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Kinetic Interaction of Positive-Energy Waves in a Plasma

E. Rebhan+) and H.K. Wimmel

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+) Courant Institute of Mathematical Sciences, New York University, New York, N.Y. (USA)
On leave of absence from the Institut für Plasmaphysik, Garching.

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ABSTRACT

Summational invariants, conservation theorems, equilibrium spectra, and the H-theorem associated with the kinetic equation for wave-wave interaction in a plasma are presented in their mutual relationship. If in some reference frame all waves have positive energies then the system always goes to equilibrium. It follows from conservation theorems that the final equilibrium state depends on the initial conditions. Regular equilibria, with $N_{\underline{k}} > 0$, may only occur as final states of time-varying solutions. For the "three-wave case", in which only separate wave triplets are coupled, the general solution of the initial value-problem is given, and it is shown that at sufficiently early times any time-dependent solution of the kinetic equation is either unphysical (some $N_{\underline{k}} < 0$) or degenerate (some $N_{\underline{k}} = 0$).

1. INTRODUCTION

In the theory of weak plasma turbulence, the following kinetic equation for interacting waves in a homogeneous plasma holds if the waves are undamped and stable ($\gamma_{\underline{k}} = 0$ according to linear theory):

$$\sigma_{\underline{k}} \dot{N}_{\underline{k}} = \int d^3k' \; W_{\underline{k}\underline{k}'\underline{k}''} \left(\sigma_{\underline{k}} N_{\underline{k}'} N_{\underline{k}''} + \sigma_{\underline{k}'} N_{\underline{k}''} N_{\underline{k}} + \sigma_{\underline{k}''} N_{\underline{k}} N_{\underline{k}'} \right), \qquad (1)$$

with

$$\underline{k}'' = -(\underline{k} + \underline{k}') \; ; \qquad \sigma_{\underline{k}} = \text{sign}\,(\omega_{\underline{k}}) \; ; \qquad (2)$$

the coupling coefficient $W_{\underline{k}\underline{k}'\underline{k}''}$ being of the form:

$$W_{\underline{k}\underline{k}'\underline{k}''} = \text{sign}\left(W_{\underline{k}} W_{\underline{k}'} W_{\underline{k}''} \right) \left| V_{\underline{k}\underline{k}'\underline{k}''} \right|^2 \delta\left(\omega_{\underline{k}} + \omega_{\underline{k}'} + \omega_{\underline{k}''} \right). \qquad (3)$$

The quantities $N_{\underline{k}}$ and $W_{\underline{k}}$ are explained below. There are more general versions of the kinetic equation which are valid for unstable or damped waves and/or for inhomogeneous plasmas. We shall not be concerned with these. For references, compare CAMAC et al. (1962), GALEEV and KARPMAN (1963), GALEEV et al. (1965), AL'TSHUL' and KARPMAN (1965), DIKASOV et al. (1965), AAMODT and SLOAN (1967), ROSENBLUTH et al. (1968).

One purpose of this paper is to study some basic theorems associated with eq. (1) in the case of positive-energy waves and present them in a coherent and simple way. There is in fact a close connection between summational invariants, conservation theorems, equilibrium spectra, and the H-theorem associated with eq. (1). From these theorems conclusions about nonlinear stability and time-asymptotic properties can be drawn. The other purpose is to present a thorough study of the "three-wave case", where waves are coupled only by separate triplets. The general solution of the initial-value problem is given for this case, and a simple expression is derived for the linear relaxation rate to equilibrium.

We start by giving some explanations about eq. (1). The absolute value of $N_{\underline{k}}$ represents the action density in $\underline{K}$-space of the plasma waves. The quantity $N_{\underline{k}}$ itself is defined via the energy density $W_{\underline{k}}$ of the waves by

$$N_{\underline{k}} = W_{\underline{k}} / |\omega_{\underline{k}}| . \qquad (4)$$

It should be noted that the definition of $N_{\underline{k}}$ varies in the literature. The definition of wave energy in a dispersive medium has been given many times in the literature and will not be discussed here. Perhaps this concept is most thoroughly presented by ALLIS et al. (1963), AGRANOVICH and GINZBURG (1966). Inhomogeneous plasmas were considered by JUNGWIRTH (1968). Nonlinear instabilities may result when waves of positive and negative energy interact: compare DIKASOV et al. (1965), KADOMTSEV et al. (1965), AAMODT and SLOAN (1967), ROSENBLUTH et al. (1968). In this article we shall assume that a reference frame exists in which all interacting waves possess positive energies; we shall then have

$$N_{\underline{k}} \geqq 0 \; ; \quad W_{\underline{k}} \geqq 0 \; ; \quad W_{\underline{k}\underline{k}'\underline{k}''} \geqq 0 \tag{5}$$

for all physical states.

Actually the kinetic wave equation could be written in a form that contains in essence only frame-independent quantities: compare AAMODT and SLOAN (1967); ROSENBLUTH et al. (1968). However, the question of nonlinear stability is treated more conveniently with the aid of a different form, as presented in eq. (1), containing quantities $N_{\underline{k}}$ and $\sigma_{\underline{k}}$ that are not invariant individually, whereas the products $N_{\underline{k}}\sigma_{\underline{k}}$ are. In order to prove stability a frame should exist in which all waves have the same sign of energy.

For given $\underline{K}$ there generally exist several frequencies $\omega_{\underline{k}}^{(m)}$ that correspond to the different branches m of the dispersion relation. For any wave ($\underline{K}$, m) one has the reality conditions:

$$\omega_{-\underline{k}} = -\omega_{\underline{k}} \; ; \quad N_{-\underline{k}} = N_{\underline{k}} \, . \tag{6}$$

The coupling coefficient $W_{\underline{k}\underline{k}'\underline{k}''}$ obeys the following symmetry relations:

$$W_{\underline{k}\underline{k}'\underline{k}''} = W_{\underline{k}\underline{k}''\underline{k}'} = W_{\underline{k}''\underline{k}'\underline{k}} = W_{-\underline{k},-\underline{k}',-\underline{k}''} . \qquad (7)$$

The elementary interaction in eq. (1) is 3-wave-interaction. One essential step in deriving eq. (1) is neglecting all phase correlations between different waves (random phase approximation). Therefore eq. (1) turns out to be irreversible in time, as is easily seen by inspection. As a measure of irreversibility an H-function may be defined (Section 3).

For consistency, it is required that, whenever $N_{\underline{k}} \geqq 0$ initially, eq. (1) exclude negative values of $N_{\underline{k}}$ at later times. This requirement is satisfied, as can be seen by substituting $N_{\underline{k}} = 0$ in eq. (1), whence it follows that $\dot{N}_{\underline{k}} > 0$. This means that unphysical values, $N_{\underline{k}} < 0$, may develop into physical ones, but not vice versa.

The range of the K-integration in eqs. (1), (12) - (14), (21) - (24), etc. is defined by the range of wave interaction, $W_{\underline{k}\underline{k}'\underline{k}''} \neq 0$. The K-integration will give rise to divergence difficulties at large values of K if the interaction is not limited to a bounded region in K-space. The divergence might occur - at least in principle - in the kinetic equation itself as well in the equations defining conserved quantities and the H-function. The H-function will diverge also in another case, e.g. when $N_{\underline{K}} \equiv 0$ in a $\underline{K}$-region of non-zero measure. Then again the problem to be studied should be reformulated in order to avoid this divergence.

2. INVARIANTS AND EQUILIBRIA

Before the H-theorem is derived, summational invariants, conservation laws, and stationary spectra ("equilibria") will be considered.

Summational invariants are well known in connection with BOLTZMANN's equation; compare CHAPMAN and COWLING (1958). By analogy, we may define a quantity $A_{\underline{K}}$ as a summational invariant by requiring that

$$A_{\underline{K}} + A_{\underline{K}'} = A_{\underline{K}+\underline{K}'} \quad ; \quad A_{-\underline{K}} = - A_{\underline{K}} \tag{8}$$

whenever $W_{\underline{K}\underline{K}'\underline{K}''} \neq 0$. Obviously, if $A_{\underline{K}}$ and $B_{\underline{K}}$ are summational invariants, so is $C_{\underline{K}} = a A_{\underline{K}} + b B_{\underline{K}}$ (a, b = arbitrary constants). By definition the Cartesian components of $\underline{K}$ and the frequency $\omega_{\underline{K}}$ are summational invariants.

The summational invariants are connected with conservation laws. Let us consider conserved quantities that are additive:

$$I = \int d^3K \; I_{\underline{K}}(N_{\underline{K}}) \,. \tag{9}$$

Upon inserting eq. (1) and using the symmetry relations of eq. (7) the following expression for the time derivative of I is obtained:

$$\dot{I} = \frac{1}{3}\left\{\int d^3K \; \dot{N}_{\underline{K}} \frac{dI_{\underline{K}}}{dN_{\underline{K}}} + \int d^3K' \; \dot{N}_{\underline{K}'} \frac{dI_{\underline{K}'}}{dN_{\underline{K}'}} + \int d^3K'' \; \dot{N}_{\underline{K}''} \frac{dI_{\underline{K}''}}{dN_{\underline{K}''}}\right\}$$

$$= \frac{1}{3}\int d^3K \int d^3K' \; W_{\underline{K}\underline{K}'\underline{K}''} \, N_{\underline{K}} N_{\underline{K}'} N_{\underline{K}''} \left(\frac{\sigma_{\underline{K}}}{N_{\underline{K}}} + \frac{\sigma_{\underline{K}'}}{N_{\underline{K}'}} + \frac{\sigma_{\underline{K}''}}{N_{\underline{K}''}}\right)$$

$$\cdot \left(\sigma_{\underline{K}} \frac{dI_{\underline{K}}}{dN_{\underline{K}}} + \sigma_{\underline{K}'} \frac{dI_{\underline{K}'}}{dN_{\underline{K}'}} + \sigma_{\underline{K}''} \frac{dI_{\underline{K}''}}{dN_{\underline{K}''}}\right). \qquad (10)$$

No use has been made so far of the requirement that I should be conserved.

We now shall make I a conserved quantity by "detailed conservation", i.e. by separately equating to zero the contributions to $\dot{I}$ coming from all the different interacting wave triplets. This is done by putting the second bracket on the right-hand side of eq. (10) equal to zero. It follows that

$$\sigma_{\underline{K}} \frac{dI_{\underline{K}}}{dN_{\underline{K}}} = A_{\underline{K}} \qquad (11)$$

with $A_{\underline{K}}$ = an arbitrary summational invariant. Therefore, all quantities I of the form

$$I = I_0 + \int d^3K \; \sigma_{\underline{K}} A_{\underline{K}} N_{\underline{K}} \qquad (12)$$

are conserved. In particular, total momentum and total energy,

$$\underline{P} = \int d^3K \; \sigma_{\underline{K}} \, \underline{K} \, N_{\underline{K}} \quad , \tag{13}$$

$$W = \int d^3K \; |\omega_{\underline{K}}| \, N_{\underline{K}} \quad , \tag{14}$$

are conserved. Of course, it is required that the integrals in eqs. (9) to (14) converge (compare Section 1).

In considering stationary spectra, we shall restrict ourselves to "regular" equilibria, i.e. those with $N_{\underline{K}} > 0$. It then follows from eq. (1) in a straight-forward manner that the most general regular equilibrium arising from detailed balancing is given by

$$\sigma_{\underline{K}} / N_{\underline{K}} = \tilde{A}_{\underline{K}} \; . \tag{15}$$

Here $\tilde{A}_K$ is the most general summational invariant consistent with $N_{\underline{K}} > 0$, i.e. with the property:

$$\operatorname{sign}\left(\tilde{A}_{\underline{K}}\right) = \sigma_{\underline{K}} \; . \tag{16}$$

"Detailed balancing", of course, means that the bracket on the right-hand side of eq. (1) is assumed to vanish identically. We shall prove from the H-theorem to be derived that all regular equilibria come about by detailed balancing, i.e. are of the form given in eq. (15). The RAYLEIGH-JEANS spectrum,

$$N_{\underline{K}} = T / |\omega_{\underline{K}}| \quad , \tag{17}$$

is only one special case of such a regular equilibrium. It is associated with the special values

$$\underline{P} = T \int d^3K \left(\underline{K} / \omega_{\underline{K}} \right) ,$$

$$W = T \int d^3K$$

for total momentum and total energy. Of course, convergence of these integrals can be achieved only by limiting the system to waves with finite values of $|\underline{K}|$.

We shall show in Section 4 that the positive-energy wave system always goes to an equilibrium state. Given any initial state the values of total momentum and energy follow from eqs. (13) and (14). Since these quantities are conserved it is obvious that initial states with different momentum and/or energy must lead to different equilibria. On the other hand initial states with arbitrary values of $\underline{P}$ and W can be constructed except under singular conditions. Hence there must also exist a continuum of different equilibria. Some of these may be of the form

$$N_{\underline{k}} = \sigma_{\underline{k}} T / (\omega_{\underline{k}} + \underline{c} \cdot \underline{K}) . \tag{17a}$$

While the RAYLEIGH-JEANS spectrum is isotropic in $\underline{K}$-space, the more general spectrum of eq. (17a) is not. In fact $\omega'_{\underline{k}} = \omega_{\underline{k}} + \underline{c} \cdot \underline{K}$ may be interpreted as the wave frequency as seen from a moving

The property $\dot{H} \geqq 0$ will be produced by making the second bracket in eq. (19) a positive multiple of the first one, i. e. by putting

$$\sigma_{\underline{k}} \frac{dH_{\underline{k}}}{dN_{\underline{k}}} = c \frac{\sigma_{\underline{k}}}{N_{\underline{k}}} + A_{\underline{k}} \tag{20}$$

where $A_{\underline{k}}$ is any summational invariant, and $c > 0$. Then the general expression for the H-function is given by

$$H = H_0 + I\{N_{\underline{k}}\} + c\int d^3k \ \ln N_{\underline{k}} \ . \tag{21}$$

Here I is the conserved quantity constructed from $A_{\underline{k}}$. One may use the expression

$$H_1 = \int d^3k \ \ln N_{\underline{k}} \ , \tag{22}$$

or if we assume convergence of

$$\Omega = \int d^3k \ \ln |\omega_{\underline{k}}| \tag{23}$$

the alternative expression

$$H_2 = H_1 + \Omega = \int d^3k \ \ln W_{\underline{k}} \tag{24}$$

could be used in the forthcoming discussion. The H-function of eq. (22) was used earlier by DIKASOV et al. (1965), AAMODT and SLOAN (1967).

The expression of eq. (22) differs formally from BOLTZMANN's particle H-function involving $f \ln f$. Equation (22) can also be derived from statistical considerations of microstates (compare LANDAU and LIFSHITS, 1959) if the actions of those waves which are contained in a volume element d^3k are distributed according to a BOLTZMANN distribution, the average action being proportional to the macroscopic variable $N_{\underline{k}}$. However, a local BOLTZMANN distribution in $\underline{K}$-space does not seem to be implied by the assumptions made in deriving the kinetic equation in the first place. Therefore, closer investigation of this question may be of interest. It will not be pursued in this paper.

4. CONSEQUENCES

From eqs. (19) and (20), it is seen that for regular spectra ($N_{\underline{k}} > 0$) $\dot{H} = 0$ holds only when the bracket of eq. (1) vanishes identically. This proves that all regular equilibria obey eq. (15), i.e. arise from detailed balancing.

Next, we show that the wave system tends to an equilibrium distribution for $t \rightarrow \infty$. We can prove this with the following assumptions: It is postulated that the total wave energy W of eq. (14) and the expression Ω of eq. (23) both be finite. It should be borne in mind that W is conserved in time. We now have

$$H_1 = \int d^3K \ \ln N_{\underline{K}} = \int d^3K \ \ln W_{\underline{K}} - \Omega$$

$$< \int d^3K \ W_{\underline{K}} - \Omega = W - \Omega, \tag{25}$$

i.e. H_1 does not exceed an upper bound whenever W and Ω are finite. Since, in addition, $\dot{H}_1 \geqq 0$, it follows that

$$\lim_{t\to\infty} \dot{H}_1(t) = 0 \ ; \qquad \lim_{t\to\infty} H_1(t) = H_\infty , \tag{26}$$

H_∞ being finite and, in general, dependent on the initial conditions. This proves the proposition that equilibrium will be attained. As mentioned in Section 2 the equilibrium state will depend on the initial conditions.

Finally, it shall be shown that all regular equilibria are stable with respect to "conservative variations", i.e. those obeying the conservation laws. In other words, regular equilibria may only occur as final states in time, but not as initial states of time-varying solutions. In order to prove this proposition, let us consider the change of H_1 when a finite perturbation $\Delta N_{\underline{K}} = N_{\underline{K}} - N^0_{\underline{K}}$ is applied to an arbitrary regular equilibrium $N^0_{\underline{K}} > 0$. We have

$$\Delta H_1 = \int d^3K \left(\ln N_{\underline{K}} - \ln N^0_{\underline{K}} \right) < \int d^3K \left(\Delta N_{\underline{K}} / N^0_{\underline{K}} \right). \tag{27}$$

On using $N^0_{\underline{K}} = \sigma_{\underline{K}} / \tilde{A}_K$ - eq. (15) - and the special conservation law

$$\Delta I = \int d^3K \;\; \sigma_{\underline{K}} \; \tilde{A}_{\underline{K}} \;\; \Delta N_{\underline{K}} = 0 \tag{28}$$

it follows that

$$\Delta H_1 < \int d^3K \left(\Delta N_{\underline{K}} / N^0_{\underline{K}} \right) = \int d^3K \;\sigma_{\underline{K}} \; \tilde{A}_{\underline{K}} \;\; \Delta N_{\underline{K}} = \Delta I = 0 \, . \tag{29}$$

Equation (29) states that H_1 has an absolute maximum for $N_{\underline{K}} = N^0_{\underline{K}}$ under the constraint $I(N_{\underline{K}}) = I(N^0_{\underline{K}})$. This proves the above proposition.

5. GENERAL THEOREMS IN THE THREE-WAVE CASE

In the remaining portion of this paper, we shall treat a simple example which can be solved rigorously and may also serve as an illustration of the general theorems derived in the previous sections.

In the case of a one-dimensional K-space, it may be possible for the equation

$$\omega_K + \omega_{K'} = \omega_{K+K'} \tag{30}$$

to possess a solution K'(K) that is unique, except for the substitution $K' \to -(K + K')$, which gives the associated solution K"(K). Then the coupling coefficient of eq. (1) connects only two more waves, viz. $K_2 = K'(K_1)$ and $K_3 = K''(K_1)$, with a given wave K_1. We may therefore choose K_1 as a continuous parameter that distinguishes all the different triplets of interacting waves. For every triplet one obviously has the relations:
$K_1 + K_2 + K_3 = 0$; $\omega_1 + \omega_2 + \omega_3 = 0$.

By carrying out the K-integration in eq. (1), with K one-dimensional, one obtains the kinetic equation for the "three-wave case":

$$\sigma_1 \dot{N}_1 / \alpha_1 \equiv \sigma_2 \dot{N}_2 / \alpha_2 \equiv \sigma_3 \dot{N}_3 / \alpha_3$$

$$= \left(\sigma_1 N_2 N_3 + \sigma_2 N_3 N_1 + \sigma_3 N_1 N_2 \right), \qquad (31)$$

with

$$\alpha_1 = 2 \left| V_{123} \right|^2 \left| \frac{d\omega_2}{dK_2} - \frac{d\omega_3}{dK_3} \right|^{-1} > 0 . \qquad (32)$$

Cyclic permutations of the indices in eq. (32) give α_2 and α_3. In this Section we shall list, as far as necessary, the general theorems in the special form they adopt in the three-wave-case. The exact solution of eq. (31) will then be given in Section 6.

To simplify notation, we introduce the quantities $M_i = N_i/\alpha_i$. As may be shown by using eq. (31) the conserved quantities analogous to those of eq. (12), but associated with a single wave triplet, are given by

$$I = I_0 + \sum_{i=1}^{3} \sigma_i A_i M_i \,, \tag{33}$$

where the A_i are again summational invariants, with eq. (8) replaced by

$$A_1 + A_2 + A_3 = 0 \; ; \quad A_{-i} = A_i \,. \tag{34}$$

In particular, the total energy of the three waves is a constant and proportional to

$$W = \sum_i |\omega_i| \; M_i \,. \tag{35}$$

Regular equilibria are still given by eq. (15) in the three-wave case. The H-function for a wave triplet is given by

$$H = H_0 + c \sum_i \frac{\ln N_i}{\alpha_i} = \tilde{H}_0 + c \sum_i \frac{\ln M_i}{\alpha_i} \; ; \; c > 0. \tag{36}$$

The results of Section 4 equally apply in the three-wave case. In addition, one may also demonstrate by the linear perturbation method that regular equilibria can occur only as final states of time-varying solutions. Thus M_i is expanded about a regular equilibrium $\mu_i > 0$:

$$M_i = \mu_i + m_i \quad ; \quad N_i = \nu_i + n_i \quad ; \tag{37}$$

with $|m_i| \ll \mu_i$. At some initial or final time the solution should coincide with the equilibrium state ($m_i = n_i = 0$). Hence one obtains from eq. (31):

$$\sigma_1 m_1 \equiv \sigma_2 m_2 \equiv \sigma_3 m_3 \, . \tag{38}$$

On using the equilibrium condition for μ_i and the ansatz $m_i \propto \exp(-\gamma t)$, one finds from the linearized form of eq. (31) the following damping rate:

$$\gamma = (\nu_2 \nu_3 / \mu_1) + (\nu_3 \nu_1 / \mu_2) + (\nu_1 \nu_2 / \mu_3) \, . \tag{39}$$

(Because of the linear dependence of the three N_i there is only one solution for γ). For regular equilibria all μ_i, ν_i , and γ are positive, which proves our above proposition.

Perturbation theory is applicable also to degenerate equilibria which are characterized by at least two of the N_i being equal to zero. As an example, for the equilibrium $\mu_2 = \mu_3 = 0$ one obtains the following damping rate:

$$\gamma = - \sigma_2 \sigma_3 \left(\alpha_2 + \alpha_3 \right) \nu_1 \gtrless 0 \, . \tag{39a}$$

Thus degenerate equilibria may occur as initial or as final states of time-varying solutions. The question whether the corresponding solutions of eq. (31) are physical or not will be answered in Section 6.

6. SOLUTION OF THE THREE-WAVE CASE

It is instructive to give the general solution of eq. (31) in two different forms, viz. a for given final equilibrium state, and b for given initial state (at t = 0). Form a has the advantage of formal simplicity, but is less appropriate for discussing the initial value problem.

(6a) Final Equilibrium State Given

When the final equilibrium state, assumed to be regular ($\mu_i > 0$), is known, one may use the representation of eqs. (37) and (38). Now the m_i are not assumed to be small quantities, however. The kinetic equation of the three-wave case, eq. (31) is then transformed to

$$\dot{m}_i = -\gamma m_i + \sigma_j \sigma_k \alpha m_i^2 , \qquad (40)$$

with (i, j, k) = (1,2,3), (2,3,1), (3,1,2). The three equations

(40) are linearly dependent by virtue of eq. (38). The abbreviation

$$\alpha = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 \qquad (41)$$

is used, and γ is given by eq. (39). It follows that $\dot{m}_i = 0$ not only for the final-state equilibrium, $m_i = 0$, but also for the "conjugate equilibrium", $m_i = C_i$, where

$$C_i = \sigma_j \sigma_k \gamma / \alpha . \qquad (42)$$

By transforming eq. (40), one finds that the conjugate equilibrium is "unstable" ($\gamma_c = -\gamma < 0$) and, hence, unphysical ($\mu_1^c \mu_2^c \mu_3^c < 0$) or degenerate (only one $\mu_i^c \neq 0$).

The solution of eq. (40) is

$$m_i = - C_i \left[K_o e^{\gamma t} - 1 \right]^{-1} \rightarrow \begin{cases} C_i & \text{for } t \rightarrow -\infty \\ 0 & \text{for } t \rightarrow +\infty \end{cases} \qquad (43)$$

The constant K_0 may be expressed by the initial values (at t = 0):

$$K_o = (m_{io} - C_i)/m_{io} \; ; \qquad (44)$$

by virtue of eq. (38) K_0 is independent of the index i . For $K_0 > 0$ the solution becomes singular for $\gamma t = \ln K_o$. However, the singularity occurs in the unphysical region of variable space and thus presents no difficulty. This follows by observing that the constants C_i do not all have the same sign (one of them is positive, two are negative). From this observation and from eq. (43) it is seen that any time-dependent solution that is regular at large times is unphysical or degenerate at sufficiently early times.

When the equilibrium state μ_i is assumed to be degenerate (for instance $\mu_2 = \mu_3 = 0$), then γ is given by eq. (39a) or by a similar expression. The conjugate equilibrium is then regular and "stable" ($\gamma_c > 0$) if and only if the degenerate equilibrium is "unstable" ($\gamma < 0$). Inspection shows that no time-varying physical solution (with $N_i \geqq 0$) exists for which a "stable" degenerate equilibrium is the final state. However, an "unstable" degenerate state can occur as the initial state of time-varying physical solutions. If $\mu_1 = \mu_2 = \mu_3 = 0$, then the only physical solution is $m_i \equiv 0$ (i = 1, 2, 3).

(6b) Initial State Given

When the initial state (at t = 0) assumed to be physical ($M_{i0} \geqq 0$), is given, it is useful to transform eq. (31) to the following form:

$$\sigma_i \dot{M}_i \equiv \sigma_j \dot{M}_j \equiv \sigma_k \dot{M}_k = a_i M_i^2 + b_i M_i + c_i , \qquad (45)$$

with the constants

$$\left.\begin{aligned} a_i &= \sigma_j \sigma_k \alpha \\ b_i &= \sigma_j \sigma_k \left[\alpha_j(\alpha_i+\alpha_k) D_{ij} + \alpha_k(\alpha_i+\alpha_j) D_{ik}\right] \\ c_i &= \sigma_j \sigma_k \alpha_j \alpha_k D_{ij} D_{ik} \end{aligned}\right\} \tag{46}$$

where D_{ij}, D_{ik} are determined by the initial conditions:

$$\left.\begin{aligned} D_{ij} &= \sigma_i \sigma_j M_{j0} - M_{i0} \\ D_{ik} &= \sigma_i \sigma_k M_{k0} - M_{i0} \end{aligned}\right\} \tag{47}$$

Again $(i, j, k) = (1,2,3), (2,3,1), (3,1,2)$.

It is possible to show that the discriminants of the quadratic polynomials occuring in eq. (45) are non-negative; so we have two real roots for every value of i ($i = 1,2,3$):

$$\mu_i^{\pm} = \frac{1}{2a_i}\left(-b_i \mp \sqrt{b_i^2 - 4a_i c_i}\right). \tag{48}$$

The two are equal ($\mu_i^+ = \mu_i^-$) for a fixed value of i if and only if $D_{ij} = D_{ik} = 0$, i.e. for $M_{10} = M_{20} = M_{30} = 0$. Then, as mentioned, the only physical solution is $M_i \equiv 0$. Henceforth, we shall exclude this special degenerate initial state. The discriminants are then positive.

Inspection of eq. (48) shows that for any physical initial conditions, μ_i^+ is non-negative. It follows that the state $M_i = \mu_i^+$ (i = 1,2,3) is physical. The solution of eq. (45) for the initial condition $M_i(0) = M_{io}$ is then

$$M_i = \left(\mu_i^+ K_o e^{\gamma t} - \mu_i^-\right) / \left(K_o e^{\gamma t} - 1\right), \tag{49}$$

with γ and K_o independent of i :

$$\gamma = \sqrt{b_i^2 - 4a_i c_i} > 0 \tag{50}$$

$$K_o = \left(M_{io} - \mu_i^-\right) / \left(M_{io} - \mu_i^+\right). \tag{51}$$

The quantities γ and K_o are identical with the expressions given in eqs. (39) and (44). It follows that for any physical initial condition the solution approaches or coincides with a physical equilibrium state; the equilibrium is: $M_i = \mu_i^+$; it depends on the initial conditions. It follows from our general theorems and from Section (6a) that the singularity of the solution eq. (49) that exists for $K_o > 0$ must occur at negative $t = t_s$. Indeed, the fact that always $t_s < 0$ may also be proved direct. This completes the solution of the initial-value problem for the three-wave case. The three-wave case has also been considered, though in less detail than here, by DIKASOV et al. (1965).

7. CONCLUSION

We have studied some basic properties of the kinetic wave equation when applied to positive-energy waves in a homogeneous plasma. The most general form of additive conserved quantities that obey "detailed conservation" has been given. The H-theorem for the kinetic equation is constructed from simple considerations. The most important result following from the H-theorem and from conservation of energy is that a system of positive-energy waves always goes to an equilibrium distribution. On the other hand, the H-theorem and the general form of conservation theorems also show that an equilibrium that is not degenerate or unphysical occurs only as a final state, never as an initial state of a time-varying solution.

There exists a continuum of equilibrium states; which equilibrium state will be actually reached, depends on the initial state of the system. Equilibrium spectra and conserved quantities are linked together by summational invariants. This connection is important in proving the "final state property" of regular equilibria and in deriving equilibrium spectra more general than the RAYLEIGH-JEANS spectrum.

Concerning the "three-wave case" (wave interaction only within separate wave triplets) we have derived a general expression for the linear damping rate by which a solution relaxes to its final equilibrium. The general solution of the initial-value problem is given for this special case, and the general theorem of relaxation to equilibrium for all initial conditions is confirmed. It seems interesting that any time-dependent solution of the three-wave case corresponds to unphysical or degenerate states at some (early) times.

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