INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Non-linear Instability in Two-dimensional Plasma Configurations

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IPP 6/68 August 1968

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Institut für Plasmaphysik GmbH und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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ABSTRACT:

Two-dimensional collisionless plasmas in which the distribution functions depend only on the energy and canonical momentum of the third space coordinate ($f = f(E, p_3)$) can become unstable due to the tearing mode instability. In the first part of our paper we extend the linear theory of the collisionless tearing mode [2], [3] to two-dimensional configurations. If the distribution function $f(E, p_3, \alpha)$ depends continuously on a parameter α there can be several solutions for the fields at the same α and a bifurcation point $\alpha = \alpha_0$ of the equilibrium owing to the nonlinearity of the equilibrium equations.
The linear theory shows that the bifurcation point is also a transition point of the stability ($\omega = 0$).
Using an amplitude expansion we investigate the stability of the equilibria in the neighbourhood of the bifurcation point. We find that in this case a linearly stable system may be unstable with respect to finite perturbations.

In the second part we apply the theory to a plasma configuration with an X-type neutral point of the magnetic field. The electric field is neglected. We find instability if there is a minimum of the pressure on the separatrix. It exists a critical minimum pressure below which the system is linearly unstable, but it is already unstable for finite perturbations above this critical pressure. Such explosive behaviour of an instability may be of interest in some astrophysical phenomena (see [4]).

1) Introduction

This paper is an extension of investigations conducted by H. Furth [1], D. Pfirsch [2], and G. Laval et al. [3] on mirror-type microinstabilities. D. Pfirsch and G. Laval et al. considered configurations in which all equilibrium parameters depend only on one space coordinate (e.g. q_1), while the permissible perturbations are functions of two space coordinates (q_1, q_2). As part of a perturbation calculation, Pfirsch was then able to conclude from the existence of a neighbouring equilibrium ($\omega = 0$) that the Vlasov equation has unstable solutions. The collisionless tearing mode of a plane Z-pinch is an example of such an instability. Pfirsch's general approach was used by Laval [3] to calculate the growth rate of this instability, and this was then taken by B. Coppi et al. [4] to explain the presence of accelerated particles in the earth's magnetic tail. What this paper does is extend the theory from one-dimensional configurations to two-dimensional configurations and investigate the non linear behaviour of such systems in the neighbourhood of a bifurcation point.

For this purpose we consider a set of two-dimensional equilibria which are described by a system of Cartesian coordinates and depend continuously on a parameter α. This means that the distribution function is a function of energy, canonical momentum in the 3-direction, and the parameter α. Examples of such a parameter α are provided by the boundary values of the magnetic or electric field which may also vary slowly relative to the time scales of the instabilities. In one application of the general theory, the minimum pressure of the plasma is chosen as variable parameter.

As Sturrock has shown [5], such a parameter dependence may also involve an explosive onset of the instability. This can happen if several solutions exist for a given α owing to the non-linearity of the equilibrium equations and then coincide at a bifurcation point   . It can be shown with the linear stability theory that such a bifurcation point is also the transition point of the stability.

A mechanical example of such behaviour is an anharmonic oscillator with the Hamiltonian:

$$H = \frac{1}{2m} p^2 + V(q) \quad ; \quad V(q) = \alpha q^2 + \beta q^3 + \gamma q^4$$

where β and γ are fixed and α is variable.

This system is conservative, and the stability of the equilibrium positions:

$$q_1(\alpha) = 0$$

$$q_2(\alpha) = -\frac{3\beta}{8\gamma} \pm \sqrt{\left(\frac{3\beta}{8\gamma}\right)^2 - \frac{\alpha}{2\gamma}}$$

is governed by the behaviour of the potential energy at the equilibrium positions. The equilibrium positions $q_i(\alpha)$ are plotted in Fig. 1a for $\beta > 0$, $\gamma < 0$.

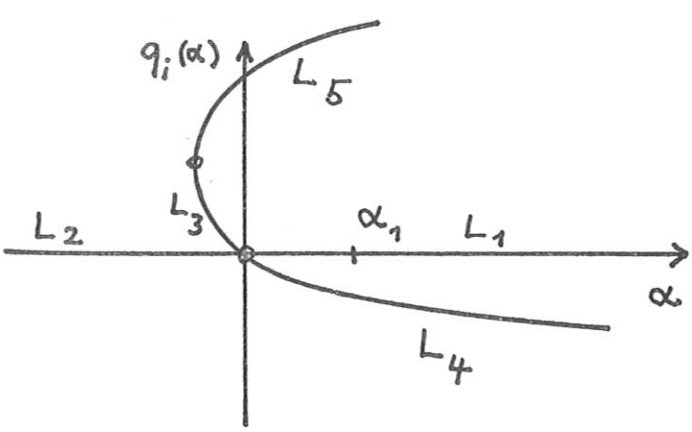

Fig. 1a. Equilibrium positions.

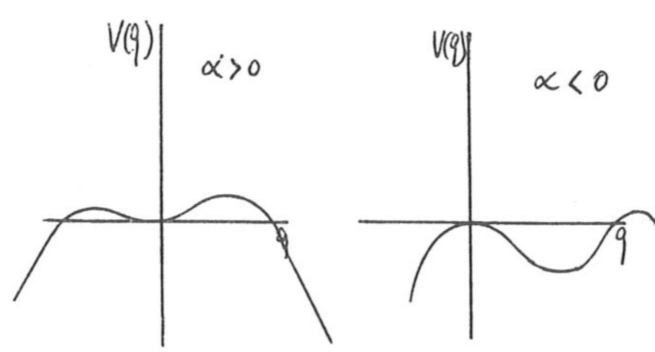

Fig. 1b. Typical Potential profiles.

In this example, the branches L_1 and L_3 are linearly stable and the branches L_2, L_4, and L_5 are linearly unstable. If the system is on L_1, e. g. at the point $\alpha = \alpha_1$, a sufficiently large perturbation can move the system to branch L_4 or L_5, whence the perturbation continues to grow owing to the instability of these branches.

As a second step of the whole procedure, conditions for such explosive behaviour in two-dimensional plasmas are derived. For this purpose the neighbourhood of the bifurcation point is investigated by means of an amplitude expansion. In a third part the theory is applied to a plasma in multipole geometry.

2) Linear theory.

As in [2], we consider here an equilibrium configuration in which none of the parameters depend on the space coordinate q^3. Let the magnetic field B_o and the electric field E_o lie in the q^1,q^2-plance. Let q^1,q^2,q^3 be Cartesian coordinates, and j indicate the particle species.

The time independent constants of motion in this configurations are H_o^j, P_3^j (energy and canonical momentum in the q^3-direction). Let ϕ_o, A_{o3} be the potentials of the electromagnetic equilibrium field ($\vec{A} = (0,0,A_{o3}(q^1,q^2))$, which are calculated in a self-consistent manner with a distribution function. Because of our complete freedom in choosing f_o^j, we can consider, for example, a one-parameter set of distribution functions $f_o^j = f_o^j(H_o^j, P_3^j, \alpha)$.

If a neighbouring equilibrium ϕ_1, A_{13} to the equilibrium ϕ_o, A_{o3} exists, it has to satisfy the following equations in accordance with formula (7) in [2]:

$$\mathrm{div}\,\mathrm{grad}\,\phi_1 = -4\pi\left(\phi_1 \frac{\partial}{\partial\phi_o} + A_{13}\frac{\partial}{\partial A_{o3}}\right)\rho_o(\phi_o, A_{o3})$$

$$(\operatorname{curl}\operatorname{curl}\vec{A}_1)_3 = \frac{4\pi}{c}\left(\phi_1\frac{\partial}{\partial\phi_0} + A_{13}\frac{\partial}{\partial A_{03}}\right)\dot{j}_{03}(\phi_0, A_{03}), \tag{1}$$

$\phi_1 = A_{13} = 0$ on the boundary of the region. According to [6], ϱ_0 and $\mathring{j}_{03}$ can be derived from a generating function:

$$\varrho_0 = -\frac{\partial W(\phi_0, A_{03})}{\partial\phi_0}\ ;\ \dot{j}_{03} = c\,\frac{\partial W(\phi_0, A_{03})}{\partial A_{03}} \tag{2}$$

Let α_0 be a discrete value of α for which the system (1) has a solution. There are then no solutions of (1) in the neighbourhood of $\alpha = \alpha_0$. At the point $\alpha = \alpha_0 + \delta\alpha$ the Vlasov equation can therefore have only unsteady-state solutions with infinitesimal amplitudes, apart from the one solution A_{03}, ϕ_0. Steady-state solutions with finite amplitudes are, however, possible.

We now write the Vlasov equations in the form

$$\frac{\partial f}{\partial t} + [H, f] = 0 \tag{3}$$

where H = Hamiltonian, $[H, f]$ = Poisson bracket. (The superscript denoting the particle will be omitted from here onwards). In order to solve this equation, we make the following ansatz:

$$f = f_0(H, p_3) + \varphi$$

and obtain for φ in linear approximation:

$$\frac{\partial \varphi}{\partial t} + \frac{\partial H_1}{\partial t}\frac{\partial f_0}{\partial H_0} + [H_0, \varphi] = 0 \tag{4}$$

with $\quad H_1 = q\phi_1 - \frac{q}{c}\vec{A}_1 \cdot \vec{v} \quad ; \quad q = \pm e$

We use $[H_0, \varphi] = i\mathcal{H}_0 \varphi$ to define the linear anti-Hermitian operator $i\mathcal{H}_0$. In the special geometry discussed this operator is of the form:

$$i\mathcal{H}_0 \varphi = v_1 \frac{\partial \varphi}{\partial q^1} + v_2 \frac{\partial \varphi}{\partial q_2} + \frac{q}{m}\Big[E_1^0 \frac{\partial \varphi}{\partial v_1} + E_2^0 \frac{\partial \varphi}{\partial v_2}$$

$$+ (v_1 B_2^0 - v_2 B_1^0)\frac{\partial \varphi}{\partial v_3} + v_3 \Big(B_1^0 \frac{\partial \varphi}{\partial v_2} - B_2^0 \frac{\partial \varphi}{\partial v_1}\Big)\Big]$$

where $\vec{E}^0, \vec{B}^0$ are the equilibrium fields.

The ansatz $\varphi, \phi_1, A_{13} \sim e^{i\omega t}$ yields a formal solution of the linearized Vlasov equation:

$$\varphi = -\omega(\omega + \mathcal{H}_0)^{-1} H_1 \frac{\partial f_0}{\partial H_0} \tag{5}$$

The perturbing potentials ϕ_1, A_{13} are then calculated from:

$$\Delta\phi_1 = -4\pi\Big(\phi_1 \frac{\partial}{\partial \phi_0} + A_{13}\frac{\partial}{\partial A_{03}}\Big)\rho_0$$

$$+ 4\pi \sum_{j=1}^{2} q_j \int \omega(\omega + \mathcal{H}_0^j)^{-1} H_1^j \frac{\partial f_0^j}{\partial H_0^j}\, d^3 v \tag{6}$$

$$\Delta A_{13} = -\frac{4\pi}{c}\left(\phi_1 \frac{\partial}{\partial \phi_0} + A_{13}\frac{\partial}{\partial A_{03}}\right) j_{03}$$

$$+ \frac{4\pi}{c}\sum_{j=1}^{2} q_j \int \omega\left(\omega + \mathcal{H}_0^j\right)^{-1} H_1^j \frac{\partial f_0^j}{\partial H_0^j} v_3 \, d^3v$$

It can be seen that for $\omega = 0$ the equations (1) are again obtained for the neighbouring equilibrium. This linear system (6) is of the kind:

$$L(\omega, \alpha)\vec{\xi} = 0 \quad ; \quad \vec{\xi} = \begin{pmatrix} \phi_1 \\ A_{13} \end{pmatrix}$$

We now have to calculate the frequency ω in the neighbourhood of $\alpha = \alpha_0$. The perturbation calculation (see also eq. (28) in [2]) yields:

$$\delta\omega = -\frac{\left(\vec{\xi}_0, \frac{\partial L}{\partial \alpha}\vec{\xi}_0\right)}{\lim\limits_{\omega \to 0}\left(\vec{\xi}_0, \frac{\partial L}{\partial \omega}\vec{\xi}_0\right)} \delta\alpha \qquad (7)$$

where $(\vec{a}, \vec{b})$ is the scalar product defined by scalar multiplication of vectors $\vec{a}, \vec{b}$ and integration over q, v. As can be seen from the system (6) and from (2), the operator $\partial L/\partial\alpha \big|_{\omega=0}$ is a Hermitian matrix:

$$\left.\frac{\partial L}{\partial \alpha}\right|_{\omega=0} = 4\pi \begin{pmatrix} \dfrac{\partial^2 \rho_0}{\partial \alpha \partial \phi_0} & \dfrac{\partial^2 \rho_0}{\partial \alpha \partial A_{03}} \\[2ex] -\dfrac{1}{c}\dfrac{\partial^2 j_{03}}{\partial \alpha \partial \phi_0} & -\dfrac{1}{c}\dfrac{\partial^2 j_{03}}{\partial \alpha \partial A_{03}} \end{pmatrix} \qquad (8)$$

The denominator in formula (7) is of the form:

$$N(\omega)_{\omega\to 0} = \sum_{j=1}^{2} q_j^2 \iint \left(\phi_1^* - \frac{V_3}{C} A_{13}^*\right)\left(\omega + \mathcal{H}_0^j\right)^{-1}_{\omega\to 0} \frac{\partial f_0^j}{\partial H_0^j}\left(\phi_1 - \frac{V_3}{C} A_{13}\right) d^3v\, dq^2 \quad (9)$$

The numerator in formula (7) is real, and we will now show that the denominator can only be imaginary. Depending on the sign of $\delta\alpha$ the instability will be damped monotonically or will grow monotonically. There is no overstability.

This fact has already been pointed out in [2] and [3], and it will now be proved in detail. The denominator in formula (7) is written in the abbreviated form:

$$N(\omega) = \sum_{j=1}^{2}\left(g,\, R^j(\omega)\frac{\partial f_0^j}{\partial H_0^j}\, g\right); \quad g = \phi_1 - \frac{V_3}{C} A_{13} \quad (10)$$

$R(\omega) = (\omega + \mathcal{H}_0)^{-1}$ is the resolvent of the operator $\mathcal{H}_0$

Since $i\mathcal{H}_0$ is an anti-Hermitian operator, the resolvent exists for all ω, except values on the real axis.

First we prove the property:

$$\overset{*}{N}(\omega) = N(\omega^*) \quad (11)$$

$\frac{\partial f_0}{\partial H_0}$ is a function of the integrals of motion, hence

$$\mathcal{H}_0 \frac{\partial f_0}{\partial H_0} = 0 \quad \rangle \quad R(\omega)\frac{\partial f_0}{\partial H_0} g = \frac{\partial f_0}{\partial H_0} R(\omega) g$$

The resolvent has the property $\overset{+}{R}(\omega) = R(\omega^*)$ (see [7]).
It thus follows that

$$\overset{*}{N}_{\dot{j}}(\omega) = \left(g, \overset{\dot{j}}{R}(\omega)\frac{\partial f_0}{\partial H_0}\overset{\dot{j}}{g}\right)^* = \left(\overset{\dot{j}}{R}(\omega^*)\frac{\partial f_0}{\partial H_0}\overset{\dot{j}}{g}, g\right)^* = \left(g, \overset{\dot{j}}{R}(\omega^*)\frac{\partial f_0}{\partial H_0}\overset{\dot{j}}{g}\right)$$

$$= N_{\dot{j}}(\omega^*) \qquad \text{q.e.d.}$$

Furthermore, $N_{\dot{j}}(\omega)$ has the property:

$$N(-\omega) = -N(\omega) \tag{12}$$

In order to prove this, we take the Vlasov equation as our starting point:

$$(\omega + \mathcal{H}_0)\varphi = -\omega H_1 \frac{\partial f_0}{\partial H_0} \tag{13}$$

Explicit representation of the operator yields the transformation property:

$$\mathcal{H}_0 \longrightarrow -\mathcal{H}_0 \qquad \text{for} \qquad V_1, V_2 \longrightarrow -V_1, -V_2$$

In this transformation the functions H_1 and $\frac{\partial f_0}{\partial H_0}$ remain invariant.

Let $\varphi(V_1, V_2)$ be the solution of eq. (13). We then use this to define the two functions $\varphi_+ = \varphi(V_1, V_2)$ and $\varphi_- = \varphi(-V_1, -V_2)$. φ_- satisfies the equation:

$$(\omega - \mathcal{H}_0)\varphi_- = -\omega H_1 \frac{\partial f_0}{\partial H_0} \tag{14}$$

The formal representation of these two functions is:

$$\varphi_{\pm} = \mp\,\omega(\pm\omega + \mathcal{H}_0)^{-1} H_1 \frac{\partial f_0}{\partial H_0} = \mp\,\omega\, R(\pm\omega) H_1 \frac{\partial f_0}{\partial H_0} \qquad (15)$$

In forming the scalar products in the v -space we integrate from $-\infty$ to $+\infty$, and so it holds that $(g, \varphi_+) = (g, \varphi_-)$ or, after substituting eq. (15), that:

$$-\left(g, R(\omega)\frac{\partial f_0}{\partial H_0}\, g\right) = \left(g, R(-\omega)\frac{\partial f_0}{\partial H_0}\, g\right) \qquad (16)$$

This is just the property described by formula (12).

With the expressions (11) and (12) it can readily be shown now that the real part of $N(\omega)$ vanishes in the limiting case $\omega \to 0$:

We have

$$2\,\mathrm{Re}\,N(\omega) = N(\omega) + \overset{*}{N}(\omega) = N(\omega) + N(\omega^*)$$

$$\omega = x + iy$$

or

$$2\,\mathrm{Re}\,N = N(x+iy) - N(-x+iy) \longrightarrow 0$$

$$x \to 0 \qquad (17)$$

$N(\omega)$ can thus have only imaginary values in the limiting case $\omega \to 0$.

If the imaginary part of $N(\omega)$ also vanishes, the perturbation has to be taken an order further, which yields a relation of the type $(\delta\omega)^2 \sim \delta\alpha$. In this case there exists always an unstable solution and in general one deals with overstability.

A special situation arises if the equilibrium parameters depend periodically on a coordinate $q^2 = y$ (period L). The following ansatz can then be written for solving the equations (1):

$$[\phi_1, A_{13}] \sim [\phi_1(k,x,y), A_{13}(k,x,y)]\, e^{iky} \tag{18}$$

(k real, arbitrary).

The wave vector k here takes the place of the parameter α discussed in the foregoing. The periodic functions in y, ϕ_1, A_{13} are then determined from:

$$\left(\Delta + 2ik\frac{\partial}{\partial y} - k^2\right)\phi_1 = -4\pi\left(\phi_1\frac{\partial}{\partial\phi_0} + A_{13}\frac{\partial}{\partial A_{03}}\right)\rho_0$$

(19)

$$\left(\Delta + 2ik\frac{\partial}{\partial y} - k^2\right)A_{13} = -\frac{4\pi}{c}\left(\phi_1\frac{\partial}{\partial\phi_0} + A_{13}\frac{\partial}{\partial A_{03}}\right)j_{03}$$

The reasoning that follows is completely analogous to that above, and in the vicinity of the transition point $k_0 \neq 0$ this yields as in [2] frequencies of the type $\delta\omega \sim ik_0\delta k$.

3) Equilibrium and Bifurcation point

The two-dimensional equilibrium is described by the equations:

$$- \Delta \phi_0 = + 4\pi \rho_0 (\phi_0, A_{03}, \alpha)$$

$$\Delta A_{03} = - \frac{4\pi}{c} j_{03} (\phi_0, A_{03}, \alpha) \tag{20}$$

The values of ϕ_0, A_{03} on the boundary of the system are given. By combining ϕ_0, A_{03} to a vector $\vec{f}$, eq. (20) can be written

$$H\vec{f} \qquad N[\alpha, \vec{f}] \tag{21}$$

$$H = \begin{pmatrix} -\Delta & 0 \\ 0 & +\Delta \end{pmatrix}$$ is a linear operator and

$$N[\vec{f}] = \begin{pmatrix} +4\pi \rho_0 (\phi_0, A_{03}) \\ -\frac{4\pi}{c} j_{03} (\phi_0, A_{03}) \end{pmatrix}$$ a non-linear operator.

For physical reasons, ρ_0 and j_{03} are chosen as bounded and continuous functions, $N[\vec{f}]$ being in this case a bounded and continuous operator. It is assumed that a solution of this Dirichlet problem exists. An existence theorem for the scalar equation $\Delta f = g(f)$ (g bounded) was demonstrated in [8].

But, in general, the solution is not unique.

If $\vec{f}_1, \vec{f}_2$ are two solutions of eq. (21) with the same boundary conditions and $\vec{\psi} = \vec{f}_2 - \vec{f}_1$ is the difference, we obtain the following equation :

$$H\vec{\psi} = N[\alpha, \vec{f}_1 + \vec{\psi}] - N[\alpha, \vec{f}_1] \tag{22}$$

$$\vec{\psi} = 0 \qquad \text{at the boundary.}$$

If the problem (22) has a non-trivial solution, the solution of eq. (21) is not unique. Since $\vec{f}_1$ is also a solution of eq. (21), it depends implicitly on α .

The bifurcation point α_0 is defined as that point where the two solutions $\vec{f}_1, \vec{f}_2$ coincide:

$$\| \vec{\psi}(\alpha, x, y) \| \longrightarrow 0 \quad \text{if} \quad \alpha \longrightarrow \alpha_0 \tag{23}$$

As shown in the theory of non-linear operators [9], the bifurcation point is calculated from the linearized version of eq. (22):

$$H\vec{\psi} = \frac{\partial N}{\partial \psi}\Big|_{\psi=0} \vec{\psi} \tag{24}$$

$$\frac{\partial N}{\partial \psi}\vec{\psi} = \begin{Bmatrix} +4\pi\left(\dfrac{\partial \rho_0}{\partial \phi_0}\phi_1 + \dfrac{\partial \rho_0}{\partial A_{03}} A_{13}\right) \\ -\dfrac{4\pi}{c}\left(\dfrac{\partial J_{03}}{\partial \phi_0}\phi_1 + \dfrac{\partial J_{03}}{\partial A_{03}} A_{13}\right) \end{Bmatrix}$$

This is the same equation as eq. (1), and so the bifurcation point is also the transition point of the stability. By expanding the right-hand side of eq. (22) in a Taylor series, we can write this equation in the symbolic form:

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{\partial^n N}{\partial \psi^n}\right|_{\psi=0} \vec{\psi}^{\,n}$$

The amplitude of the solution $\vec{\psi}$ is not yet determined by the linear equation (24), this being possible only if higher-order terms are taken into account. We consider the solution $\vec{\psi}$ in the neighbourhood of the bifurcation point and look for a solution of the form:

$$\vec{\psi} = \varepsilon \vec{\psi}_0 + \varepsilon^2 \vec{\psi}_1 + \varepsilon^3 \vec{\psi}_2 + \dots$$

$$\alpha = \alpha_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \dots \qquad (25)$$

$$\varepsilon \ll 1$$

$\vec{\psi}_0$ is the normalized solution of eq. (24) ($\| \psi_0 \| = 1$).

It should be noted that, in general, nothing can be said about the existence of the bifurcation point (or existence of the neighbouring equilibrium). The existence depends on the special problem, i.e. on $N[\alpha, \vec{f}_1(\alpha, x, y)]$.

Combining the quadratic terms we obtain the equation for $\vec{\psi}_1$:

$$\left[H - \frac{\partial N}{\partial \psi}\right] \vec{\psi}_1 = \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right) \alpha_1 \vec{\psi}_0 + \frac{1}{2} \frac{\partial^2 N}{\partial \psi^2} \vec{\psi}_0^{\,2} \qquad (26)$$

where $\frac{\partial^\nu N}{\partial \psi^\nu}$ stands here and in the following for $\left.\frac{\partial^\nu N}{\partial \psi^\nu}\right|_{\psi=0}$.

The condition of integrability yields

$$\alpha_1 = - \frac{\frac{1}{2}\left(\vec{\psi}_0, \frac{\partial^2 N}{\partial \psi^2} \vec{\psi}_0^{\,2}\right)}{\left(\vec{\psi}_0, \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right) \vec{\psi}_0\right)} \tag{27}$$

The scalar product is defined as

$$(u, v) = \int (u_1^* v_1 + u_2^* v_2)\, dx\, dy$$

If the coefficient α_1 vanishes, we obtain α_2 from the third-order equation

$$\alpha_2 = - \frac{\left(\vec{\psi}_0 \frac{\partial^2 N}{\partial \psi^2} \vec{\psi}_0 \vec{\psi}_1\right) + \frac{1}{6}\left(\vec{\psi}_0, \frac{\partial^3 N}{\partial \psi^3} \vec{\psi}_0^{\,3}\right)}{\left(\vec{\psi}_0, \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right) \vec{\psi}_0\right)} \tag{28}$$

The amplitude function $\varepsilon = \varepsilon(\alpha)$ as given in Fig. 2 is analogous to the equilibrium solution $q_i(\alpha)$ of the anharmonic oscillator considered at the beginning. If the coefficient α_1 vanishes, the point P coincides with the bifurcation point.

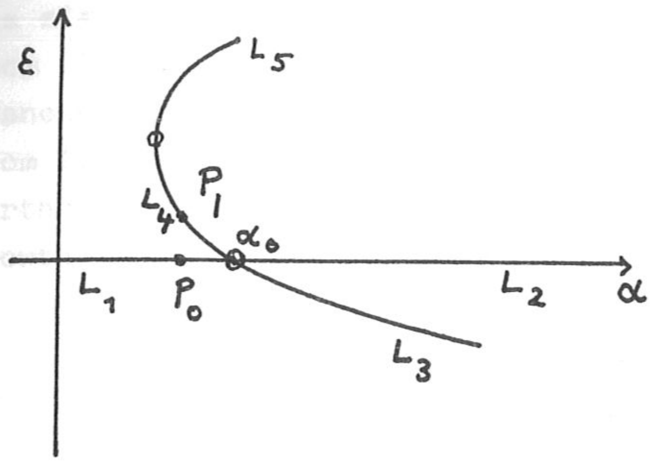

Fig. 2 Possible amplitude function for a two dimensional plasma.

4) Non-linear instability

In this section, the linear theory is used to investigate the stability of the branches L_1 ..... L_4 in the neighbourhood of the bifurcation point α_o.

The branches L_1, L_2 correspond to the equilibrium $\vec{f}_1$, and the branches L_3, L_2 are determined by the function:

$$\vec{f}_2 = \vec{f}_1 + \varepsilon \vec{\psi}_0 + \varepsilon^2 \vec{\psi}_1 + \ldots$$

If we pass from branch L_1 to L_2, the growth rate changes its sign (see formula (7)). The same holds for the transition from L_3 to L_4. In order to compare the stability of the branches L_2, L_3, we start from formula (7). As can be seen from (6), the neighbouring equilibrium $\vec{\xi}_0$ is identical with $\vec{\psi}_0$. Furthermore, it holds that $\frac{dL}{d\alpha} = \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right)$. The growth rates of the branches L_1, L_2 can be written:

$$\delta\omega = -\frac{\left(\vec{\psi}_0, \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right)\vec{\psi}_0\right)}{\lim_{\omega \to 0}\left(\vec{\psi}_0, \frac{\partial L}{\partial \omega}\vec{\psi}_0\right)}\,\delta\alpha \tag{29}$$

In order to calculate the growth rates of the branches L_3, L_4, we have to substitute $\vec{f}_1 + \varepsilon\vec{\psi}_0$ for $\vec{f}_1$ in $N[\alpha, \vec{f}_1]$.

We obtain $\quad N = N[\alpha, \vec{f}_1 + \varepsilon\vec{\psi}_0 + ..]\quad$ and

$$\frac{d}{d\alpha}\left(\frac{\partial N[\alpha, \vec{f}_1 + \varepsilon\vec{\psi}_0 + ..]}{\partial \psi}\right) = \frac{d}{d\alpha}\frac{\partial N}{\partial \psi} + \frac{\partial^2 N}{\partial \psi^2}\frac{\vec{\psi}_0}{\alpha_1} + \cdots .$$

The numerator of eq. (29) is changed to

$$\left(\vec{\psi}_0, \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right)\vec{\psi}_0\right) + \frac{1}{\alpha_1}\left(\vec{\psi}_0, \frac{\partial^2 N}{\partial \psi^2}\psi_0\right)$$

Taking α_1 from formula (27) we obtain the result

$$\delta\omega = +\frac{\left(\vec{\psi}_0, \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right)\vec{\psi}_0\right)}{\lim_{\omega \to 0}\left(\vec{\psi}_0, \frac{\partial L}{\partial \omega}\vec{\psi}_0\right)}\,\delta\alpha \tag{30}$$

It is thus shown that the growth rates of the branches L_1, L_4 and L_2 , L_3 are opposite in sign. Consequently, the system can exhibit an explosive instability. If, for example, the branch L_1 is stable, the branch L_4 is unstable. The system may be at the point P_o (see Fig. 2). If a perturbation shifts the system to P_1 , it does not return to P_o but the perturbation continues to grow. If the statistical fluctuations of the system are too small to reach the point P_1, the configuration is stable. But if we increase α slowly, i.e. if we change the system adiabatically, the stability limit grows smaller and smaller, and beginning from a certain $\alpha < \alpha_o$ the fluctuations are big enough to make the system unstable.

It is a typical property of such non-linear instabilities that they can be triggered. [5]

If the coefficient α_1 vanishes, higher-order terms have to be taken into account in the asymptotic series (25). must be replaced by $\vec{f}_1 + \varepsilon \vec{\psi}_0 + \varepsilon^2 \vec{\psi}_1$ and $\frac{\partial N}{\partial \psi}$ by $\frac{\partial N}{\partial \psi} + \frac{\partial^2 N}{\partial \psi^2} \varepsilon \vec{\psi}_0 + \frac{\partial^2 N}{\partial \psi^2} \varepsilon^2 \vec{\psi}_1 + \frac{1}{2} \frac{\partial^3 N}{\partial \psi^3} \varepsilon^2 \vec{\psi}_0^{\,2}$

Similarly, we obtain

$$\delta\omega \underset{\omega \to 0}{=} -\frac{1}{3}\,\varepsilon^2 \frac{\left(\vec{\psi}_0, \frac{\partial^3 N}{\partial \psi^3}\, \vec{\psi}_0^{\,3}\right)}{\left(\vec{\psi}_0, \frac{\partial L}{\partial \omega}\, \vec{\psi}_0\right)}\ ; \quad \varepsilon^2 = \frac{\delta\alpha}{\alpha_2} > 0 \qquad (31)$$

The sign of α_2 determines in which direction the parabola $\varepsilon^2 = \frac{\delta\alpha}{\alpha_2}$ is open. In the special case $\frac{\partial^2 N}{\partial \psi^2} \equiv 0$, α_2 is

$$\alpha_2 = -\frac{1}{3}\, \frac{\left(\vec{\psi}_0, \frac{\partial^3 N}{\partial \psi^3}\, \vec{\psi}_0^{\,3}\right)}{\left(\vec{\psi}_0, \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right) \vec{\psi}_0\right)} \qquad (32)$$

and therefore

$$\delta\omega = + \frac{\left(\psi_0, \frac{d}{d\alpha}\left(\frac{\partial N}{\partial \psi}\right) \vec{\psi}_0\right)}{\omega \to 0 \left(\vec{\psi}_0, \frac{\partial L}{\partial \omega} \vec{\psi}_0\right)} \delta\alpha \tag{33}$$

In this case, the growth rate does not change its sign if the system passes from branch L_3 to L_4.

But there is a change of sign between the branches L_3, L_4 and that branch which lies in the "interior" of the parabola

$$\varepsilon^2 = \frac{\delta\alpha}{\alpha_2}$$

Therefore a system at the point P_0 can also be unstable due to finite perturbations.

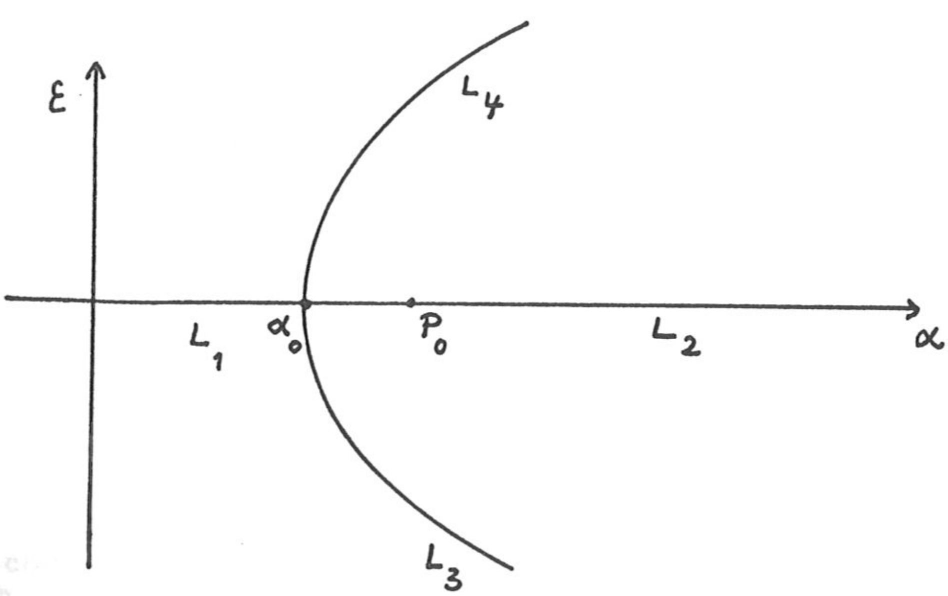

Fig. 3
Amplitude function with $\alpha_1 = 0$

5) X-type neutral point

As an example we consider a magnetic field produced by two parallel currents. An essential property of this magnetic field is the occurrence of an X-type neutral point between the currents. The magnetic field lines are described by the vektor potential $A_{03}(x,y)$ = const. $A_{03} = 0$ may be the separatrix.

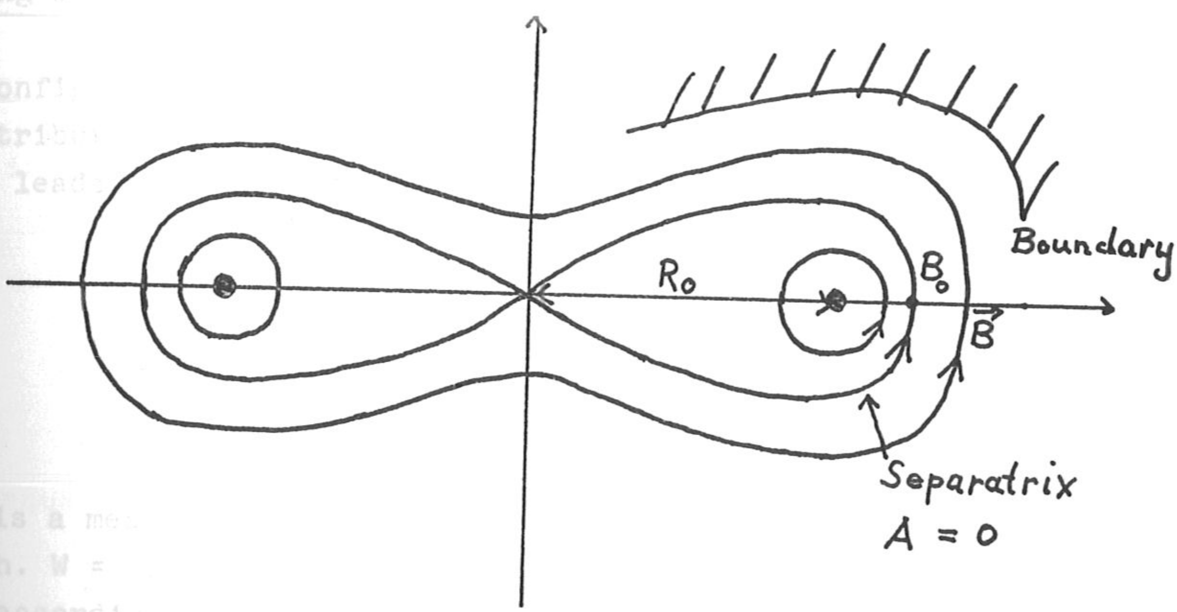

Fig. 4

X-type field configuration

We consider a plasma with a pressure minimum on the separatrix. $\beta = \frac{8\pi p}{B^2}$ may be very small, the influence of the plasma currents on the magnetic field is very small.

For simplicity, we also neglect the electric fields ($\phi_0 \equiv 0$, $\rho_0 \equiv 0$).

This situation may seem unrealistic because in space plasmas an essential part of the magnetic field is produced by the plasma currents, and so we have to consider a self-consistent equilibrium. It will be found, however, that the instability is localized to the neighbourhood of the separatrix. Thus the structure of the equilibrium far away from the separatrix has no significance for the instability, and so our approximation is justified. In laboratory plasmas (multipole machines), the magnetic field is produced by external conductors ($\beta \ll 1$). But here the plasma pressure has a maximum on the separatrix, and therefore, this configuration is stable with respect to tearing modes.

The configuration described above is shown in Fig. 4. We take a distribution function of the form $f = f(H)(1-\gamma \exp[-\delta p_z^2]$ which leads to a density profile (or pressure profile) of the form

$$p = p_\infty - p_0 \exp\left(-\frac{A_{03}^2}{d^2}\right); \quad p_\infty > p_0$$

d is a measure of the dimensions of the current-carrying sheath. W = +p is the generating function for the current, i.e. according to (2)

$$j_{03} = +\frac{\partial p}{\partial A_{03}} = +p_0 \frac{2A_{03}}{d^2} \exp\left[-\frac{A_{03}^2}{d^2}\right]$$

$\beta_{min} = \frac{2p_{min}}{B_0^2}$ is defined by the minimum pressure p_{min} on the separatrix and the maximum magnetic field on the separatrix. We define $\beta_\infty = \frac{2p_\infty}{B_0^2}$ in a similar way, and $\beta = \beta_\infty - \beta_{min} = \frac{2p_0}{B_0^2}$ is a measure of the modification of the magnetic field by the plasma currents.

The vacuum field produced by the two external currents is

$$A_{03} = \frac{B_0 R_0}{4\sqrt{2}} \ln\left[\left(\frac{r}{R_0}\right)^4 - 2\left(\frac{r}{R_0}\right)^2 \cos 2\varphi + 1\right] \tag{34}$$

The total vector potential is $A = A_{03} + o(\beta)$.

In the vicinity of the neutral point the pressure profile looks like

$$p \approx p_\infty - p_0 \exp\left[-\frac{r^4}{r_0^4} \cos^2 2\varphi\right] \tag{35}$$

r_0 is the decay length of the pressure and is connected with d by

$$d^2 = \frac{r_0^4 B_0^2}{8 R_0^2} \tag{36}$$

The equation for the neighbouring equilibrium is (see eq. (24))

$$-\Delta\psi + \beta V(A)\psi = 0 \tag{37}$$

$$\psi = 0 \qquad \text{on the boundary}$$

$$V(A) = \frac{4R_0^2}{r_0^4}\left(\frac{2A^2}{d^2} - 1\right)\exp\left[-\frac{A^2}{d^2}\right]$$

It is assumed that the boundary of the system is at a finite distance from the origin. Since $\beta \ll 1$, we neglect the corrections of the order β in A and use A_{03} (see eq. (34)). The problem (37) is an eigenvalue problem for β , and in the following we will show that there exists a solution. In order to do so, we modify the problem (37) slightly.

$$-\Delta\psi + \beta V^{*}(A_{03})\psi = E\psi \tag{38}$$

with $V^{*} = V + V_0$; $E = \beta V_0$; $V_0 = |V_{min}| = \frac{4R_0^2}{r_0^4}$

$$\psi = 0 \qquad \text{on the boundary}$$

We consider β as a given parameter and $E(\beta)$ as the unknown eigenvalues. The critical β is then determined by $E(\beta) = \beta V_0$. As is shown in the mathematical literature [10], the positive and Hermitian operator $-\Delta + \beta V^{*}(A_{03})$ has a discrete spectrum of positive eigenvalues $E_K(\beta)$.

The lowest eigenvalue $E_0(\beta)$ is the absolute minimum of the functional

$$F[\psi, \beta] = \frac{\int \left(|\nabla\psi|^2 + \beta V^{*}|\psi|^2 \right) dx\,dy}{\int |\psi|^2\, dx\,dy} \tag{39}$$

For every test function ψ_T we obtain

$$E_0(\beta) \leq \frac{\int \left(|\nabla\psi_T|^2 + \beta V^{*}|\psi_T|^2 \right) dx\,dy}{\int |\psi|^2\, dx\,dy} \tag{40}$$

The right-hand side is an upper limit of $E_0(\beta)$. Since V^{*} is positive, we also obtain a lower limit of $E_0(\beta)$:

$$E_0(\beta) \geq \min \frac{\int |\nabla\psi|^2\, dx\,dy}{\int |\psi|^2\, dx\,dy} = E_0(0) \tag{41}$$

In a finite region, the lowest eigenvalue of $-\Delta$ is positive ($E_0(0) > 0$). Since $F[\psi, \beta]$ is a continuous function of β, the $E_0(\beta)$ depends continuously on β.

From the two inequalities (41) and (40) we obtain a lower and upper limit for β_c.

The lower limit is given by $E_0(0) = \beta_1 V_0$, and the upper limit is determined by

$$F[\psi_T, \beta_2] = \beta_2 V_0$$

In order to show that the upper limit β_2 is finite, we write the condition $F[\psi_T, \beta] = \beta V_0$ in the form

$$F[\psi_T, \beta] = \frac{\int |\nabla \psi_T|^2 dx dy}{\int |\psi|^2 dx dy} + \beta \frac{\int V |\psi_T|^2 dx dy}{\int |\psi|^2 dx dy} + \beta V_0 \qquad (42)$$
$$= \beta V_0$$

The function V is negative near the separatrix, and we can always localize ψ_T to this region in order to make $\int V |\psi_T|^2 dx dy$ negative. The negative region of V is broad near the neutral point, and so we localize ψ_T to the neighbourhood of the neutral point. Thus it is always possible to find ψ_T in order to make $\int |\nabla \psi_T|^2 dx dy + \beta \int V |\psi_T|^2 dx dy = 0$ with finite β. The curves $F[\psi_T, \beta]$, $E_0(\beta)$ and βV_0 are shown in Fig. 5.

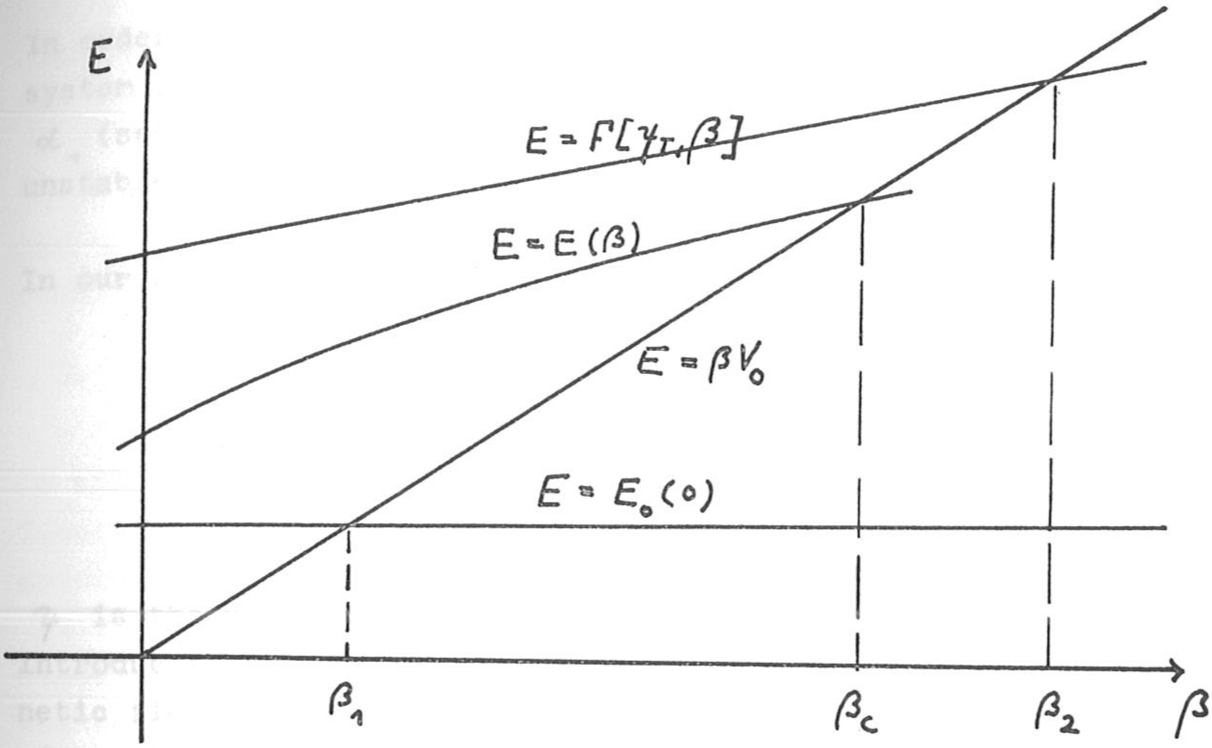

Fig. 5
Lower and upper bounds of the eigenvalue E(ß)

As an example, we consider the test function $\psi_T = e^{-\lambda(x^2+y^2)}$ ($\lambda > 0$ arbitrary). In the case $\lambda \gg \frac{1}{r_0{}^2}$, we obtain approximately

$$F[\psi_T, \beta] \approx 2\lambda + \beta V_0 \left(\frac{1}{r_0{}^4 (2\lambda)^2} - 1 \right) \tag{43}$$

If we minimize $F[\psi_T, \beta]$ with respect to λ and put $F[\psi_T, \beta] = \beta V_0$, the result is

$$\beta_2 \approx \frac{1}{\sqrt{8}} \left(\frac{r_0}{R_0} \right)^2 \tag{44}$$

This is an upper limit for the critical β_c, which is also the bifurcation point of the equilibrium.

In order to investigate the non-linear stability of the system for $\beta < \beta_c$, we have to calculate the coefficient α_1 (see 27). If $\alpha \neq 0$, the configuration is non-linearly unstable for $\beta < \beta_c^1$.

In our case, α_1 is found to be

$$\alpha_1 = - \frac{1}{2} \frac{\int \frac{\partial^2 j_0}{\partial A_{03}^2} \psi^3 \, dx\,dy}{\int |\nabla \psi|^2 \, dx\,dy} \tag{45}$$

ψ is the neighbouring equilibrium and a solution of eq. (37). Introducing A_{03}, ϕ_0 (vector and scalar potentials of the magnetic field) as new coordinates, we can write the numerator of eq. (45) as follows:

$$Z = \frac{1}{2} \int \frac{\partial^2 j_0}{\partial A_{03}^2} \left[\oint \frac{\psi^3 \, d\phi_0}{B^2(A_{03}, \phi_0)} \right] dA_{03} \tag{46}$$

$B(A_{03}, \phi_0)$ is the magnetic field strength and the Jacobian of the transformation $x, y \longleftrightarrow A_{03}, \phi_0$.

$j_0''(A_{03})$ is an antisymmetric function of A_{03}, but, in general, $\psi(A_{03}, \phi_0)$ has no symmetries. $\psi(A_{03}, \phi_3)$ is a solution of

$$- B^2(A_{03}, \phi_0) \left(\frac{\partial^2}{\partial A_{03}^2} + \frac{\partial^2}{\partial \phi_0^2} \right) \psi + \beta_c V(A_{03}) \psi = 0$$

Since $B^2(A_{03}, \phi_0)$ is not symmetric in A_{03}, $\psi(A_{03}, \phi_0)$ is not symmetric. This also holds for $G(A_{03}) = \oint \psi_0^3 / B^2 \, d\phi_0$.

Therefore $\int \mathcal{J}_0''(A_{03})\, G(A_{03})\, dA_{03}$ generally does not vanish.

But a situation with these symmetries can easily be constructed. Let us consider an X-type neutral point where the separatrix consists of two straight lines. The vector potential A_{03} is

$$A_{03} = \mathrm{Const} \times (y^2 - x^2)$$

In this case, $\psi(A_{03}, \phi_0)$, $B^2(A_{03}, \phi_0)$ are symmetric functions in A_{03} and the coefficient α_1 vanishes. The two solutions for the equilibrium field are A_{03} and $A_{03} + \varepsilon\psi$.

Since ψ is not zero on the separatrix ($A_{03} = 0$), we see that in the second equilibrium $A_{03} + \varepsilon\psi = A$ the minimum of the plasma pressure is not on the separatrix. The minimum is always on the line $A = 0$, in the second case this line differs from the line $A_{03} = 0$. If we approximate A_{03} by $A_{03} \approx y^2 - x^2$ (in the neighbourhood of the neutral point) and $\psi \approx \psi_T = \exp[-\lambda(x^2+y^2)]$, the equilibrium branches L_3 , L_4 are described by the vector potential $A \approx y^2 - x^2 + \varepsilon e^{-\lambda(x^2+y^2)}$. This field is sketched in Fig. 6

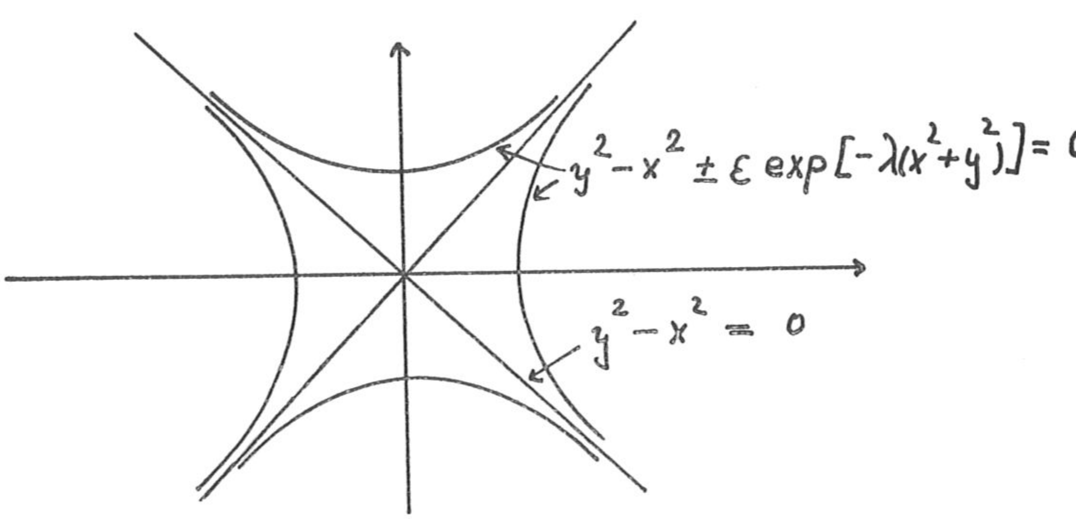

Fig. 6

Perturbation in the neighbourhood of the neutral point.

6) Conclusions

It has been shown that in a plasma there can exist a bifurcation point of the equilibrium. The bifurcation point is also the transition point of the stability. As an example we considered a plasma in a magnetic quadrupole field. If there is a minimum of the plasma pressure on the separatrix, there exists a critical β_c at which the equilibrium bifurcates.

Owing to the asymmetry of the magnetic field with respect to the separatrix, the equilibrium is already unstable at $\beta < \beta_c$ for non-linear perturbations. Since we have only investigated the neighbourhood of the bifurcation point, we cannot say where the lower bound of this unstable region is.

In multipole machines used in fusion research, the pressure has a maximum on the separatrix, and a critical β_c for the collisionless tearing mode does not exist.

The instability mentioned above may possibly explain some explosive phenomena in the magnetic tail of the earth and it might perhaps also occur in its polar regions. In the paper of Coppi et al. [4], the authors consider an infinite sheath with inverse magnetic field as a model for the magnetic tail. Since the infinite sheath is always unstable, no explanation for the stable periods and the sudden bursts of the instability was given. In a realistic model one should take into account the transition region between the earth's magnetic field and the magnetic tail, this being at least a two-dimensional problem.

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