INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Solution of the Linear Portion
of Dupree's Perturbation Theory
for Plasma Turbulence

H. Gratzl

IPP 6/65 March 1968

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Institut für Plasmaphysik GmbH und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

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ABSTRACT:

For a turbulent plasma Dupree has given equations somewhat different from the quasilinear equations, using particle orbits perturbed by the fields for integration. Special assumptions about the solution are made by Dupree for his explicit kinetic equations.
In the case of a small diffusion coefficient these assumptions are unnecessary and, in general, one of them even has to be abandoned if energy conservation is to be valid. The differences to the quasilinear equations are in terms that are nonlinear in the field energy.
But in the case of a very large diffusion coefficient, first considered by Dupree, a good approximate expansion is no longer obtained, if the time variation of the distribution function is taken into account.
As Orzag and Kraichman have explained, the terms containing initial values can be added. In the case of small diffusion it is shown here that these terms yield familiar results without the secularities of other methods.

C o n t e n t s :

Introduction

The most easily treated case of turbulence in a plasma is probably that of a homogeneous plasma in a strong magnetic field in which longitudinal electric fields are excited by an electron beam of superthermal velocity in the tail of the distribution function (bump in tail), which is parallel to the magnetic field.

The q u a s i l i n e a r t h e o r y [1] describes the variation of the distribution function and the wave spectrum when the instability is not too strong and the spectrum is sufficiently wide. The electron distribution function thereby (in the one-dimensional case) asymptotically attains a level plateau in the unstable region, while the wave spectrum forms a kind of bell curve [1, 6] . The magnetic field makes the problem to a simple one-dimensional, whereas the three-dimensional problem without magnetic field has been considered only in a topological way [6] .

The c o r r e c t i o n s to the quasilinear theory are:

1) Those made by non-linear terms in the wave energy that follow iteratively from the Vlasov equation [1 - 3] .

2) The electron distribution function varies according to the quasilinear theory only adiabatically with time. When the times are not too large the non-adiabatic corrections [4, 5, 9] are particularly important for narrow spectra.

3) Dupree takes into account perturbation of the particle trajectories by the waves [7] , which is not necessarily the case in the Vlasov equation (cf. [10]). With the assumptions of the quasilinear theory his calculation yields corrections of the order of 2), as will be shown.

4) Spontaneous emission [11 - 13] and Lenard-Balescu collisions [12], [13] are mostly ignored.

The most recent correction of these is 3), which describes the trapping of the particles by the waves. Here Dupree considers the phase change of the density fluctuations that is due to repeated interaction with the waves.

The result is derived in [10] as well in a slightly different way, but again by expanding in the same operator, namely an ensemble average Green's function.

The lowest order equations (linear portion of the operator) are used in the following as initial equations.

Dupree evaluates them only after making simplifying assumptions, but recognizes the need to adopt a more accurate treatment. In his final equations the energy is conserved, but only because he approximates roughly a complicated function (by a rectangular function) to make his equations easier to follow. For the case of a large diffusion coefficient, which is what Dupree considers, accurate treatment is not possible in the following study either (i.e. analytical treatment is not, but numerical treatment is); but for a small diffusion coefficient this can be done.

Dupree confines himself to deriving kinetic equations when the distribution function varies only adiabatically with time and the distribution function and diffusion coefficient depend only slightly on the velocity. The latter approximation, for instance, is very poor for the bell-shaped wave spectrum that is obtained in the quasilinear theory.

If these assumptions are abandoned, the trapping effect turns out to be even greater in Dupree's case of a *large diffusion* coefficient. - The terms due to the time dependence of the distribution function become so large here, however, that the expansion for the solution diverges.

For a s m a l l coefficient, as in the quasilinear theory, one obtains in addition to the small Dupree correction in this case others of similar magnitude. Only corrections that are linear in the diffusion coefficient are taken into account.

Even if the diffusion coefficient is independent of velocity and the distribution function is first assumed to be adiabatic, there still remains another correction besides Dupree's in the equation for the distribution function. The energy equation can, in general, only be satisfied with both of these corrections (Appendix I). By transformation it can be shown, furthermore, that the correction for a time dependent distribution function is of the same order as Dupree's, and so the latter is pointless without the former. So the corrections to the quasilinear result turn out to be much more complicated functions in the case of a not too large diffusion coefficient than Dupree's approximation would suggest. They should be compared with the other nonlinear corrections, which are not dealt with in detail in [7] and not at all here. As shown in [10], the initial value problem can be treated by an appropriate extension of Dupree's equations. There follow the known inhomogeneous terms of the corrections 4) [13] with just the correct time dependence (in the diffusion approximation for the distribution function), which otherwise must be obtained by summation of secular terms in higher order corrections [14].

Dupree's initial equations

If the corrections 1) are not used, attention can be confined in Dupree's method [7] to the linear portion of the propagator $\mathcal{U}$. In this case it should be calculated more exactly than in [7] in order to obtain all possible corrections of the types 2) and 3).

The following is based on Dupree's equation from the beginning of his sixth chapter (his eq. 6.5), which is also derived in [10] (as eq. 3.1) in another way. This equation for the ensemble average distribution function $\langle f(v,w,t)\rangle$, abbreviated to $\langle f(t)\rangle$, which in a homogeneous plasma no longer depends on space, is:

$$\left(\frac{\partial}{\partial t} + w\frac{\partial}{\partial v}\right)\langle f(t)\rangle = \left\langle \frac{e}{m}\mathcal{E}\frac{\partial}{\partial w} f(t)\right\rangle =$$

$$= \frac{\partial}{\partial w}\left(\frac{e}{m}\right)^2 \int_{t_0}^{t} d\tau \sum_k \mathcal{E}_{-k}(t)\mathcal{E}_k(\tau) e^{-ikv} \langle U(t,\tau)\rangle e^{ikv} \frac{\partial}{\partial w}\langle f(\tau)\rangle$$

Here $U(t,t_0)$ is defined by $f(t) = U(t,t_0)\cdot f(t_0)$ if $f(t)$ is the solution of the Vlasov equation at time t for the initial condition $f(t_0)$, $U(t_0,t_0) = 1$.

For the ensemble average $\langle U\rangle$ it follows that $\langle f(t)\rangle = \langle U(t,t_0)\rangle f(t_0)$. This is because the initial distribution function is chosen independent of the phases of the waves from which the average was taken (< >). On substituting $\tau'' := t-\tau$ it follows that

$$\left(\frac{\partial}{\partial t} + w\frac{\partial}{\partial v}\right)\langle f(t)\rangle =$$

(5) $$= -\frac{\partial}{\partial w}\left(\frac{e}{m}\right)^2 \int_{t-t_0}^{0} d\tau'' \sum_k \mathcal{E}_{-k}(t)\mathcal{E}_k(t-\tau'') e^{ikv} \langle U(t,t-\tau'')\rangle e^{ikv} \frac{\partial}{\partial w}\langle f(t-\tau'')\rangle$$

Let us write for this equation the diffusion ansatz:

(5') $$= \frac{\partial}{\partial w} \underline{D}(w,t) \frac{\partial}{\partial w}\langle f(t)\rangle$$

In the following we investigate whether an approximative solution of this diffusion form can be found.

If the operator $\langle U(t,t_0)\rangle$ is applied to an arbitrary function $F(v,w,t_0)$, then the equation for $F(v,w,t) := \langle U(t,t_0)\rangle F(v,w,t_0)$ can be written in the form

(6) $$\left(\frac{\partial}{\partial t} + w\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\underline{\underline{D}}\frac{\partial}{\partial w}\right) F(v,w,t) = 0$$

If the equation (27) for the field is also applied it is possible to show the validity of the energy equation for the asymptotic solution (Appendix II).

Integration without Dupree's additional assumptions

To solve (5) in the one-dimensional problem, Dupree [7] makes the approximation:

$$(7) \quad \langle U(t,t-\tau')\rangle e^{ikx} \frac{\partial}{\partial v} \langle f(t-\tau')\rangle \approx \langle \frac{\partial f(t)}{\partial v} \rangle \cdot \langle U(t,t-\tau')\rangle e^{ikx}$$

It is shown in the following that, in general, this approximation is inconsistent if more than the quasilinear result is required. On the further assumption that D in (5') - in this approximation (7) there follows from (5) a diffusion equation of the form (5') - depends on v to a negligible extent, Dupree then gets (his equation 7.2):

$$(8) \quad D(v,t) \approx D_1(v,t) \qquad D_1(v,t) := \left(\frac{e}{m}\right)^2 \int_0^\infty d\tau' \sum_k |E_k(t)|^2 e^{-i\int_{t-\tau}^{t} (kv-\omega_k)d\tau - \int_0^{\tau'} k^2 D\tau^2 d\tau}$$

and on the assumption that (9) $\left(\frac{D}{\omega_p v^2}\right)^{1/3} \ll \frac{\Delta v}{v}$

(Δv = width of the unstable region, ω_p = plasma frequency) he obtains from it (his equation 7.8) for resonance particles:

$$(10) \qquad D_1(v,t) \approx D_0(v,t) \qquad \text{with} \quad D_0(v,t) := \frac{\pi e^2}{m^2 v} \left|E_{\frac{\omega_k}{v}}(t)\right|^2$$

For simplicity, only the time dependence of D is sometimes written explicitly as argument, as is also done for $\langle U \rangle$ and $\langle f \rangle$. Instead of the approximation (7) we require an approximate solution of the complete expression from (5):

$$(11) \qquad \langle U(t,t-\tau')\rangle e^{ikx} \frac{\partial}{\partial v} \langle f(t-\tau')\rangle \qquad \tau_0 := t-\tau'$$

or $$\langle U(\tau_0+\tau',\tau_0)\rangle e^{ikx} \frac{\partial}{\partial v} \langle f(\tau_0)\rangle := L(v,\tau_0+\tau';\tau_0)$$

with velocity dependent D.
The solution L is written as a product:

(12) $$L = A(x,v,\tau_0+\tau') \cdot B(v,\tau_0+\tau') \cdot C(v,\tau_0+\tau')$$

Here $A(x,v,\tau_0+\tau')$ is the partial solution of the approximation (7):

$$A(x,v,\tau_0+\tau') := \langle \mathcal{U}(\tau_0+\tau',\tau_0)\rangle \, e^{ikx}$$

It can be given by

(13) $$A(x,v,\tau_0+\tau') = e^{ikx - ikv\tau' + \sum_{\nu=1}^{\infty} P_\nu}$$ (see (31))

where every P_ν contains the diffusion coefficient D to be determined or its velocity derivative ν times.

Let us also define $$C(v,\tau_0+\tau') := \langle \mathcal{U}(\tau_0+\tau',\tau_0)\rangle \langle \frac{\partial f(\tau_0)}{\partial v} \rangle$$

Since C does not depend on x it has only to satisfy the diffusion equation:

(14) $$\left(\frac{\partial}{\partial \tau'} - \frac{\partial}{\partial v} D \frac{\partial}{\partial v} \right) C(v,\tau_0+\tau') = 0$$

An explicit solution of this equation can be given by means of the variable $\langle f(\tau_0+\tau')\rangle$ and in powers of D by iteration for short times. But from this solution only the integral in (5) or (27) is interesting. There one gets a good approximation in the case of a small diffusion coefficient, while the difficulties in the opposite case are explained in section b.

$$C(v,\tau_0+\tau') = \langle \frac{\partial f(\tau_0)}{\partial v} \rangle + \int_0^{\tau'} \frac{\partial}{\partial v} D(\tau_0+\tau) \langle \frac{\partial^2 f}{\partial v^2}(\tau_0+\tau) \rangle + \sum_{n=2}^{\infty} K_n(\tau')$$

(15)

with $$K_1(\tau') := \frac{\partial}{\partial v} \int_0^{\tau'} \frac{\partial D(\tau_0+\tau)}{\partial v} \langle \frac{\partial f}{\partial v}(\tau_0+\tau) \rangle d\tau$$

$$K_n(\tau') := \int_0^{\tau'} \frac{\partial}{\partial v} D(\tau_0+\tau) \frac{\partial K_{n-1}(\tau_0+\tau)}{\partial v} d\tau \qquad n = 2,3,4,\ldots$$

For the first step (n = 2) here the equation (5') is used for conversion, from which it follows that

$$\left\langle \frac{\partial f}{\partial v}(\tau_0)\right\rangle + \int_0^{\tau'} \frac{\partial}{\partial v} D \left\langle \frac{\partial^2 f}{\partial v^2}\right\rangle d\tau = \left\langle \frac{\partial f}{\partial v}(\tau_0+\tau')\right\rangle + K_1$$

From the ansatz (12) - which has to satisfy (6) (F being set equal to L) - there then follows for B ($v, \tau_0+\tau'$) the equation:

(16) $$\frac{\partial B}{\partial \tau'} = 2D \frac{\partial A}{\partial v}\frac{\partial C}{\partial v}\frac{B}{AC} + 2D\left(\frac{\partial A}{\partial v}\frac{\partial B}{\partial v}A^{-1} + \frac{\partial B}{\partial v}\frac{\partial C}{\partial v}C^{-1}\right) + \frac{\partial D}{\partial v}\frac{\partial B}{\partial v}$$

except for the zeros of C.

If A and C are given by an expansion of the form (13) and (15) respectively, we can put up for B the ansatz:

(17) $B = e^{\sum_1^\infty Q_\nu} + \dots$ and calculate the various terms from (16). (D or a velocity derivative of D should occur ν times in Q_ν.) It follows from (13) and (16) that

(18) $$Q_1 = -2ik\int_0^{\tau'} D\tau \, \frac{\left\langle \frac{\partial^2 f}{\partial v^2}\right\rangle}{\left\langle \frac{\partial f}{\partial v}\right\rangle} d\tau$$

a) Small diffusion coefficient

If the diffusion coefficient is so small that we can confine ourselves in the expansions (13), (15) and (17) to the linear terms in D (see eq. (35)), it is possible to state the solution L of (11):

(19) $$L(v,\tau_0+\tau';\tau_0) \approx e^{ikx - ikv\tau' - \int_0^{\tau'} k^2 D\tau^2 d\tau - ik\int_0^{\tau'} \frac{\partial D}{\partial v}\tau \, d\tau - 2ik\int_0^{\tau'} D\tau \frac{\left\langle \frac{\partial^2 f}{\partial v^2}(\tau_0+\tau)\right\rangle}{\left\langle \frac{\partial f}{\partial v}(\tau_0+\tau)\right\rangle} d\tau} \cdot$$
$$\cdot \left(\left\langle \frac{\partial f}{\partial v}\underbrace{(\tau_0+\tau')}_{t}\right\rangle - \frac{\partial}{\partial v}\int_0^{\tau'} \frac{\partial D}{\partial v}\left\langle \frac{\partial f}{\partial v}(\tau_0+\tau)\right\rangle d\tau \right)$$

Here $D = D(v, \tau_0+\tau)$. The zero of $\left\langle \frac{\partial f}{\partial v}\right\rangle$ in the denominator of the exponent is not critical since this denominator is discarded in the investigated approximation (terms that are only linear in D).

Equation for the distribution function

If (12) is substituted into (5): $\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) \langle f(t)\rangle \approx$

$$\approx \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \int_0^{t-t_0} d\tau' \sum_k E_{-k}(t)\, E_k(t-\tau')\, e^{-ikv\tau' - \int_0^{\tau'} k^2 D \tau^2 d\tau - ik\int_0^{\tau'} \frac{\partial D}{\partial v}\tau\, d\tau - 2ik\int_0^{\tau'} D\tau \frac{\langle \frac{\partial^2 f}{\partial v^2}\rangle}{\langle \frac{\partial f}{\partial v}\rangle} d\tau} \cdot$$

$$\cdot \frac{\partial}{\partial v}\left(\langle f(t)\rangle - \int_0^{\tau'} \frac{\partial D}{\partial v} \left\langle \frac{\partial f}{\partial v}(t-\tau'+\tau)\right\rangle d\tau\right)$$

With the ansatz (20) $E_k(t) = E_k(t_0)\, e^{-i\int_{t_0}^{t} \omega_k(\tau)d\tau}$ it follows that

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f(t) \approx \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \int_0^{t-t_0} d\tau' \sum_k |E_k(t)|^2\, e^{-i\int_{t-\tau'}^{t}(kv-\omega_k(\tau))d\tau - \int_0^{\tau'} k\left(kD\tau - i\frac{\partial D}{\partial v}\right)\tau d\tau - 2ik\int_0^{\tau'} D\tau \frac{\langle \frac{\partial^2 f}{\partial v^2}\rangle}{\langle \frac{\partial f}{\partial v}\rangle} d\tau} \cdot \frac{\partial f(t)}{\partial v}$$

$$- \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \int_0^{t-t_0} d\tau' \sum_k |E_k(t)|^2\, e^{-i\int_{t-\tau'}^{t}(kv-\omega_k(\tau))d\tau} \cdot \frac{\partial}{\partial v}\int_0^{\tau'} \frac{\partial D}{\partial v}\left\langle \frac{\partial f}{\partial v}(t-\tau'+\tau)\right\rangle d\tau$$

(21)

if quadratic terms in D are again ignored in the second sum term.

Apart from quadratic terms in D and γ_k, both terms of the sum can be transformed by partial integration: $(\gamma_k > 0)$

$$(21) \approx \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \sum_k |E_k(t)|^2 \left[\left(\frac{\gamma_k}{(kv-\omega_k)(kv-\omega_k^*)} - (P)\frac{2k^2 D}{(kv-\mathrm{Re}\,\omega_k)^4} + (P)\frac{k\frac{\partial D}{\partial v}}{(kv-\mathrm{Re}\,\omega_k)^3}\right)\left\langle \frac{\partial f(t)}{\partial v}\right\rangle + \right.$$

$$(22) \quad + (P)\frac{2kD}{(kv-\mathrm{Re}\,\omega_k)^3}\left\langle\frac{\partial^2 f(t)}{\partial v^2}\right\rangle + \frac{(P)}{(kv-\mathrm{Re}\,\omega_k)^3}\frac{\partial \mathrm{Re}\,\omega_k}{\partial t}\left\langle\frac{\partial f(t)}{\partial v}\right\rangle + \qquad \text{(p = principal value)}$$

$$\left. + \frac{(P)}{(kv-\mathrm{Re}\,\omega_k)^2}\frac{\partial}{\partial v}\left(\frac{\partial D}{\partial v}\left\langle\frac{\partial f(t)}{\partial v}\right\rangle\right)\right]$$

This is exactly the same as solving for D as perturbation instead of integrating in closed form (which is necessary for a greater D) and expanding afterwards. If (5') is used for transforming the last expression and a few terms are combined, we obtain forms that are more favourable for comparison and for proving the energy balance in detail (Appendix I):

$$(22) = \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \sum_k |E_k(t)|^2 \left[\left(\frac{\gamma_k}{(kv-\omega_k)(kv-\omega_k^*)} + \frac{k}{kv-\mathrm{Re}\,\omega_k}\left(\frac{\partial}{\partial v}\frac{D}{(kv-\mathrm{Re}\,\omega_k)^2}\right)\right)\left\langle\frac{\partial f}{\partial v}\right\rangle - \right.$$

$$(23) \quad \left. - \frac{\partial}{\partial v}\frac{D}{(kv-\mathrm{Re}\,\omega_k)^2}\left\langle\frac{\partial^2 f}{\partial v^2}\right\rangle + \frac{1}{(kv-\mathrm{Re}\,\omega_k)^3}\frac{\partial \mathrm{Re}\,\omega_k}{\partial t}\left\langle\frac{\partial f}{\partial v}\right\rangle + \frac{1}{(kv-\mathrm{Re}\,\omega_k)^2}\frac{\partial}{\partial t}\left\langle\frac{\partial f}{\partial v}\right\rangle\right] =$$

$$(23') \quad = \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \sum_k |E_k(t)|^2 \left[\frac{\gamma_k}{(kv-\omega_k)(kv-\omega_k^*)} - \frac{(P)}{kv-\mathrm{Re}\,\omega_k}\left(\frac{\partial}{\partial v} D \frac{\partial}{\partial v}\frac{\left\langle\frac{\partial f}{\partial v}\right\rangle}{kv-\mathrm{Re}\,\omega_k} + \frac{\partial}{\partial t}\frac{\left\langle\frac{\partial f}{\partial v}\right\rangle}{kv-\mathrm{Re}\,\omega_k}\right)\right]$$

Another way of condensing again allows the time dependence of the distribution function to be discussed better (Appendix I):

$$(23) = \frac{\partial}{\partial v}\left(\frac{e}{m}\right)^2 \sum_k |E_k(t)|^2 \left[\left(\frac{\gamma_k}{(kv-\omega_k)(kv-\omega_k^*)} - \frac{2k^2 D}{(kv-\mathrm{Re}\,\omega_k)^4}\right)\left\langle\frac{\partial f}{\partial v}\right\rangle - \right.$$

$$(23'') \quad - \frac{2}{kv-\mathrm{Re}\,\omega_k}\frac{\partial}{\partial v}\frac{1}{kv-\mathrm{Re}\,\omega_k}\left\langle\frac{\partial f}{\partial t}\right\rangle + \frac{1}{kv-\mathrm{Re}\,\omega_k}\frac{\partial}{\partial v}\frac{1}{kv-\mathrm{Re}\,\omega_k}\frac{\partial D}{\partial v}\left\langle\frac{\partial f}{\partial v}\right\rangle +$$

$$\left. + \frac{1}{(kv-\mathrm{Re}\,\omega_k)^3}\frac{\partial \mathrm{Re}\,\omega_k}{\partial t}\left\langle\frac{\partial f}{\partial v}\right\rangle + \frac{2}{(kv-\mathrm{Re}\,\omega_k)^2}\frac{\partial}{\partial t}\left\langle\frac{\partial f}{\partial v}\right\rangle\right.$$

For the distribution function in the region of the resonance particles the term $\sim \frac{\partial \mathrm{Re}\,\omega_k}{\partial t}$ can be neglected.

The contributions of the upper limit $t-t_0$ are neglected here. The initial values are not taken into account either.

This means that collisions and spontaneous emission [13, 12] are ignored. The latter is only important for small (thermal) wave energy density and in the region of marginal stability [12]. In the last chapter we shall extend the method to these effects.

Dupree [7] neglects them by transferring to the limit $t-t_0 \rightarrow \infty$.

Because of the small coefficient D assumed (according to the inequality (35) later) it is sufficient to use the approximation (10) for the expression D in the unstable region. From (23) and the ansatz (5') D would have to be determined more exactly, however, if investigation is to be made of corrections that are non-linear in D.

Magnitude of the terms in the bump-in-tail case

The first term in (22) - (23), the diffusion term of the quasilinear theory, can be divided into the delta function and a principal value. For the resonance particles the latter term is of the order

(24) $$\frac{\gamma(t)}{\omega_p} \left(\frac{\Delta v}{v}\right)^{-1}$$

smaller, but at the boundary of the region with positive derivation of the distribution function ($\langle \frac{\partial f}{\partial v} \rangle > 0$ for $v > 0$) it may be larger. Δv is the width of this region.

The next term in (22), the Dupree correction term, is of the order

(25) $$\frac{k^2 D}{\omega_p^3 \left(\frac{\Delta v}{v}\right)^3} \approx \frac{D}{\omega_p v^2} \left(\frac{\Delta v}{v}\right)^{-3}$$

the third power of (9). In the quasilinear theory an upper limit is obtained for this expression, this being [6]:

$$\left.\frac{D(t)}{\omega_p v^2}\right/_{t-t_0 \approx \gamma_{max}(t_0)^{-1}} \cdot \left(\frac{\Delta v}{v}\right)^{-3} \approx \frac{\gamma_{max}(t_0)}{\omega_p} \left(\frac{\Delta v}{v}\right)^{-1} \tag{26}$$

The two following terms and the last one in (22) (or the two following the quasilinear one and the last one in (23)) are of similar magnitude to the Dupree term if $|\frac{\partial}{\partial v}| \approx \frac{1}{\Delta v}$ (as for (33)). But right in the centre of the unstable region their magnitude is somewhat smaller and they are not symmetric with respect to the centre of the unstable region, but the Dupree correction is, at least in sign. They are due to the velocity dependence of D and the effect of $\langle U \rangle$ on $\langle \frac{\partial f}{\partial v} \rangle$ in (11).

The second last term in (22) is about $\left(\frac{k v_{th}}{\omega_p}\right)^2$ smaller according to [9] .

Let us forget this term for a moment. If the velocity dependence of D is now ignored as in [7] , another term besides the one following from the approximation (7) of [7] is left over in (22). The energy equation can, in general, only be satisfied with both of these together (Appendix I); with the first term done the energy is conserved only in the special approximation used in [7] . +)

Comparison with the usual treatment of the Vlasov equation

It is true that the Dupree term (the second in (22)) cannot be obtained from the Vlasov equation on integrating over unperturbed paths, nor can the third and fourth terms. The latter two terms, can, however, be obtained by partial integration if the time dependence of the distribution function is taken into account more exactly [9, 13]. (In [7] they are eliminated because in the approximation (7) $\frac{\partial}{\partial v} \langle f(t) \rangle$ is substituted for $\frac{\partial}{\partial v} \langle f(t-\tau') \rangle$.)

Footnote to page 10

+) If merely the (damping) term $-\int_0^{\tau'} k^2 D \tau^2 d\tau \approx -\frac{k^2 D}{3}\tau'^3$ is considered in equations (22) and (28) as a correction to the quasilinear treatment - as is done in [7] - this term can, irrespective of the size of D, be replaced by $-\left(\frac{k^2D}{3}\right)^{1/3}\tau'$ to get the same damping of the integral kernel, in the characteristic time. Instead, in [7] the integral is replaced by a rectangular function.

In both procedures the shape of the function is changed somewhat, the result being that energy conservation is restored. With the true shape, terms other than the one above must be retained, for energy conservation, as is shown here (in Appendix I) in the case of a small diffusion term D. Dupree [7] confines his attention, however, to the case of a large diffusion term.

Besides these terms in (22), however, another term is then obtained; and these together give the last two terms in (23). They are relatively unimportant if the variation of the distribution function goes to zero, as in the neighbourhood of the asymptotic state.

The last term of (23) can already be found in [5, 9], while the appropriate corrections in the equation for the waves (see next section) are given for the first time in [4]. Thus, all that is new here are the two terms following the quasilinear one in (23).

They can be combined with the Dupree term to give the closed form in (23'), which is very similar to the usual non-linear corrections (not treated here), but has different denominators.

Equation for the waves

To obtain the dispersion relation and the equation for the square of the wave amplitude, (19) should be substituted into the Poisson equation.

Again neglecting the initial values and in the limit $t-t_0 \to \infty$, we have according to eq. 5.2 in [7]

$$(27)\qquad E_k(t) \approx \frac{4\pi e^2}{m}\,\frac{ik}{k^2}\int d^3v \int_{t_0}^{t} d\tau\, E_k(\tau)\, e^{-ikv} \langle U(t,\tau)\rangle\, e^{ikv}\, \frac{\partial}{\partial v}\langle f(\tau)\rangle$$

With (19) and (21) this gives the asymptotic dispersion relation in the one-dimensional case:

$$\varepsilon_{k\omega_k} = 1 - \frac{4\pi e^2}{m}\,\frac{i}{k}\int dv \cdot \int_0^\infty d\tau' \cdot$$

$$(28)\qquad \cdot\Big[e^{-i\int_{t-\tau'}^{t}(kv-\omega_k)\,d\tau - \int_0^{\tau'} k^2 D\tau^2 d\tau - ik\int \frac{\partial D}{\partial v}\tau\, d\tau - 2ik\int D\tau \frac{\langle \frac{\partial^2 f}{\partial v^2}\rangle}{\langle \frac{\partial f}{\partial v}\rangle} d\tau} \Big\langle \frac{\partial f}{\partial v}(t)\Big\rangle -$$

$$- e^{-i\int_{t-\tau'}^{t}(kv-\omega_k)\,d\tau}\; \frac{\partial}{\partial v}\int_0^{\tau'} \frac{\partial D}{\partial v}\Big\langle \frac{\partial f}{\partial v}(t-\tau'+\tau)\Big\rangle d\tau \Big] = 0$$

Disregarding quadratic terms in D this reads:

$$\varepsilon_{k\omega_k} \approx 1 - \frac{4\pi e^2}{m}\int dv \Big[\Big(\frac{1}{kv-\omega_k(t)} - i\,\frac{2k^2 D}{(kv-\omega_k)^4} + i\,\frac{k\frac{\partial D}{\partial v}}{(kv-\omega_k)^3}\Big)\Big\langle \frac{\partial f(t)}{\partial v}\Big\rangle +$$

$$+ i\frac{\partial}{\partial v}\frac{2kD}{(kv-\omega_k)^3}\Big\langle \frac{\partial^2 f}{\partial v^2}\Big\rangle + \frac{i}{(kv-\omega_k)^3}\frac{\partial \omega_k}{\partial t}\Big\langle \frac{\partial f}{\partial v}\Big\rangle + \frac{i}{(kv-\omega_k)^2}\frac{\partial}{\partial v}\Big(\frac{\partial D}{\partial v}\Big\langle \frac{\partial f}{\partial v}\Big\rangle\Big)\Big] =$$

$$(29)\qquad = 1 - \frac{4\pi e^2}{mk}\int dv \Big[\frac{1}{kv-\omega_k} + \frac{ik}{kv-\omega_k}\Big(\frac{\partial}{\partial v}\frac{D}{(kv-\omega_k)^2}\Big) + \frac{i}{kv-\omega_k}\frac{\partial}{\partial t}\frac{1}{kv-\omega_k}\Big]\Big\langle \frac{\partial f}{\partial v}\Big\rangle$$

The solution to this gives $\gamma_k \equiv \mathrm{Im}\,\omega_k$ for the equation of the spectral wave energy density:

$$(30)\qquad \frac{\partial \ln |E_k(t)|^2}{\partial t} = 2\gamma_k$$

If the growth (or damping) rate γ_k is divided into a component γ_k^L, i.e. the solution of the quasilinear dispersion relation ($\varepsilon^L_{k\omega_k} = 0$), and a component $\Delta\gamma_k$, it follows from the expansion of (29) for linear terms in that

$$\Delta\gamma_k \approx \left(\frac{\partial \varepsilon^L_{k\omega}}{\partial \omega}\bigg/_{\omega \approx \mathrm{Re}\,\omega_k} \right)^{-1} \frac{4\pi e^2}{mk} \int dv \; \frac{(P)}{kv - \mathrm{Re}\,\omega_k} \cdot$$

$$\cdot \left[k \left(\frac{\partial}{\partial v} \; \frac{D}{(kv - \mathrm{Re}\,\omega_k)^2} \right) + \frac{\partial}{\partial t} \; \frac{1}{kv - \mathrm{Re}\,\omega_k} \right] \left\langle \frac{\partial f}{\partial v} \right\rangle$$

What is different here from the usual treatment of the linearized Vlasov equation in higher orders [9,13] is just the new term $k \frac{\partial}{\partial v} \frac{D}{(kv - \mathrm{Re}\,\omega_k)^2}$ in the brackets.

The quasilinear approximation is sufficient for $\mathrm{Re}\,\omega_k$ since only linear terms in D are taken into account.

Energy is conserved by the equations (23) and (30) (Appendix I), but also without the new terms [9] .

Apart from terms that are $\left(k \frac{v_{th}}{\omega_p}\right)^2$ smaller, it is possible to obtain the approximation for the non-linear variation (Appendix III): $\Delta\gamma_k \approx$

$$\approx \frac{7k^2}{16mn^2} \sum_{k'} |E_{k'}(t)|^2 \frac{\mathrm{Re}\,\omega_{k'}}{|k'|\cdot k'} \left\langle \frac{\partial f}{\partial v} \bigg/ \right\rangle_{v = \frac{\omega_{k'}}{k'}} + \frac{3|k|}{4mn^2} \sum_{k'} |E_{k'}(t)|^2 \left(\frac{\mathrm{Re}\,\omega_{k'}}{k'} \left\langle \frac{\partial f}{\partial v} \bigg/ \right\rangle_{v = \frac{\omega_{k'}}{k'}} + \frac{\mathrm{Re}\,\omega_{k'}^2}{2k'^2} \left\langle \frac{\partial^2 f}{\partial v^2} \bigg/ \right\rangle_{v = \frac{\omega_{k'}}{k'}} \right)$$

In the usual treatment of the linearized Vlasov equation, as in [9], only a little more than double the first expression is obtained (Appendix III).

For $\Delta\gamma_k$ the Dupree result is obtained if $\frac{\partial D}{\partial v}$ and $\frac{\partial f}{\partial t}$, $\frac{\partial \mathrm{Re}\,\omega}{\partial t}$ are disregarded.

b) Unrestricted diffusion coefficient

First the expansion of the solution (13) is calculated in the realistic case of a velocity dependent diffusion coefficient. If the coefficient is large the correction due to the time dependence of the distribution function is investigated.

Let the following ansatz be made for (13):

$$(31)\qquad A(x,v,\tau+\tau') = e^{ikx - ikv\tau' - \int_0^{\tau'} k^2 D\tau^2 d\tau - ik\int_0^{\tau'} \frac{\partial D}{\partial v}\tau\, d\tau + H(v,\tau')}$$

Let here $H := \sum_{\nu=2}^{\infty} P_\nu$ contain only terms that are at least quadratic in D. Since A has to satisfy equation (6) it holds for the quadratic corrections in D that:

$$\frac{\partial P_2}{\partial \tau'} + \left(\int_0^{\tau'} k^2 \frac{\partial D}{\partial v}\tau^2 d\tau + \int_0^{\tau'} \frac{\partial^2 D}{\partial v^2} ik\tau\, d\tau\right)\frac{\partial D}{\partial v} + \frac{\partial D}{\partial v}(ik\tau)^2 -$$

$$- D\left(ik\tau + \int_0^{\tau'} k^2 \frac{\partial D}{\partial v}\tau^2 d\tau + \int_0^{\tau'} \frac{\partial^2 D}{\partial v^2} ik\tau\, d\tau\right)^2 + D\int_0^{\tau'} k^2 \frac{\partial^2 D}{\partial v^2}\tau^2 d\tau + D\int_0^{\tau'} \frac{\partial^3 D}{\partial v^3} ik\tau\, d\tau = 0$$

The formal solution (as a function of D) is:

$$P_2(v,\tau') = -\frac{\partial}{\partial v}\int_0^{\tau'} D\int_0^{\tau} \frac{\partial^2 D}{\partial v^2} ik\tau'' d\tau''\, d\tau - \frac{\partial}{\partial v}\int_0^{\tau'} D\int_0^{\tau} k^2 \frac{\partial D}{\partial v}\tau''^2 d\tau'' d\tau +$$

$$(32)\qquad + \underline{\int_0^{\tau'} D\left(\int_0^{\tau} \frac{\partial^2 D}{\partial v^2} ik\tau'' d\tau''\right)^2 d\tau} + \int_0^{\tau'} D\left(\int_0^{\tau} k^2 \frac{\partial D}{\partial v}\tau''^2 d\tau''\right)^2 d\tau +$$

$$- \int_0^{\tau'} D\left[ik\tau\int_0^{\tau} k^2 \frac{\partial D}{\partial v}\tau''^2 d\tau'' - ik\tau\int_0^{\tau} \frac{\partial^2 D}{\partial v^2} ik\tau'' d\tau'' + \int_0^{\tau} k^2 \frac{\partial D}{\partial v}\tau''^2 d\tau'' \int_0^{\tau} \frac{\partial^2 D}{\partial v^2} ik\tau''' d\tau'''\right] d\tau$$

To estimate the non-linear terms in D let the time dependence of D now be neglected (which is what Dupree does throughout in [7]) relative to the powers of τ on integrating over τ .

Let, moreover, $\left|\frac{\partial D}{\partial v}\right| \approx \frac{D}{\Delta v}$, $\left|\frac{\partial^2 D}{\partial v^2}\right| \approx \frac{D}{(\Delta v)^2}$ etc.
and then to abbreviate let $g := D/(\Delta v)^2$
(if the quasilinear theory is applicable it holds according to (26) or [6] that $g \lessapprox g/_{t \approx \frac{1}{\gamma(t_0)_{max}}} \approx \gamma_{max}(t_0)$)
With this notation the magnitudes of the sum terms in (32) are respectively ($\omega_0 :\approx kv$):

$$\frac{i}{2\cdot 3}\, g^2 \omega_0 \tau'^3 \left(\frac{\Delta v}{v}\right), \quad \frac{1}{3\cdot 4}\, g^2 \omega_0^2 \tau'^4 \left(\frac{\Delta v}{v}\right)^4,$$

$$(33) \quad \underline{\frac{1}{2\cdot 2\cdot 5}\, g^3 \omega_0^2 \tau'^5 \left(\frac{\Delta v}{v}\right)^2}, \quad \frac{1}{3\cdot 3\cdot 7}\, g^3 \cdot \omega_0^4 \tau'^7 \left(\frac{\Delta v}{v}\right)^4,$$

$$\frac{i}{3\cdot 5}\, g^2 \cdot \omega_0^3 \tau'^5 \left(\frac{\Delta v}{v}\right)^3, \quad \frac{i}{2\cdot 4}\, g^2 \cdot \omega_0^2 \tau'^4 \left(\frac{\Delta v}{v}\right)^2, \quad \frac{i}{2\cdot 3}\, g^3 \omega_0^3 \tau'^6 \left(\frac{\Delta v}{v}\right)^3$$

The underlined term causes the strongest damping in (32): the magnitude of these damping terms in P_ν $\nu = 1, 2, 3, \ldots$ is given by:

$$(34) \qquad \left(\frac{\Delta v}{v} \frac{\omega_0}{(\nu-1)!}\right)^2 \cdot \frac{g^{2\nu-1}}{2\nu+1}\, \tau'^{\,2\nu+1}$$

(In Dupree [7] only P_1 appears)

Case a

If (35) $\qquad g \ll \omega_0 \frac{\Delta v}{v}$
the non-linear terms in (34) with $\nu = 2,3,4\ldots$ can be neglected. The solution for A according to (13) is then damped by (Dupree's trapping time [7]): $\left(\frac{D\omega_0^2}{3v^2}\right)^{-1/3} = \left(\frac{k^2 D}{3}\right)^{-1/3}$

Because of (35) this time is large relative to the autocorrelation time
(36) $\qquad \left(\omega_p \frac{\Delta v}{v}\right)^{-1} = (k\Delta v)^{-1}$
from which there follows inequality (9) if $\omega_0 \approx \omega_p$).

This gives in first approximation (from (8)) the quasilinear result (namely (10)).

Case b

In the converse case:

(37) $$\frac{D}{(\Delta v)^2} = g \gg \omega_0 \frac{\Delta v}{v}$$

we have to find which of the terms in (34) $\nu = 2,3,4$ .. becomes ~ 1 first. This will also depend on the magnitude of the g .

We therefore have to find:

(38) $$\max_{\nu=2,3,4} \frac{\Delta v}{v}\omega_0 \left(\frac{g}{\omega_0 \frac{\Delta v}{v}}\right)^{\frac{2\nu-1}{2\nu+1}} \left[\left((\nu-1)!\right)^2 \cdot (2\nu+1)\right]^{-\frac{1}{2\nu+1}}$$

e.g. for $g/\omega_0 \frac{\Delta v}{v} \gtrsim 10$ the maximum is at $\nu \gtrsim 3$
for $\gtrsim 100$ the maximum is at $\nu \gtrsim 5$.

For $\nu = 3$ and 5 the magnitude of the trapping time obtained from (34) is

(39) $$\left(\omega_0 \frac{\Delta v}{v}\right)^{-2/7} \cdot g^{-5/7} \quad \text{and} \quad \left(\omega_0 \frac{\Delta v}{v}\right)^{-2/11} \cdot g^{-9/11} \quad \text{respectively.}$$

Under the condition (37) the trapping time alone gives according to eq. 7.7 of Dupree [7] the diffusion coefficient. Apart from the imaginary terms in (33), we should also know whether the other corrections in (12) are important.

If this question is put aside for the moment, we get for the magnitude of the diffusion coefficient D (expressed by the quasilinear coefficient D_0 (10)):

in the case $\nu = 5$ (see (39)): $D_5 \approx D_0^{11/20} \left(k \Delta v^3\right)^{9/20}$

in the case $\nu = 3$ (see (39)): $D_3 \approx D_0^{7/12} \left(k \Delta v^3\right)^{5/12}$

in the case $\nu = 1$ [7] (cf. (8)): $D_1 \approx D_0^{3/4} \left(k \Delta v^3\right)^{1/4}$

(40)

If these coefficients are substituted into the condition (37), it then applies for $D \approx D_o$ as well. In the estimates it is best to use a mean k.

For a large diffusion coefficient there follows according to (38) a coefficient with $\nu \gtrsim 3$. This changes the power of the field strength in the diffusion coefficient relative to that in [7].

One example of application would be ion diffusion across a magnetic field in drift wave turbulence [15] . As a result of the decreased field strength diffusion could already occur for linear growth rates about a factor of $\sqrt{\frac{m_e}{m_i}}$ smaller relative to [15] .

This is all wishful thinking, however, if we consider the correction in (12) made by the other terms of the expansion (15). These are due to the effect of the operator $\langle U \rangle$ on the distribution function, i.e. to the time dependence of the distribution function. Assuming (37) we get on integrating in (5) terms of the order of the (integral) powers of:

trapping time $\cdot \dfrac{D}{(\Delta v)^2}$

(whereas in case a (35) the autocorrelation time [7] $(k\Delta v)^{-1}$ takes the place of the trapping time).

Depending on the magnitude of the diffusion coefficient, we obtain for this expansion parameter according to (39), (40) and (37):

$$\nu = 3 \qquad \frac{D}{(\Delta v)^2} \cdot \frac{(\Delta v)^{10/7}}{\left(\omega_o \frac{\Delta v}{v}\right)^{2/7} D^{5/7}} = \left(\frac{D}{k\Delta v^3}\right)^{2/7} \approx \left(\frac{D_o}{k\Delta v^3}\right)^{1/6} > 1$$

$$\nu = 5 \qquad \left(\frac{D}{k\Delta v^3}\right)^{2/7} \approx \left(\frac{D_o}{k\Delta v^3}\right)^{1/10} > 1$$

Thus, irrespective of the special magnitude of g or D, each of the terms in (15) makes a larger contribution than the previous one. The trapping time and hence the (always positive) exponent of the expansion parameter would be even smaller if the expansion (17) were to be considered as well. Thus, in order to put up kinetic equations when the diffusion coefficient is large, a better approximation would have to be found for the distribution function than (5') and (15), where it is expanded with $\frac{\partial}{\partial v} \langle f(t) \rangle$ as first term. This is one of the difficulties that occur for large D and that are not considered in [7] (cf. [16]).

But the nonlinear portion cannot be disregarded either, in general.

Inclusion of terms containing initial values of the fluctuations.

In Dupree's method of obtaining equations the initial values for the fluctuating part of the distribution function (which vanishes in the average) are discarded.

Orszag and Kraichnan [10] proposed to include them (for small D) and called the resulting equations conservative pseudolinear equations; they obey the energy conservation law (Appendix II).

We shall specify here the way in which these equations lead in the kinetic equations to the known inhomogeneous terms of [12, 13] which describe Lenard-Balescu collisions and spontaneous emission. Special attention is given to the time dependence in these terms, since it clarifies what is entailed in Dupree's method. We recall first the result without allowance for the diffusion term in the integration path:

The fluctuations are then $f_1(t) := f(t) - \langle f(t) \rangle \approx$

(40) $$\frac{e}{m} \sum_k \int_{t_0}^{t} dt' \; E_k(0) \, e^{ikx - i \int_{t_0}^{t'} \omega_k dt''} \left\langle \frac{\partial f(t)}{\partial v} \right\rangle + \sum_k g_k^{(v)} e^{ikx - ikv(t-t_0)}$$

where

(41) $$f_1(t_0) = f(t_0) - \langle f(t_0) \rangle := \sum_k g_k^{(v)} e^{ikx}$$

We now use the parallelism of the Vlasov to the Klimontovitch-Hierarchy. It is stated in [17] that the former is equivalent to the latter except for certain terms describing particle discreteness, but these do not contribute to our problem as calculated in [12] .

Hence we obtain the inhomogeneous terms as in [13] , if we choose the same appropriate space average relation as in the Klimontovitch - formalism [13] :

(42) $$\langle f_1(x,v,t') f_1(x',v',t') \rangle = \frac{V}{n} \langle f(x,v,t') \rangle \cdot \delta(x-x') \; \delta(v-v')$$

(+two particle correlations)

The ensemble average in the sense of [13] reduces to the space average in our homogeneous case. Thus we write $\langle\ \rangle$ for it, what cannot lead to confusion here. In [13] this relation is applied to the Fourier transform of $f - \langle f \rangle$ at time $t' = t$:

(43) $$\langle f_{k_1}(v_1,t) f_{k_2}(v_2,t) \rangle \approx \langle g_{k_1} g_{k_2} \rangle \approx \frac{(2\pi)}{n} \delta(k_1+k_2) \, \delta(v_1 - v_2) \cdot \langle f(v_1, t_0) \rangle$$

The two-particle correlations are damped out in the equations.

The time $t' = t_0$ is afterwards shifted to $t' = t$ by the argument that the difference would be small. But in the higher-order calculation [14] secular terms must be summarized to fill this gap.

In Dupree's method we have instead of the fluctuations (40) others of which the asymptotic part is used in the prece ding chapters and the corresponding initial value part is from (19) and (41)

$$(44)\quad f_1(t)\Big/_{in.} := \sum_k \left(f_k + \int_{t_0}^{t} \frac{\partial}{\partial v} D(t') \frac{\partial f_k}{\partial v}\, dt' \right) e^{ikx - ikv(t-t_0) + \dots}$$

According to (42) we then have instead of (43) the following terms in the kinetic equations apart from the destroyed two-particle correlations:

$$(45)\quad \langle f_{k_1}(v_1,t) f_{k_2}(v_2,t) \rangle \approx \langle f_{k_1}(v_1,t)\big/_{in.} \cdot f_{k_2}(v_2,t)\big/_{in.} \rangle \approx \frac{2\pi}{n} \langle f(v_1,t) \rangle\, \delta(k_1+k_2)\, \delta(v_1-v_2)$$

where the r.h. function $\langle f(v_1,t') \rangle$ obeys in the approximation of (44):

$$\frac{\partial}{\partial t} \langle f(v_1,t) \rangle = \frac{\partial}{\partial v} D(v_1,t) \frac{\partial}{\partial v} \langle f(v,t_0) \rangle$$

The higher-order terms of the expansion in D (like (15)) would give the correct time t instead of t_0 . Thus one obtains the true time dependence in the inhomogeneous term without secular terms, but this is only the true time dependence as long as the diffusion approximation is valid, i.e. the inhomogeneous terms remain small (far from the steady state).

In the asymptotic part of the fluctuations, however, the appearance of the diffusion coefficient D in Dupree's method has no correspondence to other methods.

Hence in comparison with other work the main advantage of Dupree's method for small D is that he gives the true time dependence for spontaneous emission and Lenard-Balescu collisions in the diffusion approximation for the distribution function. For long times secular divergences, which grow otherwise, can thus be avoided.

Summary

Dupree's method thus avoids complications which arise otherwise at higher orders [14]. For a more detailed comparison we should also solve the non-linear portion of Dupree's perturbation theory.
In the case of a small diffusion coefficient, the linear portion is shown here to be equivalent to the quasilinear treatment including the effects of the time dependence of the distribution function [9] , if we ignore certain processes which have a similar structure to the known non-linear corrections to the linearized Vlasov equation.
One of these processes was estimated by Dupree to broaden, in k-space, the kernel of the diffusion coefficient for the particle distribution function if this coefficient should become large, i.e. for strong turbulence. It is then not clear whether the linear portion could be separated from the non-linear one. But on this assumption it is shown here that the time dependence of the distribution function does not obey a diffusion equation: In this case the distribution function can no longer be treated as varying slowly with time in integration over time.

Acknowledgement

The author should like to thank Dr. K.U. von Hagenow for his critical remarks and Dr. H. Völk for encouraging discussions.

Appendix I

Energy equation for the solution a

The energy law postulates:

$$\text{(A1)} \qquad \int \frac{m}{2} v^2 \frac{\partial}{\partial t} \langle f(t) \rangle \, d^3v = - \frac{1}{8\pi} \frac{\partial}{\partial t} \int |\mathcal{E}_k(t)|^2 d^3k$$

Substituting (23') into the left-hand side, we get:

$$\text{(A2)} \qquad \frac{e^2}{m} \int dv \, v \sum_k |E_k(t)|^2 \cdot [\ldots] = \frac{e^2}{m} \sum_k |E_k(t)|^2 \int dv \left(v - \frac{Re\,\omega_k}{k}\right) [\ldots] +$$
$$+ \frac{e^2}{m} \sum_k |E_k(t)|^2 \int dv \, \frac{Re\,\omega_k}{k} [\ldots]$$

The second expression disappears according to the dispersion relation (29) since it is necessary that $\operatorname{Im} \varepsilon_{k\omega_k} = 0$.

From the first term in the first expression, i.e. the quasi-linear term in the square brackets (eq. (23)), there follows as in [1, 6, 9] :

$-\frac{1}{8\pi} \sum_k 2\gamma_k |E_k(t)|^2$ which according to (A1) and (30) is the total result.

It therefore remains to be shown that the following quantity disappears from (A2):

$$\int dv \left(v - \frac{Re\,\omega_k}{k}\right) \left[\left\{ \frac{k}{kv - Re\,\omega_k} \left(\frac{\partial}{\partial v} \frac{D}{(kv - Re\,\omega_k)^2} \right) - \frac{\partial}{\partial v} \frac{D}{(kv - Re\,\omega_k)^2} \frac{\partial}{\partial v} + \right.\right.$$
$$\left.\left. + \frac{1}{kv - Re\,\omega_k} \frac{\partial}{\partial t} \frac{1}{kv - Re\,\omega_k} \right\} \left\langle \frac{\partial f}{\partial v} \right\rangle \right] =$$

(A3)

$$= \int dv \left[\left(\frac{\partial}{\partial v} \frac{D}{(kv - Re\,\omega_k)^2} \right) \left\langle \frac{\partial f}{\partial v} \right\rangle + \frac{D}{(kv - Re\,\omega_k)^2} \left\langle \frac{\partial^2 f}{\partial v^2} \right\rangle + \frac{\partial}{\partial t} \frac{\langle \frac{\partial f}{\partial v} \rangle}{k(kv - Re\,\omega_k)} \right]$$

The last expression in (A3) is discarded because the derivative of the dispersion relation with respect to time vanishes (the quasilinear dispersion relation is sufficient owing to the restriction to linear corrections in D). The first two expressions in (A3) together give zero.

Even if D is assumed to be independent of the velocity and $\langle f \rangle$ is perhaps assumed to be adiabatically time dependent, we would have to retain, in addition to the correction according to Dupree, the first term of the expansion of B (17) in order to satisfy the energy law.

This term results from the fact that (unlike the approximation [7]) we take into account here the first velocity derivative of $\langle f \rangle$ in applying the operator $\langle \mathcal{U} \rangle$.
This immediately raises the question whether the other derivatives, particularly with respect to time, can then still be neglected. Insofar as these are important for the energy equation, this can be verified from the following:

If the form (23") is used instead of (23'), we get instead of (A3) the integral:

$$\int dv \left[- \frac{2kD}{(kv - \mathrm{Re}\,\omega_k)^3} \left\langle \frac{\partial f}{\partial v} \right\rangle + \frac{1}{k(kv - \mathrm{Re}\,\omega_k)} \frac{\partial}{\partial v} \left\langle \frac{\partial f}{\partial t} \right\rangle + \frac{\partial}{\partial t} \frac{\left\langle \frac{\partial f}{\partial v} \right\rangle}{k(kv - \mathrm{Re}\,\omega_k)} \right] = 0$$

$$\left\langle \frac{\partial f}{\partial t} \right\rangle = \frac{\partial}{\partial v} D \left\langle \frac{\partial f}{\partial v} \right\rangle$$

Here the contribution of the Dupree correction is cancelled by an expression which vanishes with $\left\langle \frac{\partial f}{\partial t} \right\rangle \to 0$. The time variation of the distribution may therefore not be neglected.

Appendix II

Energy equation for the asymptotic solution of Dupree's initial equations; inclusion of initial values of the fluctuations

In the ansatz (21) for the field the energy equation can be satisfied for the asymptotic solutions ($t-t_0 \to \infty$) of Dupree's initial equations, just as a Maxwell equation not used hitherto can.

With the ansatz (20) for $E_k(t)$, eq. (5) for a uniform distribution function can also be written in the form:

$$\frac{\partial}{\partial t}\langle f(t)\rangle = \frac{\partial}{\partial w}\left(\frac{e}{m}\right)^2 \cdot \sum_k \mathcal{E}_{-k}(t)\,\mathcal{E}_k(t)\cdot \int_0^{t-t_0} d\tau\; g_k(w,t;t-\tau) \qquad (t-t_0 \to \infty) \tag{A4}$$

where $\left(\frac{\partial}{\partial \tau} + i\left(kw - \omega_k(\tau_0+\tau)\right) - \frac{\partial}{\partial w}\underline{\underline{D}}(\tau_0+\tau)\frac{\partial}{\partial w}\right) g_k(w,\tau_0+\tau;\tau_0) = 0$

$$\tau_0 := t-\tau \qquad g_k(w,\tau_0+\tau;\tau_0)\Big/_{\tau=0} = \frac{\partial}{\partial w}\langle f(\tau_0)\rangle \tag{A5}$$

In the quasilinear theory D would here be zero, which otherwise alters nothing of the following proof of the energy equation. With this definition the relevant dispersion relation following from (27) is:

$$\varepsilon_{k\omega_k} = 1 - \frac{4\pi e^2}{m}\frac{ik}{k^2}\int d^3v \int_0^{t-t_0} d\tau\; g_k(w,t,t-\tau) = 0 \qquad (t-t_0 \to \infty) \tag{A6}$$

We now have to substitute (A4) into the energy equation (A1):

$$\frac{m}{2}\int w^2 \frac{\partial}{\partial t}\langle f(t)\rangle\, d^3v = -\frac{e^2}{m}\int d^3v\; w\cdot \sum_k \mathcal{E}_{-k}\mathcal{E}_k \cdot \int_0^{t-t_0} g_k(w,t,t-\tau) \tag{A7}$$

Since $\mathcal{E}_k(t) = \frac{k}{|k|} \cdot |\mathcal{E}_k(t)|$, assuming

(A8) $$\sum_k \mathcal{E}_{-k}(t) \cdot \mathcal{E}_k(t) \cdot \frac{e^2}{m k^2} \int d^3v \, (\omega_k(t) - k v) \, k \cdot g_k(v, t; t-\tau) = 0$$

it follows for the quantity (A7):

$$- \frac{4\pi e^2}{m} \int d^3v \sum_k |\mathcal{E}_k(t)|^2 \frac{2\omega_k(t)}{8\pi} \int_0^{t-t_0} d\tau \, \frac{k}{k^2} \, g_k(v, t; t-\tau) =$$

with (A6):

$$= (Re) \; i \sum_k 2\omega_k \frac{|\mathcal{E}_k(t)|^2}{8\pi}$$

The energy equation (A1) is thus satisfied if (A8) is valid. The latter holds if (A4) and (A6) are assumed, i.e. for Dupree's initial equations, just as does the form of the energy equation following from the Maxwell equations:

(A9) $$\sum_k \frac{\partial}{\partial t} \frac{\mathcal{E}_k(t) \mathcal{E}_{-k}(t)}{8\pi} = \sum_k j_k \mathcal{E}_{-k} = - \sum_k e \int v \, f_k \, d^3v \, \mathcal{E}_{-k}$$

For it follows from the dispersion relation (A6) that:

(A10) $$- \sum_k \omega_k(t) \frac{4\pi e^2}{m k^2} \int d^3v \int d\tau \, k g_k(v, t; t-\tau) \cdot \frac{|\mathcal{E}_k(t)|^2}{4\pi} = + \sum_k \mathrm{Im} \, \omega_k(t) \frac{|\mathcal{E}_k(t)|^2}{4\pi}$$

while we get for the right-hand side of (A9):

(A11) $$- \sum_k \frac{e^2 k}{m k^2} \cdot \int d^3v \int_0^{t-t_0} d\tau \, (k v) \, g_k(v, t; t-\tau) \, |\mathcal{E}_k(t)|^2 = + \sum_k j_k \mathcal{E}_k$$

Substituting (A10) and (A11) into (A9) we get (A8).

For the integral kernel of (A8) we obtain with (A5):

$$-\frac{e^2}{m}\int d^3v \int_0^{t-t_0} d\tau \,\left(k v - \omega_k(t)\right) \frac{k}{k^2}\, \mathcal{G}_k(v,t,t-\tau) =$$

$$= -i\,\frac{e^2}{mk^2}\int d^3v \int_0^{t-t_0} d\tau \; k \left(\frac{\partial \mathcal{G}_k}{\partial \tau} - \frac{\partial}{\partial v}\, \underline{\underline{D}}\, \frac{\partial \mathcal{G}_k}{\partial v} \right) =$$

$$= -i\,\frac{e^2 k}{mk^2}\int d^3v \left(\mathcal{G}_k(v,t;t_0) - \mathcal{G}_k(v,t;t) \right)$$

Only the last part is left over asymptotically ($t-t_0 \to \infty$), and it vanishes in the integral over the velocities because of the initial condition in (A5).

As pointed out in [10] initial values of the fluctuations could be added in the equations for the particle distribution function and the wave energy density. We note, that the energy equation here given can be extended to these "conservative pseudolinear equations" [10].

Appendix III

Approximation of the non-linear change of the wave energy density

Disregarding terms of the order $\left(k\frac{v_{th}}{\omega_p}\right)^2$ (viz. in the approximation $|\omega_k| \approx \omega_p$), we obtain for the non-linear growth rate from (30) et seq. if the coordinate system is chosen such that for resonant particles: v > 0:

$$\Delta\gamma_k \approx \frac{1}{2\omega_p}\,\frac{4\pi e^2}{m}\int dv\,\Bigg(\frac{1}{|k|v-\omega_p}\left\langle\frac{\partial f}{\partial v}\right\rangle\frac{\partial}{\partial v}\,\frac{D}{\left(\frac{|k|v}{\omega_p}-1\right)^2}\,+$$

$$+\,\frac{1}{Re\,\omega_k}\,\frac{\omega_p^3}{k\,(kv-Re\,\omega_k)^3}\cdot\frac{\partial Re\,\omega_k}{\partial t}\cdot\left\langle\frac{\partial f}{\partial v}\right\rangle+\frac{1}{|k|\left(\frac{|k|v}{\omega_p}-1\right)^2}\cdot\frac{\partial}{\partial t}\left\langle\frac{\partial f}{\partial v}\right\rangle\Bigg)=$$

$$=\frac{1}{2\omega_p}\,\frac{4\pi e^2}{m}\int dv\Bigg(\frac{-2|k|D}{\omega_p^2\left(\frac{|k|v}{\omega_p}-1\right)}\left\langle\frac{\partial f}{\partial v}\right\rangle+\frac{\omega_p^2\,\frac{\partial D}{\partial v}}{(|k|v-\omega_p)^3}\left\langle\frac{\partial f}{\partial v}\right\rangle+$$

$$+\,\frac{\omega_p^3}{k\,(kv-Re\,\omega_k)^3}\,\frac{1}{Re\,\omega_k}\,\frac{\partial Re\,\omega_k}{\partial t}\left\langle\frac{\partial f}{\partial v}\right\rangle+\frac{\partial}{\partial t}\,\frac{1}{|k|\left(1-\frac{|k|v}{\omega_p}\right)^2}\left\langle\frac{\partial f}{\partial v}\right\rangle\Bigg)=$$

$$=\frac{\omega_p}{4}\,\frac{4\pi e^2}{m|k|}\int dv\,\frac{1}{(|k|v-\omega_p)^2}\,\frac{\partial}{\partial v}\left(\frac{\partial D}{\partial v}\left\langle\frac{\partial f}{\partial v}\right\rangle\right)+$$

$$+\,\frac{\omega_p^2}{2}\,\frac{4\pi e^2}{mk}\int dv\,\frac{1}{(kv-Re\,\omega_k)^3}\,\frac{\partial \ln Re\,\omega_k}{\partial t}\left\langle\frac{\partial f}{\partial v}\right\rangle+\frac{4\pi e^2}{3\omega_p|k|m}\,\frac{\partial}{\partial t}\int dv\,\frac{1}{\left(1-\frac{|k|v}{\omega_p}\right)^2}\left\langle\frac{\partial f}{\partial v}\right\rangle\approx$$

(The last term is transformed as in [9] , Appendix 11.)

$$\approx -\frac{3}{2\omega_p^3}\,\frac{4\pi e^2}{m}\,|k|\int du\; v\,\frac{\partial D}{\partial v}\left\langle\frac{\partial f}{\partial v}\right\rangle +$$

$$+\frac{\omega_p^2}{2k}\left(1-\frac{16}{9}\right)\frac{4\pi e^2}{m}\cdot\frac{\partial \ln \omega_k}{\partial t}\int du\,\frac{1}{(kv-\mathrm{Re}\,\omega_k)^3}\left\langle\frac{\partial f}{\partial v}\right\rangle \approx$$

$$\approx -\frac{7}{6}\,\frac{\partial \ln \mathrm{Re}\,\omega_k}{\partial t} + \frac{3}{2}\,\frac{|k|}{\omega_p}\,\frac{1}{n}\int du\; D\,\frac{\partial}{\partial v}\left(v\left\langle\frac{\partial f}{\partial v}\right\rangle\right)$$

(A12)

With the solution of the (quasi-) linear dispersion relation for $k \ll \frac{\omega_p}{v_{th}}$, with the energy equation (A1) for the quasilinear terms and with the result obtained below (A14) for the last term of (A12) it follows that:

$$\text{(A12)}\quad \frac{7k^2}{4\,\omega_p^2\,mn\cdot 2\pi}\sum_{k'}\gamma_{k'}\,|E_{k'}(t)|^2 + \frac{3|k|\omega_p}{4mn\cdot 2\pi}\sum_{k'}|E_{k'}(t)|^2\left(\frac{4\gamma_{k'}k'}{\omega_{k'}^3}+\pi\,\frac{\omega_p}{k'^2}\left\langle\frac{\partial^2 f}{\partial v^2}\right\rangle\right)\Bigg|_{v=\frac{\omega_p}{k'}}$$

$$\text{(A13)}\approx \frac{7\omega_p k^2}{16\,mn^2}\sum_{k'}|E_{k'}(t)|^2\frac{1}{k'^2}\left\langle\frac{\partial f}{\partial v}\right\rangle\Bigg|_{v=\frac{\omega_p}{|k'|}} + \frac{3|k|\omega_p}{4mn^2}\sum_{k'}|E_{k'}(t)|^2\left(\frac{1}{|k'|}\left\langle\frac{\partial f}{\partial v}\right\rangle\Bigg|_{v=\frac{\omega_p}{|k'|}}+\frac{\omega_p}{2k'^2}\left\langle\frac{\partial^2 f}{\partial v^2}\right\rangle\Bigg|_{v=\frac{\omega_p}{|k'|}}\right)$$

The last term is somewhat less than half of the result (A15) obtained in the usual treatment of the Vlasov equation. For $k' \approx k$ we get $\frac{19}{15}$ of (A15) and also the term with the second derivative of the distribution function.

For a Maxwell distribution this term would be much larger in magnitude than the other terms if a constant square of the amplitude is assumed. In the bump-in-tail case, however, the square of the amplitude is small where the most pronounced curvatures of the distribution function occur, viz. at the boundary of the unstable region.

(If D were assumed to be independent of velocity both in and outside the unstable region, with a jump occurring between these, this would only make the following calculation more difficult.)

With (22) the second part of (A12) is approximately:

$$\frac{3|k|\omega_p}{2mn^2}\,\frac{1}{4\pi}\int dv \sum_{k'} |E_{k'}(t)|^2 \,\frac{\gamma_{k'}}{(k'v-\mathrm{Re}\,\omega_{k'})^2+\gamma_{k'}^2}\,\frac{\partial}{\partial v}\left\langle v\frac{\partial f}{\partial v}\right\rangle \approx$$

$$\approx \frac{3|k|\omega_p}{8\pi m n^2}\sum_{k'}|E_{k'}(t)|^2\cdot\left[\frac{\pi}{|k'|}\left\langle \frac{\partial f}{\partial v}\Big/_{v=\frac{\omega_p}{k'}}\right\rangle - 2\gamma_{k'}k'\int\frac{\langle f\rangle}{(k'v-\mathrm{Re}\,\omega_{k'})^3}\,dv \,+\right.$$

$$\left.+\frac{\pi\omega_p}{k'^2}\left\langle\frac{\partial^2 f}{\partial v^2}\Big/_{v=\frac{\omega_p}{k'}}\right\rangle + 6\gamma_{k'}\int\frac{v\langle f\rangle}{(k'v-\mathrm{Re}\,\omega_{k'})^4}\,dv\right]\approx$$

$$\approx \frac{3|k|\omega_p}{8\pi m n^2}\sum_{k'}|E_{k'}(t)|^2\left[\frac{\pi}{|k'|}\left\langle\frac{\partial f}{\partial v}\Big/_{..}\right\rangle + \frac{\pi\omega_p}{k'^2}\left\langle\frac{\partial^2 f}{\partial v^2}\Big/_{..}\right\rangle + 2\gamma_{k'}\frac{k'}{\omega_{k'}}\,n\right]\approx$$

$$\approx \frac{3|k|\omega_p}{8\pi m n^2}\sum_{k'}|E_{k'}(t)|^2\left[\frac{4\gamma_{k'}k'}{\omega_{k'}^3}+\frac{\pi}{n}\cdot\frac{\omega_p}{k'^2}\left\langle\frac{\partial^2 f}{\partial v^2}\Big/_{v=\frac{\omega_p}{|k'|}}\right\rangle\right]$$

(A14)

Without the first term in the expression for $\Delta\gamma_k$ ((30) et seq.) we should get (according to Appendix 11 of [9]) instead of (A13):

(A15) $$\Delta\gamma_k\Big/_{vl.} = \frac{15}{8\pi}\cdot\frac{k^2}{\omega_p^2 mn}\sum_{k'}\gamma_{k'}\,|E_{k'}(t)|^2$$

The unstable waves ($\gamma_{k'} > 0$) would accordingly grow even more rapidly, because $\Delta\gamma_k > 0$

References:

1 Drummond, Pines Nucl.Fus.Suppl:Conf.Salzb. 3, 1049 (62) and Ann.of Phys. 28, 478 (64)

2 Aamodt, Drummond Phys.Fl. 7, 1816 (64)

3 Frieman, Rutherford Ann. of Phys. 28, 134 (64)

4 Baldwin Phys. Fl. 7, 782 (64)

5 Altshul, Karpman JETP 49, 515 (65)

6 Bernstein, Engelmann Phys. Fl. 9, 937 (66)

7 Dupree Phys. Fl. 9, 1773 (66)

8 Pfirsch Nucl. Fus. 6, 304 (66)

9 Gratzl Dissertation, IPP Garching 6/56 (67)

10 Orszag, Kraichnan Phys.Fl. 10, 1720 (67)

11 Pines, Schieffer Phys. Rev. 125, 804 (62)

12 Laval, Pellat FC 316, Fontenay (65)

13 Rogister, Oberman Journ. Pl. Phys. 2, 33 (68)

14 Rogister, Oberman Matt 583 (67); Am.Phys. Soc., Sect.Plasm. Phys. 9. Ann. Meet. 1967 Austin, Tex.

15 Dupree Phys.Fl. 10, 1049 (67)

16 Aamodt Phys. Fl. 10, 1245 (67)

17 Davidson Phys. Fl. 10, 1707 (67)