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Low Frequency Interchanges +)

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ABSTRACT

Unstable low frequency interchanges, i.e. electrostatic modes with growth rates of the order of the guiding centre drift frequencies and thus conserving the longitudinal adiabatic invariant J, are shown to exist in a wide class of self-consistent low β guiding centre equilibria. This class is defined by some restrictions, the most important of which are symmetry, vanishing of the electric field, and stability on the faster time scale of the longitudinal guiding center motion. Open-ended as well as toroidal configurations, with or without a rotational transform, and arbitrarily anisotropic distribution functions are admitted. Some relationships to the MHD description are derived for the sub-class of equilibria having isotropic distribution functions. The well-known interchange stability criterion of the guiding centre theory, for example, is shown to be identical with the well-known V"-criterion of the MHD theory. Since low frequency interchange stability (J conserved) turns out to be a more restrictive requirement than interchange stability (J violated), the confusion arising from earlier work on this subject is removed. The present theory is a generalization to a much wider class of equilibria of Rosenbluth's theory of the trapping instability in axisymmetric multipoles containing isotropic Maxwellians. This generalization shows that the stability condition, as far as it concerns the confining magnetic field, is relaxed by allowing for anisotropy.

1. Introduction

Microscopic stability of real confinement systems is a fairly tractable problem if one considers only certain limiting cases of electrostatic motions (interchange motions) in low β quasineutral collisionless plasmas, i.e. if one takes as a model the equations

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{e}{m}(-\nabla\varphi + \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}} = 0, \qquad (1.1)$$

$$\sum e \int d^3v \, f = 0, \qquad (1.2)$$

and then assumes certain scaling laws involving mainly the Larmor radius. There is one Vlasov equation for each particle species, and in the charge neutrality condition the summation runs over these species. $\underline{B}$ is any confining static vacuum field, and φ is the electrostatic potential, the electric field $\underline{E}$ being given by

$$\underline{E} = -\nabla\varphi. \qquad (1.3)$$

Equations (1.1-3) may be regarded as an approximation of the full system of the Vlasow-Maxwell equations.

There are also other models which should not be confused with eqs. (1.1-3). A onefluid model, for example, in which the parallel component of the electric field as well as the neutrality condition are ignored yields similar results. One could also use the Poisson equation instead of the neutrality condition without altering some of the essential features of the theory. We emphasize, however, that the results of this paper are based only on eqs. (1.1-3), and also that they are not at all proven to be identical with the low β limit of any corresponding results holding for finite β.

Guiding centre equilibria are stationary solutions of eqs. (1.1-3), the scale lengths of which are large compared with the Larmor radius, i.e. for which the quantity

$$\varepsilon \sim \frac{mv}{eB}\frac{\partial}{\partial x} \tag{1.4}$$

may be used as an expansion parameter, and in which the electric field is a first-order quantity, i.e.

$$E \sim \varepsilon v B . \tag{1.5}$$

The distribution functions of the most general guiding centre equilibria have, as proven by Hastie et al [1], to lowest order the form

$$f_0(\underline{x},\underline{v}) = G(\mu, J, K), \tag{1.6}$$

where the G's are almost arbitrary functions. K is the ratio of particle energy and mass,

$$K = \frac{1}{2}v^2 + \frac{e}{m}\varphi , \tag{1.7}$$

μ is the magnetic moment,

$$\mu = \frac{v_\perp^2}{2B} , \tag{1.8}$$

and J is the lowest-order longitudinal adiabatic invariant,

$$J(K,\mu,\psi,\chi) = \oint ds \sqrt{2\left(K - \mu B(\psi,\chi,s) - \frac{e}{m}\varphi(\psi,\chi,s)\right)} \tag{1.9}$$

where ψ and χ are flux functions satisfying

$$\underline{B} = \nabla\psi \times \nabla\chi , \tag{1.10}$$

and s is the arc length along the magnetic field lines.

There are several distinct approaches to the stability problem, depending on the way in which the perturbing quantities are formally ordered in terms of the parameter ε, and thus considering different classes of motions. In this paper we treat only low frequency interchanges, i.e. motions conserving the J-invariant. J-invariance requires that the perturbations satisfy the same ordering laws as the equilibrium, and that the frequencies be of the same order as the drift frequencies,

$$\omega \sim \varepsilon^2 \frac{eB}{m} . \qquad (1.11)$$

The theory using the scaling laws (1.4-5) and (1.11) may be called "drift theory" because the first-order guiding centre drifts play an essential role, zero-order electric drifts being forbidden by the scaling (1.5). This scaling requires that the electric drift velocity be small compared with the thermal longitudinal guiding centre velocity. The fact that it is necessary for the J-invariant to be conserved seems to be quite unknown though it has frequently been used as a sufficient condition. It follows from the requirement that the zero-order guiding centre arbits be closed. This requirement is, in general, not fulfilled if there are zero-order drifts removing the guiding centres to finite distances during one bounce period.

All that is known about low frequency interchange stability of general guiding centre equilibria is a sufficient criterion. It was recently derived by Rutherford and Frieman [2] and consists of the two sets of conditions

$$\frac{\partial G}{\partial K} < 0 , \qquad (1.12)$$

$$\frac{\partial G}{\partial K} + \frac{\partial G}{\partial J}\frac{\partial J}{\partial K} < 0 . \qquad (1.13)$$

It should be mentioned that these conditions can always be satisfied by an appropriate choice of the functions G, if the fields are given. A problem arises only from the fact that the conditions (1.12) have to be combined with the requirement of confinement, i.e. with the requirement that the distribution functions f_0 drop to zero at the boundaries of a bounded region in space. The conditions (1.13) require that the distribution functions, when expressed in terms of the quantities K, μ, ψ, and χ, are monotonic in K, and thus are monotonicity conditions in velocity space.

Since the criterion (1.12-13) is only sufficient for stability against a sub-class of all possible motions, it cannot provide any conclusions with respect to absolute stability. Conclusions with respect to instability, on the other hand, can be drawn by considering only special motions, if the equilibrium violates a <u>necessary</u> stability criterion. In this paper a condition will be derived which is necessary for stability to low frequency interchanges, and hence sufficient for instability. Necessary conditions are, in general, much more difficult to derive than sufficient conditions because this involves some proof of existence. Accordingly, in order to get through with the calculations we shall have to adopt some restrictions concerning the equilibria to be considered. But nevertheless there will remain a fairly general class of guiding centre equilibria to which our theory will apply.

Hitherto, a necessary low frequency interchange stability condition was only derived for the much more special cases of isotropic Maxwellians with constant temperatures in axisymmetric multipoles. This theory was recently done by Rosenbluth [3] (see also B. Kadomtsev and O. Pogutse [4], S.E. Rosinskij et al [5], Rutherford and Frieman [6], Rutherford et al [7], J.B. Taglor and R.G. Hastie [8]).

Since any Maxwellians satisfy the monotonicity conditions (1.13) of Rutherford and Frieman, the stability condition of Rosenbluth may be compared with the set (1.12). For the special distribution functions considered this set reduces to one single inequality which is identical with Rosenbluth's condition. The unstable mode which occurs if this condition is violated was termed "trapping instability". In contrast to the conditions of Rutherford and Frieman which apply to the general case, and in which the confining magnetic field and the distribution functions are equal partners, Rosenbluth's condition is an essential restriction only with regard to the magnetic field. The reason is, of course, the special choice of distribution functions.

The theory to be developed in this paper is a generalization of Rosenbluth's theory to a much wider class of equilibria, which include, in particular, arbitrarily anisotropic distribution functions. The stability condition to be derived will accordingly be a generalization of the trapped particle condition. It will coincide with the general low frequency condition only for a special sub-class of equilibria, being less restrictive in general. Only for isotropic distribution functions will it essentially refer only to the magnetic field. The generalization to anisotropic equilibria enables us to treat also open-ended configurations, which because of the inevitable loss cone cannot contain any isotropic distribution functions.

One of our assumptions will be that the equilibria considered are stable according to the guiding centre theory. This theory was extensively described by H. Grad [9]. We shall now say a few words about it because there is much confusion in the literature about its relation to the drift theory. The guiding centre theory assumes conservation of the magnetic moment, but not of the $\mathcal{J}$-invariant. Hence the appropriate scaling is

$$\omega \sim \varepsilon \frac{eB}{m} , \quad E_{\parallel} \sim \varepsilon v B . \qquad (1.14)$$

In contrast to the drift scaling (1.5) and (1.11) this scaling allows for a zero-order transverse electric field. Hence in guiding centre theory all particles have a common zero-order electric drift $\underline{U}_\perp = \underline{E} \times \underline{B}/B^2$ which equals the transverse component of the lowest order macroscopic velocity $\underline{U}$. Because of the identity $\underline{E} + \underline{U} \times \underline{B} = 0$ one may use the picture of guiding centres orbiting only along magnetic field lines, which in turn move with the velocity $\underline{U}$. Hence electrostatic motions may be thought as being built up by interchanging the particles on different infinitesimal flux tubes containing equal magnetic fluxes. This is the reason why electrostatic motions were termed "interchanges". In MHD theory a similar picture holds.
In order to avoid any misunderstanding we emphasize that it does not apply to the drift theory. Hence low frequency interchanges have nothing whatsoever to do with interchanging flux tubes.

There is a general criterion for stability to interchanges which was derived by several authors using several more or less correct methods. In its correct form this criterion is

$$\sum m \int d\mu \, d\xi \, \frac{\partial G}{\partial K} \left(\frac{\partial K}{\partial \psi} \delta\psi + \frac{\partial K}{\partial \chi} \delta\chi \right)^2 < 0, \qquad (1.15)$$

which means that this quadratic expression has to be negative for any values of the numbers $\delta\psi$ and $\delta\chi$. It was first derived by Andreoletti [10] from the energy principle of Kruskal and Oberman [11] (see also Kulsrud [12] and Grad [13]). It is necessary and sufficient for interchange stability according to a one-fluid guiding centre theory. But since the inclusion of the self-consistent electric field, as shown by Andreoletti [14] (see also Kulsrud (Ref.12) and Gred (Ref.13)), relaxes any stability condition, the criterion (1.15) is at least sufficient according to the self-consistent guiding centre theory. It has, however, to be combined with the monotonicity condition (1.13), which is necessary for the Kruskal-Oberman principle to be valid.

The condition (1.13) plays still another role. As shown by Grad (Ref 13), it is necessary for local stability according to the finite β guiding centre theory, and remains necessary even at arbitrarily low β.

We have two reasons for requiring that the equilibria to be considered be stable according to the guiding centre theory. The first is the mathematical one that some of our formal conclusions break down if the local stability condition (1.13) and the interchange condition (1.15) are not both satisfied. The second is the physical one that it does not make any sense to look for low frequency instabilities of equilibria which are already unstable to faster growing, and thus dominating, modes. The necessary low frequency condition will indeed turn out to be more restrictive than the sufficient interchange condition, thus proving the existence of equilibria which are stable to interchanges and unstable to low frequency interchanges, and moreover proving that low frequency interchange stability implies interchange stability.

These results are in contradiction with the widely held opinion that low frequency interchanges are special interchanges, which gave rise to a misinterpretation of the interchange criterion. This criterion was again derived by Andreoletti (Ref 10) and by J.B. Taylor [15] using the $\mathcal{J}$-invariant, but without assuming the supplementary monotonicity condition (1.13) to be satisfied. Hence these authors interpreted it as being a low frequency criterion. Andreoletti concluded that interchange stability is more restrictive than low frequency interchange stability because using the $\mathcal{J}$-invariant he did not need the supplementary condition which he had to assume otherwise.

Besides the result of this paper, which definitely proves that these interpretations are false, there are some simple arguments which indicate that interchange stability and low frequency interchange stability are at least independent, and which explain why the above mentioned derivations of the interchange criterion are not correct. The key lies in the different scaling laws. The range of frequencies considered in the drift theory shrinks to zero in the guiding centre theory. Hence one expects the drift theory to reveal effects that are concealed in the guiding centre theory. One can formulate this fact in another way by noting that the effects considered in the drift theory play their role in time intervals of the order of the drift periods, while in the guiding centre theory one considers only times up to the order of the bounce periods. This difference can be formally understood by observing that both theories arise from the basic equations (1.1-3) by a consequent time scale expansion in which all quantities are written as formal series of the form $f = \sum \varepsilon^{\nu} f_{\nu}(\tau_0, \tau_1, \tau_2, \cdots)$, where $\tau_{\ell} = \varepsilon^{\ell} t$. The guiding centre theory assumes $\partial/\partial\tau_0 = 0$, while in the drift theory the additional assumption $\partial/\partial\tau_1 = 0$ is made. Thus, in order to get any time dependence, the drift theory has proceeded one order further than the guiding centre theory. The former thus deals with the second order of the Vlasov equations, the lower orders being trivial, while the latter deals with its first order. The effects which enter the drift theory, but not the guiding centre theory, are of course the first order guiding centre drifts. There are other effects which enter the guiding centre theory, but which are assumed to vanish in the drift theory. This corresponds to the difference in the scaling laws envolving the electric field. In the guiding centre theory the zeroth order of the macroscopic velocity is an essential quantity, while in the drift theory it is assumed to vanish. One may sum up all these arguments by stating that any motion or at least almost any motion, which appears in the guiding centre theory violates the J-invariant.

As to the derivations of the interchange stability criterion which use the J -invariant, we first note that they more or less explicitly use the picture of interchanging flux tubes. Hence the underlying equations, though never explicitly stated, are in fact the equations of the guiding centre theory, which are not consistent with $\mathcal{J}$ -invariance. This is the reason why the resulting stability criterion must not be interpreted as being relevant to low frequency motions. The question why this obviously unjustified use of the $\mathcal{J}$ -invariant nevertheless led to the correct interchange criterion was excellently answered by the concept of the "pessimistic variation" of H. Grad (Ref 13). Here the quantity $\mathcal{J}$ is used only as a coordinate in phase space, but not as an integral of the particle motion.

Before going into the details we should mention a serious objection to the use of the $\mathcal{J}$ -invariant in any stability theory. As pointed out by Grad [16)] (see also Ref 13), there are always perturbations which create internal mirrors, causing sudden changes of the $\mathcal{J}$ -invariant of some particles which drift to such critical regions. A related problem arises from particles the $\mathcal{J}$ -invariant of which, when considered as a function in phase space, does not change smoothly from the equilibrium state to the perturbed state. The class of perturbations avoiding such effects is too awkward to be relevant to any reasonable stability considerations. The present theory (as well as any other existing theories using the $\mathcal{J}$ -invariant) does not take into account these effects. Hence its relation to the basic equations (1.1-3) is not quite clear, and it should only be regarded as a model which is nevertheless hoped to yield reasonable results.

In the present paper full allowance is made, however, for the presence of internal mirrors in the equilibrium state, i.e. for the presence of several types of trapped particles, the $\mathcal{J}$ -invariants of which may have entirely different analytical forms. This feature, which is not fully discussed in the papers of Rutherford and Frieman and of Rosenbluth, necessitates some slight modifications. There are also some other points in Rosenbluth's paper which will, by the way, be somewhat refined or more fully justified.

In Sec. 2 a full description is given of the class of equilibria to be considered. Some relationships to the MHD theory are derived. In Sec. 3 the integral equation is derived which serves as a dispersion relation for the normal modes of the linearized drift theory. In Sec. 4 for a certain range of phase velocities this integral equation is reduced to a variational principle from which the stability condition is derived.

2. Equilibrium

We consider any static magnetic field which allows for the longitudinal adiabatic invariant

$$J(K,\mu,\psi,\chi) = \oint d\sigma \frac{B}{D}\sqrt{2(K-\mu B)} \tag{2.1}$$

to be defined for an appropriate range of the parameters K and μ and of the flux functions ψ and χ, which together with the quantity σ are used as a set of curvilinear coordinates. The determinant

$$D = \underline{B}\cdot\nabla\sigma \tag{2.2}$$

is, of course, assumed to be bounded away from zero. We will never use the fact that $\underline{B}$ is a vacuum field, though this is implied by the low β limit which we actually consider. Hence our results will in a certain sense also apply to finite β systems. The integral in eq. (2.1) runs along the field lines, either once back and forth between two points where the integrand is zero (trapped particles), or once around a closed field line, if the integrand vanishes nowhere (untrapped particles). We allow also for toroidal fields forming magnetic surfaces ("sheared configurations"). In these the J-invariant of untrapped particles degenerates, as discussed by Kulsrud [17] (see also Hastie et al. (Ref 1)), into a surface integral.

We consider only a sub-class of the general guiding centre equilibria described by eq. (1.6). This class is defined by some restrictions, the first of which is, as already indicated in eq. (2.1), the vanishing of the electric field. More precisely, this means $E \sim \varepsilon^2 v B$. This requirement implies a certain constraint on the equilibrium distribution functions which we shall derive below. It also implies that the analytical form of the J-invariant does not depend on the particle species, and it enables us to exhibit explicitly the K-dependence of J and its derivatives if we introduce instead of μ the quantity

$$\lambda = \mu / K . \tag{2.3}$$

This quantity is related to the pitch angles of the particles, and its value decides whether a particle is trapped or not. The J-invariant is now $J = \sqrt{K}\,\bar{J}$, where

$$\bar{J}(\lambda,\psi,\chi) = \oint d\sigma \frac{B}{D}\sqrt{2(1-\lambda B)} . \tag{2.4}$$

From this the derivatives of J are easily computed to be $\partial J/\partial K = \bar{\tau}_B/\sqrt{K}$, and $\partial J/\partial \psi = -\sqrt{K}\,\bar{\tau}_B \bar{v}_\chi$, where

$$\bar{\tau}_B(\lambda,\psi,\chi) = \oint d\sigma \frac{B}{D} \frac{1}{\sqrt{2(1-\lambda B)}} , \tag{2.5}$$

and

$$\bar{v}_\chi(\lambda,\psi,\chi) = -\frac{1}{\bar{\tau}_B}\frac{\partial \bar{J}}{\partial \psi} \tag{2.6}$$

Here the notations have been chosen in order to remind that $\partial J/\partial K$ equals the bounce period τ_B, and that the time averaged change v_χ of χ due to the guiding centre drifts is given by

$$v_\chi = \left\langle \frac{d\chi}{dt} \right\rangle = - \frac{m}{e} \frac{\partial J/\partial \psi}{\partial J/\partial K} = \frac{m}{e} K \bar{v}_\chi .$$

Our second assumption concerns the confining magnetic field. We assume that it possesses some symmetry by which the flux functions can be chosen such that the J-invariant does not depend on χ. Helical symmetry, for example, which contains axial symmetry as well as plane symmetry as limiting cases, serves for this purpose. Familiar examples of configurations to which our theory will apply are axisymmetric mirrors, axisymmetric multipoles or spherators, tokamaks, and the helically symmetric "ideal stellarators". Since symmetry implies that ψ, as well as J, is an integral of the time averaged guiding centre motion, we may write the most general equilibrium distribution functions in the form

$$f_0(\underline{x}, \underline{v}) = F(K, \lambda, \psi) \tag{2.7}$$

The functions G occurring in eq. (1.6) are then given by $G(\mu, J, K) = F(K, \mu/K, \psi(\mu/K, J/\sqrt{K}))$, where the function $\psi(\lambda, \bar{J})$ is the inverse of the function $\bar{J}(\lambda, \psi)$, eq. (2.4).

Note that only in topologically simple configurations, i.e. in open-ended configurations having no internal mirrors, is $\bar{J}$ a unique function of λ and ψ, defined in the range $1/B_{max} \leq \lambda \leq 1/B_{min}$. In general there are several local maxima and minima of B on a field line. Hence one has, in general, several "types" of trapped particles and, accordingly, several different functions $\bar{J}$. Thus, there are several functions $G(\mu, J, K)$ for each particle species, though there is only one function $F(K, \lambda, \psi)$. Which one of these different functions is chosen depends, of course, on the values of the quantities λ and ψ. But it also depends on the neighbowhood of which minimum of B one considers. In closed-line or in sheared configurations there are, in addition, two types of untrapped particles (depending on the sense in which these particles circulate) for which $0 \leq \lambda \leq 1/B$ max. Note that though the function G of one particle species is, in general, discontinuous at some surfaces in (μ, J, K)- space and has different numbers of branches in the different regions separated by these surfaces, the corresponding distribution function f_0 is well-behaved in phase space, provided that the function $F(K,\lambda,\psi)$ is sufficiently smooth. Equation (2.7) implies that the two branches of the function G representing the two types of untrapped particles are equal, because K is quadratic in $v_{\parallel}$. Hence the equilibria considered are to lowest order static, the macroscopic velocity being a first-order quantity.

Without requiring symmetry it would be difficult to construct a class of guiding centre equilibria the distribution functions of which are continuously differentiable, unless one considers only topologically simple configurations. In sheared configurations one would encounter an additional difficulty concerning the J -invariant of untrapped particles. Without symmetry this J -invariant would be an extremely ill-behaved function of ψ and χ, being constant on irrational magnetic surfaces ψ= const, but depending on χ on rational surfaces. This is, as pointed out by Grad (Ref 16), precisely the same sort of difficulty as that occurring in MHD equilibrium theory. Our symmetry requirement has still the further advantage of preventing particles from drifting to critical regions and thus suddenly changing their J -invariant. This is not possible because particles stay on the flux surfaces ψ= const, and because on these surfaces all field lines are equivalent.

We thirdly assume that the monotonicity conditions

$$\frac{\partial F^*}{\partial K} < 0 \qquad (2.8)$$

be satisfied, where the functions F^* are the equilibrium distribution functions, when expressed in terms of the variables K, μ and ψ,

$$F^*(K, \mu, \psi) = F(K, \mu/K, \psi). \qquad (2.9)$$

These conditions are identical with the monotonicity conditions (1.13) of Rutherford and Frieman, which are, as mentioned in Sec. 1, also necessary for local stability according to the guiding centre theory. They have, as proven by Grad (Ref 9), the further property of being sufficient for the self-consistent electric field to be uniquely determined by the functions F, if the magnetic field is given.

We are now prepared to derive a simple constraint on the functions $F(K, \lambda, \psi)$ which is necessary and sufficient for our first assumption to be satisfied.

First let us assume that the electrostatic potential is zero. The volume element in velocity space is then

$$d^3v = 2\pi B \frac{\sqrt{K}}{\sqrt{2(1-\lambda B)}} \, d\lambda \, dK \, .$$

Hence the neutrality condition (1.2) reads

$$\int_0^{1/B} \frac{d\lambda}{\sqrt{1/B - \lambda}} \sum e \int_0^\infty dK \sqrt{K} \, F = 0 \, .$$

This relation has to hold on any field line, and for any value of the coordinate σ, i.e. in the range $B_{min} \leq B \leq B_{max}$ of the upper limit on the λ-integration. Hence in open-ended configurations, in which the integrand is zero for $\lambda < 1/B_{max}$ because of the loss cone, we are left with

$$\int_{1/B_{max}}^{1/B} \frac{d\lambda}{\sqrt{1/B - \lambda}} \sum e \int_0^\infty dK \sqrt{K} \, F = 0 \, , \qquad (2.10)$$

while in toroidal configurations we have the additional constraint

$$\int_0^{1/B_{max}} \frac{d\lambda}{\sqrt{1/B-\lambda}} \sum e \int_0^{\infty} dK \sqrt{K}\, F = 0 \qquad (2.11)$$

on the distribution functions of the untrapped particles. Equation (2.10) implies, as first noted by Persson [18], that

$$\sum e \int_0^{\infty} dK \sqrt{K}\, F = 0 . \qquad (2.12)$$

This relation has to hold for the trapped particles. Equations (2.11-12) are thus necessary for the electric field to be zero. Sufficiency follows from the uniqueness of the electric field, which is ensured by the monotonicity condition (2.8). The constraint (2.12) will later turn out to be essential in order to obtain a variational principle for stability. For this reason we assume it to hold also for untrapped particles in closed-line (but not necessarily in sheared) configurations.

We now give some relations to certain quantities occurring in MHD-theory. First let us consider closed-line configurations and evaluate the quantity

$$q(\psi) = \int_0^{1/B_{min}} d\lambda \sum \tau_B , \qquad (2.13)$$

where the summation runs over the particle types which are possible for the given values of λ and ψ. Using eq. (2.5) we write

$$q = \int_0^{1/B_{min}} d\lambda \int d\sigma \frac{B}{D} \frac{1}{\sqrt{2(1-\lambda B)}} ,$$

where the σ -integration runs over the ranges where $\lambda B \leq 1$. Reversing the order of integration we find

$$q = \oint d\sigma \frac{B}{D} \int_0^{1/B} d\lambda \frac{1}{\sqrt{2(1-\lambda B)}} .$$

Here the σ -integration runs once around the closed field lines. The λ -integration can now be performed to yield

$$q = \oint \frac{ds}{B} , \tag{2.14}$$

where s is the arc length along the field lines. Hence q is the very quantity occurring in MHD equilibrium theory, the pressure being constant on the surfaces q = const. Next let us consider the quantity

$$q' = \int_0^{1/B_{min}} d\lambda \sum \bar{\tau}_B \bar{v}_\chi . \tag{2.15}$$

By a similar procedure we find the identity

$$q' = \frac{dq}{d\psi} . \tag{2.16}$$

Hence q' is the very quantity occurring in low β MHD stability theory, the inequality

$$p'q' < 0 \tag{2.17}$$

being a necessary and sufficient interchange stability criterion. The identities (2.13-16) do not hold in open-ended configurations, because in these the quantities $\bar{\tau}_B$ and $\bar{v}_\chi$ are not defined in the range $0 \leq \lambda \leq 1/B_{max}$. In sheared configurations they are valid if one interprets the σ -integrations in the appropriate way as integrations over magnetic surfaces. The relation to be discussed next deals with a moment of the distribution functions.
Hence it holds for any configuration, the distribution functions being zero whenever the J -invariant is not defined. Considering the moment

$$\bar{p}(\lambda,\psi) = \sum m \int_0^\infty dK \sqrt{K}^3 F , \tag{2.18}$$

we find the identity

$$\int_0^{1/B_{min}} d\lambda\, \bar{p} \sum \bar{\tau}_B = \oint \frac{ds}{B} p \,, \tag{2.19}$$

where p is the mean pressure,

$$p = \sum m \int d^3v\, v^2 f_0 \,. \tag{2.20}$$

If the distribution functions are isotropic in velocity space, i.e. if the functions F do not depend on λ, then the pressure is also isotropic, and equals $\bar{p}$, eq. (2.19) reducing to the identity (2.13-14). The relations (2.18-20) may be generalized to other moments of the distribution functions.

Our fourth assumption reads

$$\left| \frac{\partial J}{\partial \psi} \frac{\partial F^*}{\partial K} \right| \ll \left| \frac{\partial J}{\partial K} \frac{\partial F^*}{\partial \psi} \right| . \tag{2.21}$$

This may be expressed in terms of the average drift velocity

$$v_\chi = - \frac{m}{e} \frac{\partial J/\partial \psi}{\partial J/\partial K} \tag{2.22}$$

and a generalized diamagnetic velocity

$$v_c = - \frac{m}{e} \frac{\partial F^*/\partial \psi}{\partial F^*/\partial K} \tag{2.23}$$

as

$$| v_\chi | \ll | v_c | \,. \tag{2.24}$$

The physical content of this assumption is that the scale lengths across the flux surfaces of the distribution functions are small compared with a certain average scale length of the magnetic field. This also holds,of course, for the scale lengths of any moments of the distribution functions. In the isotropic case the latter fact can be formulated more precisely by using the relations derived above. First note that in this case the inequality (2.21) may be rewritten as

$$\left| K \frac{\partial F}{\partial K} \bar{v}_\chi \right| \ll \left| \frac{\partial F}{\partial \psi} \right| .$$

Multiplying this by mK , summing over particle species, and integrating with respect to K, we obtain, after an integration by parts of the l.h.s.

$$|\bar{v}_\chi p| \ll |p'| ,$$

where the prime denotes differentiation with respect to ψ. Multiplying by $\bar{\tau}_B$, summing over particle types, and integrating with respect to λ, we finally obtain

$$|q'p| \ll |p'q| . \qquad (2.25)$$

This relation holds also for any other moments of the distribution functions.

We finally assume that the sufficient interchange stability criterion (1.15) is satisfied for those types of equilibria to which it is relevant, i.e. we assume it to be satisfied for closed-line configurations and for open-ended configurations with insulating and plates ("interchange equilibria"), but not necessarily for sheared configurations and for open-ended configurations with perfectly condicting walls (tied magnetic field lines). Using the notations introduced above and employing the inequality (2.21), we rewrite this criterion as

$$\int_0^{1/B_{min}} d\lambda \frac{\partial \bar{p}}{\partial \psi} \bar{\tau}_B \bar{v}_\chi < 0 .$$

This form applies, however, only to topologically simple configurations. In general, the appropriate form is

$$\int_0^{1/B_{min}} d\lambda \frac{\partial \bar{p}}{\partial \psi} \sum \bar{\tau}_B \bar{v}_\chi < 0 . \qquad (2.26)$$

For isotropic distribution functions this reduces to the MHD stability criterion (2.17), because $\bar{p}$ does not depend on and because of the identity (2.15-16). This equivalence of guiding centre stability theory and MHD stability theory holds only for the special class of equilibria introduced in this section. It may be used to get a further conclusion from the inequality (2.25). Regarding the pressure p as being a function of q and observing that the MHD stability condition then reads $dp/dq < 0$, we may rewrite this inequality as $-dp/dq \gg p/q$. By integration we conclude $pq \ll p_{max}\, q_{min}$, which is obviously consistent with (though independ of) the low β limit and with confinement.

3. Linearized dynamics

In the range of frequencies in which J is invariant the distribution functions have, as shown, for example by Hastie et al. (Ref 1), to lowest order the form

$$f(\underline{x},\underline{v},t) = g(\mu, J, \psi, \chi, t) \tag{3.1}$$

There is one function g for every particle type of every particle species, i.e. for particles being trapped between the same mirrors or circulating in the same sense, and having the same mass and charge. Since in the perturbed motion we have to allow for an electric field, the J-invariant is now given by

$$J(K,\mu,\psi,\chi,t) = \oint d\sigma \frac{B}{D}\sqrt{2\left(K-\mu B-\frac{e}{m}\varphi\right)} \tag{3.2}$$

The lowest non-trivial order of the Vlasov equations reads now

$$\frac{\partial g}{\partial t} = \frac{m}{e}\left(\frac{\partial g}{\partial \psi}\frac{\partial K}{\partial \chi} - \frac{\partial g}{\partial \chi}\frac{\partial K}{\partial \psi}\right), \tag{3.3}$$

where the Hamiltonian $K(\mu, J, \psi, \chi)$ is to be computed by inverting eq. (3.2).

Let us linearize eqs. (3.1-3) about some equilibrium. We then introduce a small parameter ϱ by writing

$$g(\mu,\mathcal{J},\psi,\chi,t)=G(\mu,\mathcal{J},K(\mu,\mathcal{J},\psi,\chi,t)+\varrho\, g_1(\mu,\mathcal{J},\psi,\chi,t),$$

$$\varphi(\psi,\chi,\sigma,t)=\varrho\,\varphi_1(\psi,\chi,\sigma,t),$$

and expand all quantities up to first order in ϱ. As the functions $\mathcal{J}$ and K depend on the perturbing electrostatic potential φ_1, they have to be expanded too. Care has to be taken with regard to the independent variables, which may be either μ, K, ψ, χ, t, or $\underline{x}, \underline{v}, t$. In the first case the perturbation to the equilibrium $\mathcal{J}$-invariant $\mathcal{J}_0$ is

$$\mathcal{J}_1(K,\mu,\psi,\chi,t)=-\frac{e}{m}\frac{\partial \mathcal{J}_0}{\partial K}\langle\varphi_1\rangle \tag{3.4}$$

inversion of which yields

$$K_1(\mu,\mathcal{J},\psi,\chi,t)=\frac{e}{m}\langle\varphi_1\rangle. \tag{3.5}$$

Here $\langle\cdots\rangle$ denotes the time average over the unperturbed quasi-periodic longitudinal guiding centre orbits,

$$\langle\varphi_1\rangle=\left(\frac{\partial \mathcal{J}_0}{\partial K}\right)^{-1}\oint d\sigma\,\frac{B}{D}\frac{\varphi_1}{\sqrt{2(K-\mu B)}}\,. \tag{3.6}$$

In the second case the perturbation to K is simply $K_1(\underline{x},\underline{v},t)=\frac{e}{m}\varphi_1$, which gives rise to a term to be added to the r.h.s. of eq. (3.4). The result is

$$\mathcal{J}_1(\underline{x},\underline{v},t)=\frac{e}{m}\frac{\partial \mathcal{J}_0}{\partial K}(\varphi_1-\langle\varphi_1\rangle). \tag{3.7}$$

Substitution now yields

$$f = g + \frac{e}{m}\frac{\partial F^*}{\partial K}(\varphi - \langle\varphi\rangle), \tag{3.8}$$

$$\frac{\partial g}{\partial t} = \left(\frac{\partial F^*}{\partial K}\frac{\partial K}{\partial \psi} + \frac{\partial F^*}{\partial \psi}\right)\frac{\partial\langle\varphi\rangle}{\partial \chi} - \frac{m}{e}\frac{\partial K}{\partial \psi}\frac{\partial g}{\partial \chi}. \tag{3.9}$$

Here the suffixes have been dropped, i.e. f, g, and φ are perturbing quantities, and the functions J and K refer to the equilibrium state. In the linearized Vlasov equations (3.9) some terms have vanished by symmetry.

It should be mentioned that in the derivation of eq. (3.4) the effects of the perturbing potential on the limits of the σ -integration have been neglected. There seems to be no easy way to include these effects, and it also seems to be impossible to prove them to be negligible. They actually destroy the J -invariant of some particles being reflected by any interior mirrors arising from the perturbation.

Let us look for "normal modes" of the form

$$\varphi(\psi,\chi,\sigma,t) = \tilde{\varphi}(\psi,\sigma)\, e^{i(\omega t - k\chi)}. \tag{3.10}$$

The solutions of the Vlasov equations are then

$$g = -\frac{\frac{\partial F^*}{\partial K}\frac{\partial K}{\partial \psi} + \frac{\partial F^*}{\partial \psi}}{\frac{\omega}{k} - \frac{m}{e}\frac{\partial K}{\partial \psi}}\langle\tilde{\varphi}\rangle e^{i(\omega t - k\chi)} \tag{3.11}$$

Note that in toroidal configurations one has to require that φ be a univalued function in space. This, together with the ansatz (3.10), implies that neither any choice of the

(ψ, χ, σ)-coordinates nor any choice of the Fourier coefficient $\tilde{\varphi}$ is allowed. As to the coordinates, the contours ψ = const, σ = const have to be closed once around the torus, χ increasing by 2π. Hence the quantity $2\pi\psi$ is some magnetic flux. In spherators and in tokamaks, for example, the quantities $2\pi\psi$ and σ may be chosen to be the flux and the potential of the poloidal component of the magnetic field, χ being the angle around the symmetry axis. As to the functions $\tilde{\varphi}$, the appropriate constraint arises from the identity

$$\varphi(\psi, \chi - \chi^*, \sigma + \sigma^*, t) = \varphi(\psi, \chi, \sigma, t), \tag{3.12}$$

where σ^* is the periodicity interval of the coordinate σ, and χ^* is a generalized rotational transform. Equation (3.12) implies

$$\tilde{\varphi}(\psi, \sigma) = \hat{\varphi}(\psi, \sigma) \exp\left[-ik\frac{\chi^*}{\sigma^*}\sigma\right], \tag{3.13}$$

where $\hat{\varphi}$ is an arbitrary function periodic in σ,

$$\hat{\varphi}(\psi, \sigma + \sigma^*) = \hat{\varphi}(\psi, \sigma). \tag{3.14}$$

In closed-line and in open-ended configuration χ^* is zero. In the latter $\hat{\varphi}$ need not be periodic.

Equations (3.13-14) imply that for untrapped particles in sheared configurations the time average (3.6) of φ is zero. Hence the perturbing distribution function g (but not f!) of these particles are also zero, consistent with the fact that they are constant on the field lines which cover the magnetic surfaces ergodically. For trapped particles the functions g may depend on χ because the functions f need be defined only in the regions where these particles orbit.

We must, of course, ensure that the $\tilde{J}$-invariant of untrapped particles in sheared configurations is indeed conserved by the perturbed motion. This requires that during one period of the oscillation these particles effectively sample a representative part of their flux surfaces, thus providing a restriction involving the number k, which, on the other hand, is also restricted by the requirement that the scale length of the perturbing potential be large compared with the Larmor radius. An admitted range for the number k will, in general, only be left if too large and too small values of the rotational transform are excluded. The proof depends on the special configuration considered and will not be given here.

Substitution of the solutions (3.11) of the Vlasov equations into eq. (3.8) yields

$$f = \frac{e}{m}\frac{\partial F^*}{\partial K}\exp\left[i\left(\omega t - k\chi - k\frac{\chi^*}{\sigma^*}\sigma\right)\right]\times$$
$$\left(\hat{\varphi} - \frac{\omega - k v_c}{\omega - k v_\chi}\left\langle \hat{\varphi}\, e^{-ik\frac{\chi^*}{\sigma^*}\sigma}\right\rangle e^{ik\frac{\chi^*}{\sigma^*}\sigma}\right). \qquad (3.15)$$

for the perturbing distribution functions of the diverse particle types in phase space. The velocities v_c and v_χ are defined by eqs. (2.22-23). The first-order charge neutrality condition is now formed from eq. (3.15) by summing over particle types, integrating over the whole velocity space, multiplying with the particle charge, and then summing over species. The result is

$$\sum \frac{e^2}{m}\int \frac{B\, d\mu\, dK}{\sqrt{K-\mu B}}\frac{\partial F^*}{\partial K}\times$$
$$\sum\left(\hat{\varphi} - \frac{\omega - k v_c}{\omega - k v_\chi}\left\langle \hat{\varphi}\, e^{-ik\frac{\chi^*}{\sigma^*}\sigma}\right\rangle e^{ik\frac{\chi^*}{\sigma^*}\sigma}\right) = 0. \qquad (3.16)$$

This is a non-symmetric linear integral equation for the electrostatic potential. For a given number k there will, in general, only be certain complex eigenvalues ω for which this equation possesses non-trivial solutions. Hence it serves as a dispersion relation. The boundary conditions depend on the special configuration considered. Any singularities arising from the vanishing of the denominator are handled in the usual way by approaching from below in the complex ω-plane. One proves this by performing a Fourier-Laplace transformation instead of simply considering the pure modes (3.10).

4. Stability

We regard the equilibrium as stable to the motions considered if, and only if, the integral equation (3.16) possesses no eigenvalues ω with negative imaginary parts. In order to get a self-adjoint problem we consider only the range of "phase velocities" ω/k restricted by

$$|v_x| \ll |\omega/k| \ll |v_c| . \tag{4.1}$$

We thus exclude resonances with the particle drifts and with the diamagnetic velocities, which is made possible by the assumption (2.24). The integral equation (3.16) is now expanded according to

$$\frac{\omega - k v_c}{\omega - k v_x} = -\frac{k v_c}{\omega} + 1 - \frac{k^2 v_x v_c}{\omega^2} + O\left(\frac{k v_x}{\omega}\right) .$$

The dominant part may then be written

$$\omega^2 P \hat{\varphi} = A \hat{\varphi} + \omega L \hat{\varphi} , \tag{4.2}$$

where the operators P, A, and L are given by

$$P\hat{\varphi} = -\sum \frac{e^2}{m} \int \frac{B\, d\mu\, dK}{\sqrt{K-\mu B}} \frac{\partial F^*}{\partial K} \sum \left(\hat{\varphi} - \langle \hat{\varphi}\, e^{-ik\frac{X^*}{\sigma^*}\sigma} \rangle e^{ik\frac{X^*}{\sigma^*}\sigma}\right) ,$$

$$\tag{4.3}$$

$$A\hat{\varphi} = k^2 \sum m \int \frac{B\,d\mu\,dK}{\sqrt{(K-\mu B)}} \frac{\partial F^*}{\partial \psi} \sum \frac{\partial J/\partial \psi}{\partial J/\partial K} \left\langle \hat{\varphi} e^{-ik\frac{X^*}{\sigma^*}\sigma} \right\rangle e^{ik\frac{X^*}{\sigma^*}\sigma} , \tag{4.4}$$

$$L\hat{\varphi} = -k \sum e \int \frac{B\,d\mu\,dK}{\sqrt{K-\mu B}} \frac{\partial F^*}{\partial \psi} \sum \left\langle \hat{\varphi} e^{-ik\frac{X^*}{\sigma^*}\sigma} \right\rangle e^{ik\frac{X^*}{\sigma^*}\sigma} . \tag{4.5}$$

We first show that the operator L is zero by the equilibrium charge neutrality. Using the fact that the time average of the potential does not depend on K when the variable λ is introduced, we write

$$\left\langle \hat{\varphi} e^{-ik\frac{X^*}{\sigma^*}\sigma} \right\rangle = \bar{\varphi}(\lambda, \psi) . \tag{4.6}$$

The operator L is then

$$L\hat{\varphi} = -k e^{ik\frac{X^*}{\sigma^*}\sigma} \int \frac{d\lambda}{\sqrt{1-\lambda B}} \sum \bar{\varphi} \sum e \int dK \sqrt{K} \frac{\partial F}{\partial \psi} . \tag{4.7}$$

By the version (2.12) of equilibrium neutrality, which was provided by the vanishing of the electric field, the integrand in eq. (4.7) is zero. One can show that the term $\omega L\hat{\varphi}$ in eq. (4.2) dominates over the other two terms if the equilibrium electric field does not vanish, and if it is as large as allowed by the drift scaling (1.5), and that in that case the eigenvalues do not satisfy the assumption (4.1).

Henceforth, we deal with the eigenvalue problem

$$\omega^2 P\hat{\varphi} = A\hat{\varphi} . \tag{4.8}$$

The operators P and A are easily shown to be symmetric according to the inner product

$$(\hat{\varphi}_1, \hat{\varphi}_2) = \int \frac{d\psi\, d\sigma}{D} \bar{\hat{\varphi}}_1 \hat{\varphi}_2 . \tag{4.9}$$

The (real) quadratic forms of these operators are

$$(\hat{\varphi}, A\hat{\varphi}) = k^2 \sum m \int d\psi\, d\mu\, dK \frac{\partial F^*}{\partial \psi} \sum \frac{\partial y}{\partial \psi} \left|\left\langle \hat{\varphi} e^{-ik\frac{\chi^*}{\sigma^*}\sigma} \right\rangle\right|^2 \tag{4.10}$$

$$(\hat{\varphi}, P\hat{\varphi}) = -\sum \frac{e^2}{m} \int d\psi\, d\mu\, dK \frac{\partial F^*}{\partial K} \sum \frac{\partial y}{\partial K} \left(\langle |\hat{\varphi}|^2 \rangle - \left|\left\langle \hat{\varphi} e^{-ik\frac{\chi^*}{\sigma^*}\sigma} \right\rangle\right|^2 \right). \tag{4.11}$$

The latter is positive semi-definite because of our assumption that F^* is monotonic in energy. Hence a variational principle holds which states that the eigenvalues ω^2 of eq. (4.8) are given by the stationary values of the functional

$$W[\hat{\varphi}] = \frac{(\hat{\varphi}, A\hat{\varphi})}{(\hat{\varphi}, P\hat{\varphi})} . \tag{4.12}$$

A rough estimate shows that $\omega^2 \sim k^2 v_c v_\chi$ in agreement with the assumption (4.1). The lowest eigenvalue, the sign of which decides the question of stability, is given by the absolute minimum of W . Hence we have stability if the numerator is positive definite, and instability if it is indefinite. For the marginal case no conclusion can be drawn within the present approximation. It will be shown later that the variational functional is bounded from below. This is necessary in order that the above conclusion with respect to instability, which is based on the existence of a minimum, really holds.

In order to arrive at a stability criterion we rewrite the numerator as

$$(\hat{\varphi}, A\hat{\varphi}) = -k^2 \int d\psi\, d\lambda \frac{\partial \bar{p}}{\partial \psi} \sum \tau_B \bar{v}_\chi |\bar{\varphi}|^2 , \tag{4.13}$$

where some of the relations of Sec. 1 have been used. From this it follows that

$$\frac{\partial \bar{p}}{\partial \psi} \bar{\sigma}_\chi < 0 \qquad (4.14)$$

is necessary and sufficient for stability. This inequality has of course to be satisfied only for particles that really contribute to the quadratic form (4.13). It thus need not hold for untrapped particles in sheared configurations, because for these $\bar{\varphi}$ equals zero. The sufficiency of condition (4.14) is obvious. As to its necessity, one has to show that a trial function $\hat{\varphi}$ existes for which $(\hat{\varphi}, A\hat{\varphi}) < 0$, if the condition (4.14) is violated by some particles. Suppose $\bar{\sigma}_\chi \partial\bar{p}/\partial\psi > 0$ in the neighbowhood of some values $\psi = \psi_0, \lambda = \lambda_0$. Consider a "function" $\hat{\varphi}_0$ which is proportional to $\delta(\sigma - \sigma_0)$ near ψ_0, and zero elsewhere. Here σ_0 has to be chosen such that $\lambda_0 B(\sigma_0) = 1$. For this function the numerator of the variational expression is minus infinity, while the denominator is not defined at all. The function $\hat{\varphi}_0$ is, of course, not admitted as a trial function. But one may nevertheless believe that a trial function exists for which the numerator is finite and negative, the denominator being defined, and that the corresponding unstable mode is localized at the turning points of the particles violating the stability condition. Strictly speaking, in order to complete the proof of instability, one has to show that the eigenfunction belonging to the lowest eigenvalue is consistent with all assumptions made above. One should also estimate the effects of the internal mirrors developing at the points where the unstable mode is localized. We leave these problems to further investigations.

There is another unsolved question concerning the relevance to untrapped particles in closed-line configurations of the stability condition. These particles certainly contribute to the quadratic form (4.13). But since they have no turning points, it is not clear whether there is an instability in the special situation that some untrapped particles violate the stability condition, while all trapped particles satisfy it. One may even ask whether this situation is at all possible in any interesting configuration.

We prove now that the variational functional W cannot tend to minus infinity. This could happen if the numerator had negative values in the null space of the denominator. By the Schwartz inequality this null space consists of all admitted trial functions $\hat{\varphi}$ proportional to $\exp[ik \frac{\chi^*}{\sigma^*}\sigma]$, the coefficient depending on ψ only. In sheared configurations these functions are excluded by the periodicity requirement (3.14). In other configurations the exponential factor equals one, and the null space is simply formed by all functions $\hat{\varphi}(\psi)$ i.e. by all perturbing electric fields having no component along the magnetic field lines. Hence in closed-line configurations there is always a non-trivial null space. In open-ended configurations this depends on the boundary conditions. If φ is to be constant at the boundaries (tied magnetic field lines), then there is again no null space, while in interchange equilibria purely transverse electric fields are admitted. Collecting these results we state that the denominator is positive definite in sheared configurations and in open-ended configurations with tied field lines, and that it is positive semi-definite for closed-line configurations and for open-ended interchange equilibria, being zero only for purely transverse perturbations $\hat{\varphi}(\psi)$. These trial functions are obviously identical with their time averages, which thus do not depend on λ. Hence in the null space of the denominator the numerator may be written:

$$(\hat{\varphi}, A\hat{\varphi}) = -k^2 \int d\psi \, |\hat{\varphi}|^2 \int d\lambda \, \frac{\partial \bar{P}}{\partial \psi} \sum \bar{\tau}_B \bar{\upsilon}_\chi ,$$

and the necessary and sufficient condition for the numerator to be positive whenever the denominator vanishes is identical with the interchange stability condition (2.26), which we have already assumed to be satisfied precisely by those equilibria for which the denominator has a null space.

There might be cases in which the numerator is minus infinity for some admitted trial functions because the quantity $\bar{\tau}_B \bar{v}_\chi \partial \bar{p}/\partial \psi$ is plus infinity for some values of ψ and λ. This situation occurs if the stability condition is violated by some barely trapped particles, i.e. by particles having infinite bounce period τ_B. In this case the denominator is, however, likely to diverge, too, such that the ratio remains finite. We do not try to discuss this situation further because here just those particles play a dominant role which are not correctly described within the drift theory, their $\hat{J}$ -invariant being violated by any time dependent perturbation.

The stability condition (4.14) requires that particles have favourable average drifts. More precisely, it requires that the average drifts of all particles belonging to the same species, orbiting on the same flux surfaces and having the same pitch, have the same sign, this sign being given by the sign of the gradient of the quantity $\bar{p}$, which is related to the mean pressure by eq. (2.19). For the sub-class of equilibria having isotropic distribution functions this sign does not depend on the pitch angle. Hence in this case the signs of the average drifts are required to be the same for any pitch angle. This, together with the requirement of confinement of course, is a rather severe restriction on the magnetic field. This restriction can obviously be relaxed by allowing for anisotropy.

For the class of equilibria considered in this paper the sufficient low frequency interchange stability criterion (1.7) of Rutherford and Frieman reads

$$\frac{\partial F}{\partial \psi} \bar{v}_\chi < 0 . \tag{4.15}$$

This is, as it has to be, in general more restrictive than the necessary criterion (4.14), the latter being essentially an integral over the former. The two criteria coincide if $\frac{\partial \bar{p}}{\partial \psi}\frac{\partial F}{\partial \psi} > 0$ for any values of K, λ, and ψ . This happens if the distribution functions are sufficiently smooth. The functions $F \sim p(\psi)\exp[-mK/T]$, for example, which were considered by Rosenbluth (Ref 3), have this property.

Since the sufficient interchange stability criterion (2.26) is an integral over the necessary low frequency interchange stability criterion derived in this paper, the former is less restrictive than the latter. Hence we have proven that low frequency interchange stability implies interchange stability.

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