

I N S T I T U T F Ü R P L A S M A P H Y S I K
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Non-linear Quasi-neutral Electrostatic
Plasma Waves and Shock Waves

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ABSTRACT

For the stationary one-dimensional Vlasov equation, quasi-neutrality is used instead of the Poisson equation. This allows one to obtain explicit solutions for the density as a function of the potential and to prove the positiveness of the trapped particle distribution function in a potential interval. The usefulness of this method for oscillating waves and shock-waves is obvious.

The non-linear electrostatic Vlasov equation has been studied in the stationary one-dimensional case by Bernstein, Green and Kruskal^[1] using the Poisson equation. The solutions were obtained in two different ways: 1) The full distribution function was given, the density of the ions and electrons was then calculated as a function of the electrostatic potential, and the Poisson equation allowed the potential to be determined. 2) The potential and the distribution functions for all ions (trapped and free) and free electrons were given; an integral equation then allowed the distribution function for the trapped electrons to be calculated. For each solution of 2) it was necessary to verify that this latter calculated function was positive.

This work will show that, by the second method, a wide class of solutions is obtained, namely the quasi-neutral solutions, for which the positiveness condition is automatically satisfied in a potential interval. The existence of such solutions is based on the non-monotony of the distribution functions with respect to the total energy of a particle. Furthermore, quasi-neutrality imposes a maximum on the relative amplitude of the potential.

The basic equations for stationary quasi-neutral waves are:

$$(1) \quad v \frac{\partial f_{\pm}(x, v)}{\partial x} \mp \frac{e}{m_{\pm}} \frac{\partial \psi}{\partial x} \frac{\partial f_{\pm}(x, v)}{\partial v} = 0$$

$$(2) \quad \int_{-\infty}^{+\infty} f_{\pm} dv = \int_{-\infty}^{+\infty} f_{\pm} dv \quad (\text{Quasi-neutrality})$$

f is the original distribution in v_x, v_y, v_z space integrated over v_y and v_z ; the subscript of v_x has been dropped. The general solution of eq. (1) is:

$$(3) \quad f_{\pm} = f_{\pm}(E_{\pm})$$

where
$$E_{\pm} = \frac{1}{2} m_{\pm} v^2 \pm e\phi$$

Substituting eq. (3) in eq. (2) and denoting "trapped" and "untrapped" by the subscripts t and u, one gets:

$$(4) \quad \int_{-e\phi}^{-e\phi_{\min}} dE f_{\pm t}(E) [2m_{\pm}(E+e\phi)]^{-\frac{1}{2}} = g(e\phi)$$

$$(5) \text{ where } g(e\phi) = \int_{e\phi}^{\infty} dE f_{\pm}(E) [2m_{\pm}(E-e\phi)]^{-\frac{1}{2}} - \int_{-e\phi_{\min}}^{\infty} dE f_{\pm u}(E) [2m_{\pm}(E+e\phi)]^{-\frac{1}{2}}$$

According to eq. (4) $g(e\phi_{\min}) = 0$. When $f_{\pm t}(E)$ and $f_{\pm u}(E)$ are given, eq. (4) is an integral equation of the convolution type and following B.G.K.^[1] the solution is obtained by way of a Laplace transform:

$$(6) \quad f_{-t}(E) = \frac{(2m_-)^{1/2}}{\pi} \int_{e\phi_{\min}}^{-E} dv \frac{dg(v)}{dv} [-(E+v)]^{-1/2}$$

The derivative of $g(e\phi)$, $\frac{dg}{d(e\phi)} \rightarrow +\infty$, when $e\phi \rightarrow e\phi_{\min}$. This is readily seen from eq. (5) if integration over E is replaced by integration over v .

Equation (5) could then be written

$$(7) \quad g(e\phi) = \int_{-\infty}^{+\infty} f_{+} dv - \int_{-\infty}^{\infty} \frac{f_{-u}(\sigma > 0)}{\sqrt{\frac{2e}{m_-}(\phi - \phi_{\min})}} dv - \int_{-\infty}^{-\sqrt{\frac{2e}{m_-}(\phi - \phi_{\min})}} f_{-u}(\sigma < 0) dv$$

where $\sigma = \text{sign } v$

$$(8) \quad \frac{dg}{d(e\phi)} = \int_{-\infty}^{+\infty} \frac{\partial f_{+}}{\partial(e\phi)} dv - \int_{-\infty}^{\infty} \frac{\partial f_{-u}(\sigma > 0)}{\partial(e\phi)} \frac{1}{\sqrt{\frac{2e}{m_-}(\phi - \phi_{\min})}} dv - \int_{-\infty}^{-\sqrt{\frac{2e}{m_-}(\phi - \phi_{\min})}} \frac{\partial f_{-u}(\sigma < 0)}{\partial(e\phi)} dv +$$

$$+ \frac{1}{\sqrt{2m_-}} \cdot \frac{1}{\sqrt{e(\phi - \phi_{\min})}} \int_{-u} \left(v = \pm \sqrt{\frac{2e}{m_-}(\phi - \phi_{\min})} \right)$$

Let us assume v integrability for $\frac{\partial f_{+}}{\partial(e\phi)}$ and $\frac{\partial f_{-u}}{\partial(e\phi)}$

and that $\int_{-u} f_{-u}(\sigma > 0) = \int_{-u} f_{-u}(\sigma < 0)$ for $e\phi = e\phi_{\min}$

(The latter assumption was not made in a recent work of Montgomery and Joyce^[2]).

The first three terms of eq. (8) are then finite for $e\phi \rightarrow e\phi_{\min}$. The last term goes to $+\infty$ when $e\phi \rightarrow e\phi_{\min}$, and makes the main contribution to eq. (6) when $E \rightarrow -e\phi_{\min}$. Calculating this contribution one gets $f_{-t}(E) \rightarrow f_{-u}(E)$ when $E \rightarrow -e\phi_{\min}$. This means that there must be at least a finite interval where $f_{-t} > 0$.

Finally, according to eq. (5), $g(e\phi)$ becomes negative if $e\phi$ is above a certain $e\phi_{\max}$; because of eq. (4) f_{-t} is no longer positive anywhere.

One example

Let us now illustrate the preceding remarks taking a simple example:

$$\text{for ions } f_{+t} = a_+ e^{-E/kT_i}, \quad f_{+u}(\sigma > 0) = a_+ e^{-E/kT_i}, \quad f_{+u}(\sigma < 0) = a_+ e^{-[E/kT_i + \alpha_i/kT_i](E - e\phi_{\max})}$$

(9)

$$\text{for electrons } f_{-u}(\sigma > 0) = a_- e^{-E/kT_e}, \quad f_{-u}(\sigma < 0) = a_- e^{-[E/kT_e + \alpha_e/kT_e](E + e\phi_{\min})}$$

The choice of α_i and α_e determines the average velocity of every species. Let us calculate $g(e\phi)$ for this case:

$$g(e\phi) = \frac{2a_+}{\sqrt{2m_+}} e^{-\frac{e\phi}{kT_i}} \int_0^{\sqrt{e(\phi_{\max} - \phi)}} dy e^{-\frac{y^2}{kT_i}} + \frac{a_+}{\sqrt{2m_+}} e^{-\frac{e\phi}{kT_i}} \int_{\sqrt{e(\phi_{\max} - \phi)}}^{\infty} dy e^{-\frac{y^2}{kT_i}} +$$

(10)

$$+ \frac{a_+}{\sqrt{2m_+}} \exp\left[-\frac{e\phi}{kT_i} + \frac{\alpha_i e}{kT_i} (\phi - \phi_{\max})\right] \times \int_{\sqrt{e(\phi_{\max} - \phi)}}^{\infty} dy \exp\left[-\frac{y^2}{kT_i} (1 + \alpha_i)\right]$$

$$-\frac{a_-}{\sqrt{2m_-}} \frac{e^{\phi}}{e^{\frac{\phi}{kT_e}}} \int_{\sqrt{e(\phi-\phi_{\min})}}^{\infty} dy e^{-\frac{y^2}{kT_e}} + \frac{a_-}{\sqrt{2m_-}} \exp\left[\frac{e\phi}{kT_e} + \frac{\alpha_e}{kT_e} e(\phi-\phi_{\min})\right] \times \int_{\sqrt{e(\phi-\phi_{\min})}}^{\infty} dy \exp\left[-\frac{y^2}{kT_e} (1+\alpha_e)\right]$$

The condition $g(e\phi_{\min}) = 0$ gives the following relation between a_+ and a_- .

$$(11) \quad a_- = \frac{2a_+ \sqrt{\frac{m_-}{m_+}}}{1 + \sqrt{\frac{1}{1+\alpha_e}}} \frac{e^{-\frac{e\phi_{\min}}{kT_e}}}{\sqrt{\pi kT_e}} \left\{ e^{-\frac{e\phi_{\min}}{kT_e}} \left[2 \int_0^{\sqrt{e(\phi_{\max}-\phi_{\min})}} dy \exp\left[-\frac{y^2}{kT_e}\right] + \int_{\sqrt{e(\phi_{\max}-\phi_{\min})}}^{\infty} dy \exp\left[-\frac{y^2}{kT_e}\right] \right] + \exp\left[-\frac{e\phi_{\min}}{kT_e} (1+\alpha_i) + \alpha_i \frac{e\phi_{\max}}{kT_e}\right] \times \int_{\sqrt{e(\phi_{\max}-\phi_{\min})}}^{\infty} dy \exp\left[-\frac{y^2}{kT_e} (1+\alpha_i)\right] \right\}$$

Equation (6) is now used to calculate $f_t(E)$. Changing the integration variable V in $u = \sqrt{-(E+V)}$, we get:

$$(12) \quad f_t(E) = a_- e^{\frac{e\phi_{\min}}{kT_e}} + \frac{(2m_-)^{1/2}}{\pi} \int_{\sqrt{-(E+e\phi_{\min})}}^0 du \left\{ \frac{4a_+}{\sqrt{2m_+}} \frac{e^{E/kT_e}}{kT_e} e^{\frac{u^2}{kT_e}} \int_0^{\sqrt{e\phi_{\max}+u^2+E}} dy e^{-\frac{y^2}{kT_e}} + \right.$$

$$\left. + \frac{2a_+}{\sqrt{2m_+}} \frac{e^{E/kT_e}}{kT_e} e^{\frac{u^2}{kT_e}} \int_{\sqrt{e\phi_{\max}+u^2+E}}^{\infty} dy e^{-\frac{y^2}{kT_e}} + 2a_+ \frac{(1+\alpha_i)}{\sqrt{2m_+} kT_e} \exp\left[\frac{E}{kT_e} (1+\alpha_i) + \frac{\alpha_i}{kT_e} e\phi_{\max}\right] \times \right.$$

$$\left. \exp\left[\frac{u^2}{kT_e} (1+\alpha_i)\right] \int_{\sqrt{e\phi_{\max}+u^2+E}}^{\infty} dy \exp\left[-\frac{y^2}{kT_e} (1+\alpha_i)\right] + \frac{2a_-}{\sqrt{2m_-}} \frac{e^{-E/kT_e}}{kT_e} e^{-\frac{u^2}{kT_e}} \int_{\sqrt{-(u^2+E+e\phi_{\min})}}^{\infty} dy e^{-\frac{y^2}{kT_e}} \right.$$

$$+ \frac{2a_-}{\sqrt{2m_-}} \frac{(1+\alpha_e)}{kT_e} \exp\left[-\frac{E}{kT_e}(1+\alpha_e) + \frac{\alpha_e e \phi_{\min}}{kT_e}\right] e^{-\frac{u^2(1+\alpha_e)}{kT_e}} \int_{-\infty}^{\infty} dy \exp\left[-\frac{y^2}{kT_e}(1+\alpha_e)\right] \left. \vphantom{\int_{-\infty}^{\infty}} \right\} \\ \sqrt{-(u^2 + E + e\phi_{\min})}$$

We see that for $E = -e\phi_{\min}$, $f_{-t}(E) = f_{-u}(E) = a_- e^{\frac{e\phi_{\min}}{kT_e}}$.

The integral term in (12) becomes negative when $E < -e\phi_{\min}$.

It is interesting to estimate at what value of E , f_{-t} ceases to be positive. We now substitute $E = -e\phi_{\max}$ in eq. (12), eliminate a_- using eq. (11), and expand in $\sqrt{\frac{e(\phi_{\max} - \phi_{\min})}{kT}}$ (with $T_i = T_e$).

After complicated but straight forward calculations, we obtain up to the first order in $\sqrt{\frac{e(\phi_{\max} - \phi_{\min})}{kT}}$

$$(13) \quad \sqrt{\frac{e(\phi_{\max} - \phi_{\min})}{kT}} \leq \frac{\sqrt{\pi}}{2 + \sqrt{1+\alpha_i} + \sqrt{1+\alpha_e}}$$

The equality gives the value of $e(\phi_{\max} - \phi_{\min})$ at which f_{-t} ceases to be positive.

The results are all independent of the form of $\phi(x)$. $\phi(x)$ can be an oscillating function or can have a shock-like form. The density $n(\phi)$ is given by the a_+ terms of eq. (10) in the first example. In the case of static equilibrium $\alpha_i = \alpha_e = 0$, $n(\phi) = A e^{-\frac{e\phi}{kT_i}}$, which is simply the barometric formula.

When α_i and $\alpha_e \rightarrow \infty$, $e(\phi_{\max} - \phi_{\min}) \rightarrow 0$ in which case the average velocity is approximately the thermal velocity, this seems to be a limitation for the Mach number. This is due to the thermal type of the free distribution functions (9) whose derivatives relative to $e\phi$ becomes infinite for $\alpha_{i,e} \rightarrow 0$, in contradiction with previous assumptions. It is easy to find an example which does not behave so:

$$\begin{aligned}
 f_{+u}(\sigma > 0) &= a_+ \exp - \left[\frac{E - E_{0i}}{\theta_i} \right]^2 \\
 f_{+u}(\sigma < 0) &= a_+ \exp - \left[\frac{E - e\phi_{\max}}{\theta_i} + \left(\frac{e\phi_{\max} - E_{0i}}{\theta_i} \right)^2 \right] \\
 f_{+t} &= a_+ \exp - \left[\frac{E - e\phi_{\max}}{\theta_i} + \left(\frac{e\phi_{\max} - E_{0i}}{\theta_i} \right)^2 \right] \\
 f_{-u}(\sigma > 0) &= a_- \exp - \left[\frac{E - E_{0e}}{\theta_e} \right]^2 \\
 f_{-u}(\sigma < 0) &= a_- \exp - \left[\frac{E + e\phi_{\min}}{\theta_e} + \left(\frac{e\phi_{\min} + E_{0e}}{\theta_e} \right)^2 \right]
 \end{aligned}
 \tag{14}$$

This choice allows any Mach number and the derivatives of the chosen

functions cease to be v integrable only for infinite Mach number.

Then for any Mach number there must be a finite interval $[e\phi_{\max}, e\phi_{\min}]$ where a quasi-neutral solution is possible.

CONCLUSION

The use of quasi-neutrality instead of the Poisson equation for non-linear electrostatic waves allows one to prove the positiveness of the trapped particle distribution in a finite interval $[e\phi_{\max}, e\phi_{\min}]$. It is possible to give explicitly a class of solutions with a potential which depends arbitrarily on the position, but which has to be limited, and, in order that quasi-neutrality be a good approximation, must have an inhomogeneity length much bigger than the Debye length. The correction on $g(e\phi)$ which comes from $\frac{d^2\phi}{dx^2}$ if the Poisson equation is used makes a small contribution to f_{-t} . If f_{-t} was sufficiently positive the addition of the Poisson term does not change the property. In this way, one can study static electrostatic equilibria, oscillating waves, and shock waves: The density depends on the potential in a simple way.

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