{\underline{P}}).$$

On account of the parallel component of eq. (2.5) this can be written also in the following way:

(2.12) $$n\underline{v}\cdot \mathrm{grad}\, P = n\underline{j}_\perp\cdot(\underline{E}_\perp-\underline{C}_\perp) - \frac{1}{e}\,\underline{j}_\perp\cdot(\underline{\nabla}\cdot\underline{\underline{Q}})_\perp - n\underline{v}\cdot(\underline{\nabla}\cdot\Delta\underline{\underline{P}}),$$

where $\perp$ designates the component perpendicular to $\underline{B}$. The plasma flux through an isobaric surface is given by

(2.13) $$\Gamma = \oint n\underline{v}\cdot d\underline{S} = -\oint \frac{dS}{|\mathrm{grad}\, P|}\, n\underline{v}\cdot \mathrm{grad}\, P$$

$$= \frac{d}{dP}\int d\tau\; n\underline{v}\cdot \mathrm{grad}\, P\, .$$

Hence Γ may be expressed as

(2.14) $$\Gamma = \oint \frac{dS}{|\mathrm{grad}\, P|}\left[n\underline{j}\cdot(\underline{C}-\underline{E}) + \frac{1}{e}\,\underline{j}\cdot(\underline{\nabla}\cdot\underline{\underline{Q}}) + n\underline{v}\cdot(\underline{\nabla}\cdot\Delta\underline{\underline{P}})\right]$$

or equivalently as

(2.15) $$\Gamma = \oint \frac{dS}{|\mathrm{grad}\, P|}\left[n\underline{j}_\perp\cdot(\underline{C}_\perp-\underline{E}_\perp) + \frac{1}{e}\,\underline{j}_\perp\cdot(\underline{\nabla}\cdot\underline{\underline{Q}})_\perp + n\underline{v}\cdot(\underline{\nabla}\cdot\Delta\underline{\underline{P}})\right].$$

Here and in the following all surface integrals extend over surfaces $P =$ const, if not indicated otherwise.

Equations (2.14) and (2.15) are the desired generalization of the energy balance equation eq. (1.1). They provide a means of interpreting any stationary plasma loss process, and an independent check of diffusion calculations. It is seen that in either of its two forms Γ consists of four terms that contain products of fluxes ($\underline{j}, n\underline{v}$) and forces (collisional friction, E-field, anisotropic pressure including inertia). In a self-explanatory manner eqs. (2.14) and (2.15) may be written as

(2.16) $$\Gamma = \Gamma_C + \Gamma_E + \Gamma_Q + \Gamma_{\Delta P}$$

and as

(2.17) $$\Gamma = \Gamma_{C\perp} + \Gamma_{E\perp} + \Gamma_{Q\perp} + \Gamma_{\Delta P} .$$

Let us first discuss the terms of eq. (2.14/2.16). In Γ_C the collision term $\underline{C}$ is usually written as $\eta \underline{j}$ or $\underline{\underline{\eta}} \cdot \underline{j}$ in the fluid approximation. An expansion as a sum of several terms involving different moments of f_j would be more rigorous [11] . For our purpose - discussion of ECD - it will be good enough to assume $\underline{C}$ to be of the order of $\eta \underline{j}$ or $\underline{\underline{\eta}} \cdot \underline{j}$ where $\underline{\underline{\eta}}$ contains the distinction between $\eta_\parallel$ and $\eta_\perp$.

The term

(2.18) $$\Gamma_E = \frac{d}{dP} \int d\tau \; n \underline{j} \cdot \underline{E}$$

vanishes when $\underline{j} \cdot \operatorname{grad} n \equiv 0$ and $\underline{j} \cdot \operatorname{grad} P \equiv 0$. Then $\Gamma_E = 0$ follows from $\underline{E} = - \operatorname{grad} \Phi$, $\operatorname{div} \underline{j} = 0$ by simple vector analysis. This case is realized whenever $\underline{\underline{P}}$ is isotropic and $P = P(n)$.

The term

(2.19) $$\Gamma_Q = - \frac{d}{dP} \int d\tau \; \frac{1}{e} \; \underline{j} \cdot (\underline{\nabla} \cdot \underline{\underline{Q}})$$

vanishes if $\underline{\underline{Q}}$ is isotropic, i.e. $\underline{\nabla}\cdot\underline{\underline{Q}} = \operatorname{grad} Q$ and $\underline{j}\cdot\operatorname{grad} Q \equiv 0$ and/or $\underline{j}\cdot\operatorname{grad} P = 0$. The latter is certainly true if $\underline{\underline{P}}$ is isotropic. Finally the term $\Gamma_{\Delta P}$ vanishes by definition whenever $\underline{\underline{P}}$ is isotropic. Of course, $\underline{\underline{P}}$ and $\underline{\underline{Q}}$ can only be expected to be isotropic everywhere if the inertial contributions may be neglected (compare eq. 2.3).

It is instructive to discuss eqs. (2.16) and (2.17) also in connection with PFIRSCH-SCHLÜTER diffusion ([2], [9], [12]). In this case one assumes isotropic ion and electron pressures, $P = P(n)$, $Q = Q(n)$, and $\underline{C} = \underline{\underline{\eta}}\cdot\underline{j}$, where $\underline{\underline{\eta}}$ is the resistivity tensor. Then

(2.20) $$\Gamma_E = \Gamma_Q = \Gamma_{\Delta P} = \Gamma_{Q\perp} = 0,$$

and consequently the PFIRSCH-SCHLÜTER diffusion flux is given by:

(2.21) $$\Gamma_{P.S.} = \Gamma_C = \oint \frac{dS}{|\operatorname{grad} P|}\, n\left(\eta_\perp j_\perp^2 + \eta_\parallel j_\parallel^2\right)$$

or equivalently by

(2.22) $$\Gamma_{P.S.} = \Gamma_{C\perp} + \Gamma_{E\perp} = \oint \frac{dS}{|\operatorname{grad} P|}\, n\left(\eta_\perp j_\perp^2 - \underline{E}_\perp\cdot\underline{j}_\perp\right).$$

It follows that

(2.23) $$\oint \frac{dS}{|\operatorname{grad} P|}\, n\,\eta_\parallel j_\parallel^2 = -\oint \frac{dS}{|\operatorname{grad} P|}\, n\,\underline{E}_\perp\cdot\underline{j}_\perp,$$

or in brief: $\Gamma_{C\parallel} = \Gamma_{E\perp}$ where $\Gamma_{C\parallel}$ is defined as $\Gamma_{C\parallel} = \Gamma_C - \Gamma_{C\perp}$.

For small aspect ratio $\left(\frac{r}{R} \ll 1\right)$ the following additional result holds [9]:

(2.24) $$\Gamma_{C\parallel} = \Gamma_{E\perp} \approx \frac{4\pi^2}{\iota^2} \Gamma_{C\perp}$$

where $\iota/2\pi$ is the rotational transform of the B-field. If $\iota/2\pi \ll 1$ the PFIRSCH-SCHLÜTER flux is approximately given by $\Gamma_{P.S.} \approx \Gamma_{C\parallel} = \Gamma_{E\perp}$.

Returning to the general discussion we shall interpret the first two terms of eq. (2.17). We limit the discussion to the case that the equation

(2.25) $$\underline{j} \times \underline{B} \approx c \operatorname{grad} P$$

determines $\underline{j}_\perp$ to sufficient accuracy. Then the flux $\Gamma_{E\perp}$ has a simple interpretation. From the definition of $\Gamma_{E\perp}$ and from eq. (2.25) one obtains immediately

(2.26) $$\Gamma_{E\perp} \approx \oint d\underline{S} \cdot \frac{nc}{B^2} \left(\underline{E}_\perp \wedge \underline{B}\right) = \oint n \, \underline{v}_E \cdot d\underline{S} .$$

Hence $\Gamma_{E\perp}$ arises from $\underline{E} \wedge \underline{B}$ drift, and not from any random walk process. This holds in particular for the PFIRSCH-SCHLÜTER diffusion [12].

With the aid of eq. (2.25) and $\underline{E} \approx \underline{\underline{\eta}} \cdot \underline{j}$ the flux $\Gamma_{C\perp}$ may be expressed as

(2.27) $$\Gamma_{C\perp} \approx \oint dS \, \frac{nc^2}{B^2} \eta_\perp \left|\operatorname{grad} P\right| .$$

Replacing the quantities in the integrand by averages yields the following diffusion coefficient for isothermal plasma with $P = p$:

(2.28) $$D_{C\perp} \approx \eta_\perp c^2 p / B^2 = R_e^2 \tilde{\nu}$$

Here the gyroradius R_e and the effective collision frequency $\tilde{\nu}$ are defined by

(2.29) $$R_e = \frac{c}{eB} \sqrt{2 m_e K T_e} \, ,$$

(2.30) $$\tilde{\nu} = \frac{n e^2}{2 m_e} \eta_\perp \frac{T_e + T_i}{T_e} \, .$$

In the fluid regime $D_{C\perp}$ is the total diffusion coefficient across $\underline{B}$ for configurations with $\underline{j}_\perp \cdot \underline{E}_\perp = 0$, e. g. for suitable straight configurations.

It follows from eqs. (2.24) and (2.28) that the PFIRSCH-SCHLÜTER diffusion coefficient has the following form $(r/R \ll 1)$:

(2.31) $$D_{P.S.} = D_{C\perp} + D_{E\perp} = R_e^2 \tilde{\nu} \left(1 + \frac{4\pi^2}{\iota^2}\right) ,$$

3. ENHANCED COLLISIONAL DIFFUSION

In this section we shall give a brief review of enhanced collisional diffusion and then interpret the results by means of the energy balance equation, eq. (2.15/2.17). For simplicity we limit the discussion to diffusion in an axisymmetric torus with magnetic surfaces of circular cross-section, and with $r/R \ll 1$, $\iota/2\pi \ll 1$.

The various theories on ECD in an axisymmetric torus ([3], [4], [9]) in essence agree that for electron mean free paths

(3.1) $$\lambda_e > \frac{2\pi}{\iota}\left(\frac{R}{r}\right)^{3/2} R$$

the diffusion coefficient is enhanced and obeys the GALEEV-SAGDEEV formula

(3.2) $$D_{G.S.} = 1.6\; R_e^2\, \nu_{ei}\, \frac{4\pi^2}{\iota^2}\left(\frac{R}{r}\right)^{3/2}.$$

Here R is the major radius of the torus, r is the distance from the magnetic axis, and ν_{ei} is an effective electron-ion collision frequency. The latter is sufficiently well defined by noting that according to [9] PFIRSCH-SCHLÜTER diffusion obeys [compare eq. (2.31)!] :

(3.3) $$D_{P.S.} = R_e^2\, \nu_{ei}\left(1 + \frac{4\pi^2}{\iota^2}\right).$$

It is worth mentioning that $D_{G.S.}$ does not diverge for $\nu \to 0$ because eq. (3.1) will be violated for $\nu \to 0$. Hence the largest value of $D_{G.S.}$ within the range of validity, eq. (3.1), will be

(3.4) $$D_{max} \approx 1.6 \frac{R_e^2 u_e}{R} \cdot \frac{2\pi}{\iota},$$

where u_e is the thermal velocity of electrons.

Other important features of ECD in an axisymmetric torus are the following ones:

a) The diffusion fluxes of ions and electrons through a closed magnetic surface are equal (ambipolar diffusion);
b) the diffusion fluxes are independent of the radial electric field E_r;
c) like-particle collisions do not contribute to the diffusion; therefore ν_{ei} enters eq. (3.2).

These results apparently hinge on axisymmetry and its consequence, conservation of the corresponding canonical momentum ([9], [13], [14]). Results to the contrary [3] appear to be in error ([4], [9]).

It has not been proved that the inverse holds, too, i. e. that in non-axisymmetric tori the diffusion is ambipolar only for appropriate E-fields. This though seems plausible in the case of sufficient asymmetry and has been used explicitly in [7].

For the following discussion of ECD in connection with the energy balance equation it is important that the GALEEV-SAGDEEV diffusion can be recovered from a simple-minded random walk model without drift term. This model uses the distinction between transiting and trapped particles in a torus [3]. The orbits of the trapped particles, or the projections of these orbits in a moving meridian plane, are called "bananas". By virtue of the collisions the bananas undergo a random walk process, and inspection shows that the consequent diffusion flux of particles that are trapped or nearly trapped is much higher than the one of free particles.

This is shown in the following way. The validity of the simple formula

(3.5) $$D = \frac{(\Delta r)^2}{\tau} \quad \frac{\Delta n}{n}$$

is assumed for the partial diffusion coefficient of a class of particles with particle density Δn, random walk step length Δr, and effective collision time (= time between steps) τ. The other assumption is that it is the electrons that determine the diffusion velocity. It is shown in [3] that the trapped electrons possess small parallel velocities, if $E_r = 0$ is assumed:

(3.6) $$|v_\parallel| < \Delta v_\parallel \approx v_\perp \sqrt{\frac{r}{R}} \sim u_{e\perp} \sqrt{\frac{r}{R}} .$$

Collisions change $v_\parallel$ of (nearly) trapped electrons by amounts of $\Delta v_\parallel$ in an effective collision time

$$\tau_t \approx \nu_e^{-1}\left(\frac{\Delta v_\parallel}{u_{e\perp}}\right)^2 \approx \nu_e^{-1}\,\frac{r}{R} \tag{3.7}$$

and thereby introduce displacements of the order of a banana thickness

$$\Delta r_t \mathrel{\dot{\approx}} \frac{\Delta v_\parallel}{\Omega_e}\,\frac{R}{r}\,\frac{2\pi}{\iota} \approx R_e\,\frac{2\pi}{\iota}\sqrt{\frac{R}{r}}\ . \tag{3.8}$$

In addition, by virtue of $\Delta v_\parallel \ll u_e$ one has

$$\frac{\Delta n_t}{n} \approx \frac{\Delta v_\parallel}{u_e} \approx \sqrt{\frac{r}{R}} \ll 1 \tag{3.9}$$

for trapped electrons. The diffusion coefficient D_t for (nearly) trapped electrons according to eqs. (3.5) to (3.9) is virtually identical to the one given by eq. (3.2), if ν_e is identified with ν_{ei} , as is reasonable.

On the other hand the quantities for free particles are, according to [3]:

$$\Delta r_f \approx R_e\,\frac{2\pi}{\iota}\ ;\quad \tau_f \approx \nu_e^{-1}\ ;\quad \frac{\Delta n_f}{n} \approx 1\ ; \tag{3.10}$$

$$D_f \approx R_e^2\,\nu_e\,\frac{4\pi^2}{\iota^2} \approx D_{P.S.} \ll D_t\ . \tag{3.11}$$

The conclusion to be drawn from this model is the following: Enhanced collisional diffusion ("banana diffusion") is caused by a random walk process. Effects of $\underline{E} \times \underline{B}$ drifts may be ruled out; at least insofar as they will not furnish the leading contribution to ECD. This is contrary to the situation in PFIRSCH-SCHLÜTER diffusion.

We are now prepared to discuss the ECD formula, eq. (3.2), in the light of the energy balance equation, in the form of eq. (2.15/2.17). In order to do this we assume that the collision term $\underline{C}$ is of the order of magnitude of $\underline{\underline{\eta}}\cdot\underline{j}$ and that eq. (2.25) is valid. These assumptions seem reasonable on remembering that ECD theories use near-Maxwellian velocity distributions and, thus, near-isotropic pressure tensors. Furthermore we use the assumption of small aspect ratio; consequently the enhancement factor $(R/r)^{3/2}$ is large. We ask which of the terms of eq. (2.17) can be responsible for ECD.

Starting with the term $\Gamma_{C\perp}$ it follows from eq. (2.28) and from $\tilde{\nu} \approx \nu_{ei}$ [compare eqs. (2.31) and (3.3)] that

(3.12) $$D_{C\perp} = R_e^2 \tilde{\nu} \approx R_e^2 \nu_{ei} \ll D_{G.S.} \, ,$$

if the usual perpendicular resistivity $\eta_\perp$ is used that holds for a plasma without trapped particles. A calculation of the parallel conductivity $\sigma_\parallel$ by HINTON and OBERMAN [15] for a plasma with trapped particles yields a decrease of $\sigma_\parallel$ merely of the order of the relative number of trapped particles:

(3.13) $$\sigma_\parallel^{H.O.} \approx \sigma_\parallel \left(1 - \frac{\Delta n_t}{n}\right) \approx \sigma_\parallel \left(1 - \sqrt{\frac{r}{R}}\right) .$$

We shall therefore make the assumption that the order of magnitude of $\eta_\perp$, or rather of the perpendicular collision term $C_\perp$, is not altered by the presence of trapped particles. It then follows that the term $\Gamma_{C\perp}$ cannot account for ECD.

As to the term $\Gamma_{E\perp}$ we have shown that this flux arises from $\underline{E} \times \underline{B}$ drifts (see eq. 2.26). On the other hand we saw that ECD can be explained by a random walk model without drift motions. Therefore it is obvious that $\Gamma_{E\perp}$ cannot yield the main contribution to ECD.

It follows that ECD must be explained by the fluxes $\Gamma_{Q\perp}$ and/or $\Gamma_{\Delta P}$ arising from pressure anisotropy. We discuss these terms for isothermal plasma with $T_i \approx T_e$ in order to be able to compare with eq. (3.2).

Putting

(3.14) $$\underline{j}_\perp \cdot (\underline{\nabla} \cdot \underline{\underline{Q}})_\perp = \alpha \, |j_\perp| \cdot |\mathrm{grad}\, P|$$

and employing eq. (2.25) one obtains

(3.15) $$\Gamma_{Q\perp} = \frac{e}{c} \oint \frac{\alpha}{B} \, |\mathrm{grad}\, P| \, dS .$$

The corresponding diffusion coefficient is

(3.16) $$D_{Q\perp} = \langle\alpha\rangle \frac{cKT}{eB} = \langle\alpha\rangle \frac{\lambda_e}{R_e} D_{C\perp} ;$$

where $\langle\alpha\rangle$ is a suitable average of α and an effective measure of ion pressure anisotropy. For isotropic pressure $\langle\alpha\rangle = 0$ holds. In terms of BOHM diffusion [16] one has

(3.17) $$D_{Q\perp} = 16 \, \langle\alpha\rangle \, D_{BOHM} .$$

It is seen that whenever $D_{BOHM} \gg D_{P.S.}$ a small effective anisotropy $\langle\alpha\rangle$ may suffice to give appreciable enhancement over PFIRSCH-SCHLÜTER diffusion.

Equation (3.16) will be compared with eq. (3.2) to give $\langle\alpha\rangle$. It will be seen that $\langle\alpha\rangle \ll 1$. The conclusion will be that this agrees with the assumption of near-Maxwellian velocity distribution underlying ECD theory; hence $D_{Q\perp}$ may account for ECD.

It is more difficult to obtain an estimate of

(3.18) $$\Gamma_{\Delta P} = \oint \frac{dS}{|\mathrm{grad}\, P|}\, n\underline{v} \cdot (\underline{\nabla} \cdot \Delta \underline{\underline{P}}),$$

since an estimate of $\underline{v}$ is not easily available. It is seen, however, that the component $\underline{v}_n$, normal to the isobaric surface, yields a contribution to $\Gamma_{\Delta P}$ that is too small to explain ECD. If one agrees to the estimate [compare eq. (2.5)]

(3.19) $$|\underline{v}_t| \lesssim \frac{c}{neB}|\mathrm{grad}\, Q| < \frac{c}{neB}|\mathrm{grad}\, P|$$

for the tangential component of $\underline{v}$ then it is convenient to put

(3.20) $$n\underline{v} \cdot (\underline{\nabla} \cdot \Delta \underline{\underline{P}}) = \alpha' \frac{c}{eB}|\mathrm{grad}\, P|^2$$

with $|\alpha'| \ll 1$. Here now α' not only is a measure of the anisotropy of $\underline{\underline{P}}$, but also of the magnitude of $\underline{v}_t$. Then the corresponding diffusion coefficient is

(3.21) $$D_{\Delta P} = \langle\alpha'\rangle \frac{cKT}{eB} = \langle\alpha'\rangle \frac{\lambda_e}{R_e} D_{C\perp}.$$

It is instructive to give numerical estimates for $\langle\alpha\rangle$ or $\langle\alpha'\rangle$ in the case of eq. (3.1) being valid. By equating D_{QL} of eq. (3.16) with $D_{G.S.}$ of eq. (3.2) one obtains

(3.22) $$\langle\alpha\rangle = \frac{4\pi^2}{\iota^2}\left(\frac{R}{r}\right)^{3/2}\frac{R_e}{\lambda_e}$$

Combination of eqs. (3.1) and (3.22) gives

(3.23) $$\langle\alpha\rangle < \frac{2\pi}{\iota}\,\frac{R_e}{R} .$$

In order for the ECD theories to be valid, in addition to eq. (3.1) the banana thickness must be small compared to the plasma radius,

(3.24) $$R_e\,\frac{2\pi}{\iota}\sqrt{\frac{R}{r_{max}}} < r_{max} ,$$

or

(3.25) $$R_e < \frac{\iota}{2\pi}\left(\frac{r_{max}}{R}\right)^{3/2} R .$$

It follows with eq. (3.23) that always

(3.26) $$\langle\alpha\rangle < \left(\frac{r_{max}}{R}\right)^{3/2} \ll 1 .$$

The same relations hold for $\langle\alpha'\rangle$.

We consider the following numerical examples which are of practical interest in connection with TOKAMAK experiments:

$$\frac{R}{r} = 9 ; \quad \frac{2\pi}{\iota} = 3 ; \quad R = 10^2\ \mathrm{cm} ;$$

$$B = 10^5\ \mathrm{gauss} ;$$

(a) $n = 10^{13}\ \mathrm{cm}^{-3}$; $KT = 10^3\ \mathrm{eV}$;

(b) $n = 10^{14}\ \mathrm{cm}^{-3}$; $KT = 10^4\ \mathrm{eV}$.

Then one obtains in case (a):

$$\langle \alpha \rangle = 0.69 \times 10^{-5}; \quad \lambda_e \left[\frac{2\pi}{\iota} \left(\frac{R}{r} \right)^{3/2} R \right]^{-1} = 5;$$

and in case (b):

$$\langle \alpha \rangle = 2.0 \times 10^{-6}; \quad \lambda_e \left[\frac{2\pi}{\iota} \left(\frac{R}{r} \right)^{3/2} R \right]^{-1} = 50,$$

so that eq. (3.1) is satisfied in both cases.

It is remarkable that very small effective pressure anisotropies $\langle \alpha \rangle$ or $\langle \alpha' \rangle$, of the order of 10^{-5} or 10^{-6} are sufficient for making ECD consistent with the energy balance equation.

It should be added that the induced toroidal $\underline{E}$ - field present in TOKAMAK experiments causes an additional drift counteracting outward diffusion. This drift is not taken into account in the GALEEV-SAGDEEV formula. Therefore we neglected it altogether in the above discussion of ECD theory. In TOKAMAK experiments this drift is important whenever

$$\frac{v_{G.S.}}{v_E} \sim \frac{4\pi P_e}{B^2} \frac{4\pi^2}{\iota^2} \left(\frac{R}{r} \right)^{3/2} \lesssim 1. \tag{3.27}$$

4. CONCLUSION

For the diffusion loss of plasma particles we have derived a generalized energy balance equation that applies to the case of anisotropic pressure, including inertia, and of a general collision term. It represents a general frame of interpretation for any stationary diffusion process or loss mechanism in a quasineutral plasma. In particular GALEEV-SAGDEEV diffusion in an axisymmetric torus [3] is shown to arise from small effective pressure anisotropy, of the order of 10^{-5} to 10^{-6} for TOKAMAK-like parameters. Other causes, as enhanced resistivity or enhanced $\underline{E} \times \underline{B}$ drift, appear to be ruled out. This follows from the plausible assumption that the perpendicular resistivity $\eta_\perp$ possesses the same order of magnitude, with or without trapped particles present. No inconsistency between ECD and the energy balance equation has been found. It is also shown, with the aid of the energy balance equation, that PFIRSCH-SCHLÜTER diffusion comes about by $\underline{E} \times \underline{B}$ drift, in accordance with [12].

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