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ABSTRACT: In a Q-machine with end plates heated to a uniform temperature there must exist a radial electric field proportional to the gradient of the particle density, thus being a consequence of a deviation from the equilibrium case. This radial component of the electric field has an influence on the velocity distribution of the ions and leads -for a certain range of particle densities- to increased loss rates of ions. The calculations have been performed under the assumption of an electron sheath on the end plate and by taking into account the centrifugal forces acting on the ions.

Although the "equilibrium model" [1] seems to be a good description for a collision dominated plasma in a Q-machine, there remain still a few experimental observations not in agreement with this model; p.e., the mean ion energy perpendicular to the direction of the magnetic field (measured by the Doppler-broadening of a resonance line in a barium plasma) seems to exceed considerably the temperature of the end plates as far as the mean particle density is low enough ($\ll 10^{11}\ cm^{-3}$) [2], and the origin of a factor of about 25 discrepancy in the determination of the mean particle density by means of Langmuir single probes or using the method of resonance scattering of light [3] is also not yet known.

For a low density, electron rich plasma which can no longer be assumed to be collision dominated it has already been shown [4] that the existence of both resistive diffusion and end plate diffusion or the presence of a probe within the plasma lead to a large deviation from the equilibrium situation and, consequently, to a highly disturbt velocity distribution of the ions and to a large increase of the particle loss rate as well.

The purpose of the present report is to investigate the influence of a radially directed electric field on plasma conditions, and wether or not the above mentioned discrepancies could be related to the presence of such an electric field. This will be done by considering the influence of a radial electric field on particle energy and by taking the ion inertia into account. For his case of interest, v. Goeler [1] has already pointed out that the increase of resistive diffusion due to these effects might be neglected in the Q-machine.

In the following calculations a Q-machine of rotational symmetry and of mirror symmetry about its middle plane is considered. The magnetic field is homogeneous and parallel to the axis of rotation of the device +). The two endplates are located in a distance of $\pm l_o$ from the middle plane and heated to equal and uniformly distributed temperature. Throughout the whole calculations an electron sheath is assumed to be present on the end plates because this is the case of interest for good ion confinement.

Under these conditions a radially directed electric field exists within the plasma [7] §)

$$\underline{E}_r = - \frac{U_{th}}{n} \underline{\nabla}_r n \tag{1}$$

provided that the work function is constant on the endplates.

The equations for the two fluid model read

$$\underline{j} \times \underline{B} = \underline{\nabla}(p_i + p_e) + n m_i (\underline{v}_i \underline{\nabla}) \underline{v}_i \tag{2}$$

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{j} + \frac{1}{en} \underline{\nabla} p_i + \frac{m_i}{e} (\underline{v}_i \underline{\nabla}) \underline{v}_i \tag{3}$$

where the electron inertia has been neglected due to the small electron mass and the approximate equality of ion and electron velocity for $\underline{E} \times \underline{B}$-motions. Furthermore, the inertia of any macroscopic motion in radial direction will be neglected since these motions should be slow enough. Therefore, the only

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+) By these assumptions we go around all the difficulties encountered in systems with inclined end plates. [5] [6]

§) Use will be made of the relation

$$kT = eU_{th}$$

non vanishing term of the ion inertia is given by

$$(\underline{v}_i \underline{\nabla})\underline{v}_i = -\frac{v_i^2}{r}\underline{r}^o \qquad (4)$$

(The exponent ^{o} denotes an unit vector) and eqs. (2) and (3) become

$$\underline{j} \times \underline{B} = \underline{\nabla}(p_i + p_e) - nm_i \frac{v_i^2}{r}\underline{r}^o \qquad (5)$$

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{j} + \frac{1}{en}\nabla p_i - \frac{m_i}{e}\frac{v_i^2}{r}\underline{r}^o \qquad (6)$$

yielding

$$\underline{v}_r = \frac{\eta}{B^2}(\underline{B} \times \underline{j}) \qquad (7)$$

$$\underline{v}_{i\varphi} = \frac{1}{B^2}\underline{B} \times \left[-\underline{E}_r + \frac{1}{en}\underline{\nabla}_r p_i - \frac{m_i}{e}\frac{v_{i\varphi}^2}{r}\underline{r}^o\right] \qquad (8)$$

$$v_{i\varphi} = \frac{\nabla_r \xi}{B} - \frac{1}{\omega_{ci}}\frac{v_{i\varphi}^2}{r} \qquad (9)$$

where

$$\nabla_r \xi = -E_r + \frac{1}{en}\nabla_r p_i \qquad (10)$$

$$\omega_{ci} = \frac{eB}{m_i} \qquad (11)$$

In eqs. (8) and (9) the electron contribution to the macroscopic velocity has been neglected since it is of negligible influence in the problem under consideration.

Eq.(9) has the solution

$$v_{i\varphi} = -\frac{r\,\omega_{ci}}{2}\left(1 - \sqrt{1 + \frac{4}{r\,\omega_{ci} B}\nabla_r \xi}\right). \qquad (12)$$

where the minus sign had to be taken since $v_{i\varphi} = 0$ for $\nabla_r \xi = 0$. By combining eqs.(1) and (10) it can be seen that $\nabla_r \xi$ is usually negative in a Q-machine, i.e. both forces, enE_r and $-\nabla_r p_i$, are pointed outward, in the same direction as the centrifugal forces. For this case $\nabla_r \xi$ and thus also $v_{i\varphi}$ have an upper limit for which the outward directed forces can still be compensated by the Lorentz force. If any larger electric field is applied the plasma becomes unstable due to the rapidly growing centrifugal forces.

If the electric field forces and the centrifugal forces are directed opposite to each other eq.(12) has a solution for each value of E_r. In this case $v_{i\varphi}$ becomes proportional to $\sqrt{E_r}$ and the centrifugal forces can be balanced by the electric field forces. One should not forget, however, that now the electrons play the role of the ions in determining the upper limit of the electric field by their inertia but, certainly, this limit for E_r is much higher in magnitude than the one put by the ions in the first case.

For $E_r > 0$ the maximum stable ion velocity

$$|v_{i\varphi max}| = \frac{1}{2} r \, \omega_{ci} \tag{13}$$

is obtained for

$$\nabla_r \xi_{max} = -\frac{1}{4} r \, \omega_{ci} \, B \tag{14}$$

Introducing eqs.(13) and (14) into eq.(9) shows that the system becomes unstable when the electric field plus the pressure forces become smaller than the centrifugal forces.

In order to get an impression about the importance of eq.(14) an example will be given. Therefore rewriting eq.(14)

$$|\nabla_r \xi| \leq \frac{1}{4} \frac{e}{m_i} r B^2 \tag{15}$$

$$= 2.4 \cdot 10^{11} \frac{z}{A} r B^2 \tag{16}$$

where z is the ion charge, A the atomic weight of the ions and B the magnetic field strength measured in $Vsec/cm^2$. For z = 1, A = 133 (Cs), r = 1 cm and B = 3 kGauss = $3 \cdot 10^{-5}$ $Vsec/cm^{-2}$

$$E_r - \frac{1}{en} \nabla_r p_i \leq 1.6 \frac{V}{cm} \tag{17}$$

Eq.(16) shows that the critical value for the static radial forces depends only on ion mass, position and the magnitude of the magnetic field but it does not depend on parameters such as the temperature or the mean particle density.

The next step is to evaluate the mean energy of the particles in presence of a radial electric field. This will be done under the assumption that all particles are generated on the end plate and have, when leaving the end plate, a velocity distribution according to the plate temperature. By passing the sheath the particles enter the region with a non vanishing radial electric field and will thus gain energy according to their radius of gyration. It is well known from rotating plasmas that in such a case half of the energy gained in an E x B-field is put into plasma rotation and half into the gyro-motion of the particles. The velocity of these particles is described by

$$\underline{\hat{v}} = \frac{1}{B^2}\left[\left(\underline{E}_r + \frac{m_i}{e}\,\frac{\hat{v}_{i\varphi}^2}{r}\,\underline{r}^o\right) \times \underline{B}\right] + \frac{1}{B^2}\left|\left(\underline{E}_r + \frac{m_i}{e}\,\frac{\hat{v}_{i\varphi}^2}{r}\,\underline{r}^o\right) \times \underline{B} - B^2\,\underline{v}_{p\perp}\right| e^{i(\omega t + \chi)} \qquad (18)$$

where $v_{p\perp}$ is the perpendicular component of the initial particle velocity and χ the initial phase of the motion. Eq.(18) describes the ion motion in the single particle picture and shall be used only to determine the gain of energy of the ions by their motion into an $\underline{E} \times \underline{B}$-field. If the collision time is long compared to the time the particle needs to pass the sheath the pressure gradient does not enter into eq.(18). But the velocity determined by eq.(18) is the velocity of the individual particles, i.e. the first term represents the motion of the centers of gyration which, in general, is different from the macroscopic velocity of the ions, and the second term represents the gyro-motion. Thus the term $\frac{m_i}{e}\,\frac{v_i}{r}\,\underline{r}^o$ is the centrifugal force due to the motion of the centers of gyration and not the centrifugal force due to the macroscopic motion of the ions.

The mean particle energy is obtained by taking the average over the distribution function of the second term of eq.(18)

$$\frac{m_i}{2}\overline{v_\perp^2} = \frac{m_i}{2}\,\frac{1}{B^4}\left[\left(E_r + \frac{m_i}{e}\,\frac{\hat{v}_{i\varphi}^2}{r}\,r^o\right) \times B\right]^2 + eU_{th} \qquad (19)$$

where $\overline{v_\perp^2}$ is measured in a system moving with the velocity of the centers of gyration.

It might be somewhat surprising that in those regions where a radial density gradient exists the mean particle energy in a Q-machine is larger than the temperature of the end plate, due to the non vanishing radial electric field. But it should be kept in mind that the existence of a radially directed mass flow due to resistive and end plate diffusion -so small it might be- causes a deviation from the equilibrium situation; and it is just this process which -in connection with the distribution of the particle sources- determines the distribution of particle density and thus also of potential (according to eq.(1)) within the plasma which in turn is responsible for the increase of the mean particle energy. One sees immediately that a vanishing radial mass flow leads to an equilibrium situation since it requires a vanishing gradient of particle density and thus also a vanishing radial component of the electric field.

The picture would, perhaps, become more clear if one puts himself into a coordinate system rotating with the $\underline{E} \times \underline{B}$- velocity of the plasma. In this particular system E_r vanishes. But now the end plates have to be assumed to be rotating with the same frequency as the ions before but in opposite direction. That means that all particles generated on the surface of the end plates have a net momentum due to the rotation of the plates superimposed to their thermal momentum. This situation is similar to the ones encountered in injection machines. Again the effect disappears for vanishing $\nabla_r n$.

There is more evidence for the existing deviation from an equilibrium situation. If one assumes the plasma to be in equilibrium there would be detailed balance of all individual processes. Considering volume ionization and recombination and assuming the neutrals to have plate temperature, one sees that an ion generated by volume ionization must gain energy from the radial electric field according to its final radius of gyration, i.e. all ions generated this way populate the high energy part of the distribution function. The recombination probability, however, is determined by the electron energy +) and thus equal for all ions. This is a contradiction to the assumption of detailed balance which again disappears for vanishing $\nabla_r n$. Also this example can be visualized in a rotating frame. In this case the neutral gas has to be assumed to rotate with respect to the ions. Therefore, the momentum has to be conserved for any individual ionizing process. This can be done only in the gyro-motion since the frame has been selected just this way. Again the rotation of the neutral gas vanishes for vanishing $\nabla_r n$.

Therefore, the ion pressure, p_i , perpendicular to B is obtained by multiplying eq.(19) by the mean particle density, n

$$p_{i\perp} = n \frac{m_i}{2}\left[\left(\frac{E_r}{B} + \frac{1}{r\omega_{ci}} \hat{v}^2_{i\varphi} \underline{r}^o\right) \times \underline{B}^o\right]^2 + neU_{th} \tag{20}$$

$$= n \frac{m_i}{2} \left(\frac{E_r}{B} + \frac{1}{r\omega_{ci}} \hat{v}^2_{i\varphi}\right)^2 + enU_{th} \tag{21}$$

Assuming

$$\left|\frac{E_r}{B}\right| \ll \frac{1}{4} r\omega_{ci} \tag{22}$$

yields

$$\hat{v}_{i\varphi} \approx \frac{E_r}{B} \tag{23}$$

and

$$p_{i\perp} = n \frac{m_i}{2} \left(\frac{E_r}{B}\right)^2 \left(1 + \frac{1}{r\omega_{ci}} \frac{E_r}{B}\right)^2 + enU_{th} \tag{24}$$

+) The electrons gain practically no energy from the radial electric field due to their small radius of gyration.

$$p_{i\perp} \approx n \frac{m_i}{2} \left(\frac{E_r}{B}\right)^2 \left(1 + \frac{2}{r\omega_{ci}} \frac{E_r}{B}\right) + enU_{th} \tag{25}$$

In eqs.(24) and (25) it has been assumed that the energy gained by the ions from the radial electric field will not distribute itself among the three degrees of freedom. This is the case only if the mean time for like particle collisions is longer than the mean life time of the ions, i.e. for a range of particle densities for which the mean free path for like particle collisions is comparable with or longer than the distance between the two end plates, as shown in ref.4). For these conditions an equivalent mean particle energy, $eU_\perp$, can be defined by the equation

$$U_\perp = \frac{1}{2} \frac{m_i}{e} \left[\frac{E_r}{B} + \frac{1}{r\omega_{ci}} \hat{v}^2_{i\varphi}\right]^2 + U_{th} \tag{26}$$

Simultaneously, the centrifugal forces and the increase of pressure represent increasing forces in radial direction which have to be balanced by an increase in the azimutal current density which in turn causes an increase in the particle loss flux in radial direction as seen by combining eqs.(5) and (7). But this loss flux is a consequence of the finite resistivity of the plasma, i.e. a consequence of collisions between ions and electrons and these collision times are often longer than the life time of the ions within the plasma. Thus, for the following treatment it is necessary to consider three families of ions:

(1) Ions with a life time shorter than the like particle collision time. These ions will have a mean perpendicular energy according to eq.(26).

(2) Ions with a life time longer than the like particle collision time but shorter than the ion electron collision time. These ions will randomize the energy gained within the electric field and therefore have a mean energy, eU_E

$$U_E = \frac{1}{3}\frac{m_i}{e}\left[\frac{E_r}{B} + \frac{1}{r\omega_{ci}}\hat{v}^2_{i\varphi}\right]^2 + U_{th} \qquad (27)$$

(3) Ions with a life time longer than the ion electron collision time. These ions will loose their excess energy gained in the electric field by frequent collisions with electrons which carry the excess energy back to the end plates. This family of ions will therefore have plate temperature.

In the considerations a collision of an ion with one of the end plates and a following reionization has to be treated as a new generation of this ion.

That most of the ions present within the plasma belong to family (1) will be found in low density plasmas such as treated in ref.4). Family (2) will be encountered to be mostly populated in an intermediate range of densities and family (3) is found in high density plasmas. The criterion for family (2) to be mostly populated is

$$\tau_{ii} < \tau_{fl}\, e^{-\frac{U_s}{U_E}} < \tau_{ie} \qquad (28)$$

where τ_{fl} is the time of flight for an ion over the distance of the two end plates. It should be noted that the collisions have an effect only on the mean particle energy but they have no direct effect on plasma rotation (only via a change of the profile of particle density).

Since radial particle diffusion needs several ion electron collision times for the particle to get lost whereas the particle energy approaches the plate temperature with a time constant of approximately one ion electron collision time one has to conclude that the preponderance of family (1) or (2) within the plasma is limited to such cases for which the loss rate of ions is determined by end plate recombination and end plate diffusion. This does not mean that relatively small particle losses due to resistive diffusion will not have a large influence on the ion velocity distribution and thus on the loss rate parallel to the magnetic field.

For the following calculations it is usefull to replace eq.(25) by

$$p_{i\perp} \approx n\left[eU_{th} + a\,\frac{m_i}{2}\left(\frac{E_r}{B}\right)^2\left(1 + \frac{2}{r\,\omega_{ci}}\,\frac{E_r}{B}\right)\right] \qquad (29)$$

where

$$a = \begin{cases} 1 & \text{for family (1)} \\ \frac{2}{3} & \text{" \quad " \quad (2)} \\ 0 & \text{" \quad " \quad (3)} \end{cases} \qquad (30)$$

Furthermore, using eq.(1) and defining an e-folding length, λ, for the mean particle density by

$$\frac{1}{n}\,\nabla_r n = -\frac{1}{\lambda} \qquad (31)$$

yields+)

$$E_r = \frac{U_{th}}{\lambda} \qquad (32)$$

+) It is clear that U_{th} in eq.(32) has not to be modified since the radial electric field is determined by the variation of the sheath voltage over the plate which in turn has to provide electron balance according to the distribution of the particle density within the plasma, and the electrons can be assumed to have plate temperature.

and defining the velocity w by

$$w = \frac{U_{th}}{\lambda B} \qquad (33)$$

$p_{i\perp}$ reads

$$p_{i\perp} = n\left[eU_{th} + a\,\frac{m_i}{2}\,w^2\left(1 + 2\,\frac{w}{r\omega_{ci}}\right)\right] \qquad (34)$$

$$= neU_{th}\left[1 + a\,\frac{w^2}{v_w^2}\,\left(1 + 2\,\frac{w}{r\omega_{ci}}\right)\right] \qquad (35)$$

$$= neU_{th}(1 + 2\varepsilon) \qquad (36)$$

where v_w is the most probable velocity for $U = U_{th}$

$$v_w^2 = \frac{2eU_{th}}{m_i} \qquad (37)$$

As seen by combining eqs.(33) and (37) w/v_w^2 is identical with

$$\frac{w}{v_w^2} = \frac{1}{2\lambda\omega_{ci}} \qquad (38)$$

Therefore,

$$\varepsilon = \frac{a}{4}\,\frac{w}{\lambda\omega_{ci}}\left(1 + 2\,\frac{w}{r\omega_{ci}}\right) \qquad (39)$$

The radial flux of particles due to resistive diffusion is now calculated by using eqs.(7),(5),(36) and (38)

$$\phi_r = nv_r = 2\,\frac{\eta e n^2 U_{th}}{\lambda B^2}\left(1 + \varepsilon - \lambda\nabla_r\varepsilon + \frac{1}{2w}\,\frac{1}{r\omega_{ci}}\,v_{i\varphi}^2\right) \qquad (40)$$

where $\nabla_r\varepsilon$ is given by

$$\nabla_r \varepsilon = -\frac{a}{2}\frac{w}{\lambda^2\omega_{ci}}\left(1+3\frac{w}{r\omega_{ci}}\right)\nabla_r\lambda \tag{41}$$

$v^2_{i\varphi}$ is obtained by using eqs.(10) and (12) under the additional assumption

$$\left|\frac{\nabla_r \xi}{B}\right| \ll \frac{1}{4}\, r\omega_{ci} \tag{42}$$

(which is a somewhat stronger assumption than (22))

$$v^2_{i\varphi} \approx 4\,w^2\,(1+2\varepsilon-2\lambda\nabla_r\varepsilon) \tag{43}$$

Thus one obtains for

$$\phi_r = 2\,\frac{\eta e n^2 U_{th}}{\lambda B^2}\left\{1+\varepsilon-\lambda\nabla_r\varepsilon+2\frac{w}{r\omega_{ci}}\,(1+2\varepsilon-2\lambda\nabla_r\varepsilon)\right\} \tag{44}$$

The first term in eq.(44) is the usual expression for resistive diffusion in a plasma with a temperature equal to plate temperature. The resistivity, η, entering into eq.(44) has to be taken according to the electron temperature which -under the conditions discussed here- is practically identical with plate temperature.

To give an impression about the influence of the radial electric field present in the Q-machine on resistive diffusion, the e-folding length, λ, will be assumed to be equal to the ion radius of gyration. All derivatives of λ will thus be put to zero. Such a profile of particle density is not too far from experimental observations. For this case the quantity

$$\frac{w}{\varrho_i\,\omega_{ci}} = \frac{U_{th}}{\varrho_i^2\,\omega_{ci}\,B} = 1 \tag{45}$$

Then one obtains for

$$\varepsilon\,(\lambda = \varrho_i) = \frac{a}{4}\,(1 + 2\,\frac{\varrho_i}{r}) \tag{46}$$

$$\nabla_r \varepsilon(\lambda = \varrho_i) = 0 \tag{47}$$

and thus for

$$\phi_r = 2\,\frac{\eta e n^2 U_{th}}{\lambda B^2}\left\{1 + \frac{a}{4} + 2\,\frac{\varrho_i}{r}(1 + \frac{3}{4} a)\right\} \tag{48}$$

The term $\frac{2\varrho_i}{r}$ on the right hand side of eq.(48) is the one discussed in ref. 1) and as seen also from this analysis it can be neglected when calculating the flux of diffusing particles provided that the magnetic field is sufficiently large.

It should be checked at this point if the estimation $\lambda = \varrho_i$ is compatible with the assumption (42) which reads in the new variables

$$8w\left[1 + \frac{a}{4}\,\frac{w}{\lambda\omega_{ci}}\,(1 + 2\,\nabla_r\lambda)\right] \ll r\omega_{ci} \tag{49}$$

and thus $$8w \ll r\omega_{ci} \tag{50}$$

even in case $\frac{w}{\lambda\omega_{ci}}$ being of the order of unity. For $\lambda = \varrho_i$ (50) reduces to

$$8\,\frac{\varrho_i}{r} \ll 1 \qquad (\lambda = \varrho_i) \tag{51}$$

Based on eq.(36) a mean ion energy, eU_E, for the motion perpendicular to B can be defined (for $a = \frac{2}{3}$ this energy is identical with the mean random energy)

$$U_E = \frac{1}{en} p_{i\perp} \tag{52}$$

$$= U_{th}(1+2\varepsilon) \tag{53}$$

Here the assumption $\lambda = \rho_i$ yields for U_E

$$U_E(\lambda=\rho_i) = U_{th}\left[1 + \frac{a}{2}\left(1 + 2\,\frac{\rho_i}{r}\right)\right] \tag{54}$$

The next step is to calculate the loss flux of ions, ϕ_{LE}, to the two end plates. It can be seen immediately that the radial electric field has no influence on this flux for particles belonging to family (1) or (3). In the first case this is due to the complete decoupling of the motion parallel and perpendicular to B; whereas in the second case the ion excess energy was assumed to be carried back to the end plates by frequent collisions between ions and electrons. The following calculation is thus restricted to family (2) only. Therefore, "a" will be replaced by 2/3 and ε by ε_2 which is identical with ε for a = 2/3 but 0 for a = 0. The case a = 1 is subject to ref. 4) as far as effects are concerned related to the velocity distribution of the ions parallel to B. It should be noted, however, that the upper limit of the particle density for which the calculations of ref. 4) are still applicable is increased by the existent radial electric field due to the increase of the like particle collision time.

ε_2 is obtained from eq.(39) by putting a = 2/3

$$\varepsilon_2 = \frac{1}{6}\,\frac{w}{\lambda\omega_{ci}}\left(1 + 2\,\frac{w}{r\omega_{ci}}\right) \tag{55}$$

Then the loss flux of particles to the end plates is determined by

$$\phi_{LE} = (1-\gamma)\frac{1}{\sqrt{\pi}}nv_w\sqrt{1+2\varepsilon_2}\; e^{\frac{U_s}{U_{th}}\frac{1}{1+2\varepsilon_2}} \tag{56}$$

where again use is made of the assumption of an electron sheath existing on the end plates ($U_s < 0$!). Denoting by ϕ_L the loss flux when omitting the effect of the radial electric field one gets for the relative increase of the loss rate (see fig.1)

$$\frac{\phi_{LE}}{\phi_L} = (1+\varepsilon_2)\; e^{-\frac{U_s}{U_{th}}\frac{2\varepsilon_2}{1+2\varepsilon_2}} \tag{57}$$

Remembering that the sheath voltage is determined by the electron balance condition

$$e^{\frac{U_s}{U_{th}}} = -\frac{env_{we}}{2\sqrt{\pi}j_R} \tag{58}$$

$$= C\,n \tag{59}$$

where v_{we} is the most probable velocity of the electrons which have a temperature equal to plate temperature and j_R is the electron emission saturation current density. Thus, C is only a function of plate temperature as long as the work function is constant. Introducing eq.(59) into eq.(56) yields

$$\phi_{LE} = (1-\gamma)\frac{v_{wi}}{\sqrt{\pi}}(1+\varepsilon_2)\; C^{\frac{1}{1+2\varepsilon_2}}\; n^{2\frac{1+\varepsilon_2}{1+2\varepsilon_2}} \tag{60}$$

$$\approx (1-\gamma)\frac{v_{wi}}{\sqrt{\pi}}(1+\varepsilon_2)\; C^{1-2\varepsilon_2}\; n^{2(1-\varepsilon_2)} \tag{61}$$

which gives a deviation from the n^2-law which would be obtained under equilibrium conditions.

Finally, the effect of end plate diffusion [8] shall be considered. The radial loss flux of ions due to this process is given by

$$\phi_{eE} = \frac{1}{8\sqrt{\pi}} \frac{\overline{\varrho_i^2} v_{wi} \gamma}{\lambda l_o} n \sqrt{1 + 2\varepsilon_2} \; e^{\frac{U_s}{U_{th}} \frac{1}{1+2\varepsilon_2}} \tag{62}$$

Where ϱ_i is the ion radius of gyration just in front of the end plates, i.e. for a mean ion energy eqal to plate temperature, $2 l_o$ is the distance between the two end plates and γ the surface ionization coefficient. A comparison between eqs.(62) and (56) yields immediately

$$\phi_{eE} = \frac{1}{8\sqrt{\pi}} \frac{\overline{\varrho_i^2} v_{wi} \gamma}{\lambda l_o} (1 + \varepsilon_2) \, C^{1 - 2\varepsilon_2} \, n^{2(1-\varepsilon_2)} \tag{63}$$

which means that the loss flux to the endplates is increased by the same amount as the radial flux due to end plate diffusion. Therefore, the profile of particle density is unaffected by the magnitude of the radial electric field provided that resistive diffusion is neglegibly small compared to end plate diffusion.

Again, to give an impression about the magnitude of this effect λ will be replaced by ϱ_i, thus

$$\varepsilon_2 = \frac{1}{6} \left(1 + 2 \frac{\varrho_i}{r}\right) \tag{64}$$

and

$$\frac{\phi_{LE}}{\phi_L} = \frac{\phi_{eE}}{\phi_e} = \left[1 + \frac{1}{6}\left(1 + 2\frac{\varrho_i}{r}\right)\right] e^{-\frac{U_s}{U_{th}} \frac{1 + 2\varrho_i/r}{4 + 2\varrho_i/r}} \tag{65}$$

An example is given in fig.1 for $r = 10\,\varrho_i$

The total loss rate of ions is obtained by summing up the individual contributions and by integrating over the plasma surface

$$\phi_o = 2\pi\Big\{(1-\gamma)\frac{v_w}{\sqrt{\pi}}\int_0^r r'(1+\varepsilon_2)\, C^{1-2\varepsilon_2}\, n^{2(1-\varepsilon_2)}\, dr' + \frac{r}{4\sqrt{\pi}}\,\frac{\overline{\varrho_i^2}\, v_w \gamma}{\lambda}(1+\varepsilon_2)\, C^{1-2\varepsilon_2}\, n^{2(1-\varepsilon_2)} + 4 r l_o \frac{\eta e U_{th} n^2}{\lambda B^2}\Big[1+\varepsilon_2-\lambda\nabla_r\varepsilon_2 + 2\frac{w}{r\omega_{ci}}(1+2\varepsilon_2-2\lambda\nabla_r\varepsilon_2)\Big]\Big\} \tag{66}$$

where eqs.(61), (63) and (44) have been used.

In order to find the range of particle density for which this picture is applicable numbers will be put into condition (28). Making use of the relation

$$\tau_{ie} \approx \sqrt{\frac{m_i}{m_e}}\,\tau_{ii} \tag{67}$$

yields for (28)

$$\tau_{ii} < \tau_{fl}\, e^{-\frac{U_s}{U_E}} < \sqrt{\frac{m_i}{m_e}}\,\tau_{ii} \tag{68}$$

where

$$\tau_{ii} = 1.43\cdot 10^7\, \frac{\sqrt{A}\, U_{th}^{3/2}}{n \ln}\,(1+3\varepsilon) \tag{69}$$

$$\tau_{fl}\, e^{-\frac{U_s}{U_E}} = \frac{4\sqrt{\pi}\, l_o}{v_w}(1-\varepsilon_2)\, e^{-\frac{U_s}{U_{th}}\frac{1}{1+2\varepsilon_2}} \tag{70}$$

$$v_w = 1.38\cdot 10^6 \sqrt{\frac{U_{th}}{A}} \tag{71}$$

thus

$$\tau_{fl}\, e^{-\frac{U_s}{U_E}} = 5.14\cdot 10^{-6}\; l_o \sqrt{\frac{A}{U_{th}}}\,(1-\varepsilon_2) \qquad (72)$$

Therefore, the left hand side of condition (68) results in a lower limit for the mean particle density

$$n > 2.8\cdot 10^{12}\; \frac{U_{th}^2(1+3\varepsilon+\varepsilon_2)}{l_o}\; e^{-\frac{U_s}{U_{th}}\frac{1}{1+2\varepsilon_2}} \qquad (73)$$

for an estimation assuming ε_2 to be given by eq.(64) with $r = 10\rho_i$

$$\varepsilon = \varepsilon_2 = 0.20 \qquad (74)$$

and taking typical values for $U_{th} = 0.2$ V, $A = 133$ and $l_o = 40$ cm yields

$$5\cdot 10^9\cdot e^{-0.71\frac{U_s}{U_{th}}} < n < 5\cdot 10^9\cdot e^{-0.71\frac{U_s}{U_{th}}}\; \frac{m_i}{m_e} \qquad (75)$$

which covers the range of density mostly subject to investigation ($U_s < 0$ for an electron sheath!).

Conclusions

The outward directed flux of particles due to resistive and end plate diffusion leads to non vanishing pressure gradients perpendicular to the magnetic field and thus to a deviation from the equilibrium situation (for which, by definition, no macroscopic motions are allowed to exist). As a consequence of this effect an electric field must build up itself in order to provide balance of electrons on the surface of the end plates. For the assumptions made in this report, i.e. uniform temperature and work function on the plates, magnitude and sign of this electric field are such that the diamagnetic currents are completely carried by the ions [7], thereby providing a rotation of the plasma.

Furthermore, both the radial electric field and the centrifugal forces acting on the individual ions represent an energy source from which the particles can gain energy in excess to their thermal energy which they possess just when leaving the end plates. The gain of energy is proportional to the radius of gyration and is thus essential only for the ions.

As shown in fig.1 there is also a remarkable increase in end plate diffusion and end plate recombination as a result of this radial electric field provided that the mean particle density falls in a certain range. The increase of resistive diffusion, however, is -for typical experimental parameters- less than by a factor of 2.

At the beginning of this report it was said that the "equilibrium model" seems to be a good description of a collision dominated plasma in the Q-machine. From the results of this report it must be concludet that "collision dominated" has to be referred to ion electron collisions rather than to like particle collisions in order to provide equal mean energy for ions and electrons which then will be equal to plate temperature.

With respect to the two experimental observations mentioned at the beginning of this report it has to be stated, however, that the gain in energy for the ions is of the order of $2\varepsilon_2$ and therefore by far not sufficient to explain these experimental results. On the other hand, the increase in particle loss rates found in this report has not been subject to these experimental investigations.

Therefore it shall be checked whether the assumption of a reasonable temperature variation over the plates would influence the radial electric field so much that it could be accounted for these experimental observations. That shall be done along the lines of ref.7.

Since the plate surface is a surface of equipotential the radial electric field within the plasma is determined by the sheath voltage alone

$$E_r = - \nabla_r U_s \tag{76}$$

Utilizing eq.(58) and replacing j_R by

$$j_R = - eC_R U_{th}^2 e^{-\frac{\varphi}{U_{th}}} \tag{77}$$

where

$$C_R = 2.3\cdot 10^{28}\ \mathrm{cm}^{-2}\mathrm{sec}^{-1}\mathrm{V}^{-1} \tag{78}$$

it follows

$$U_s = \varphi + U_{th}\left[\ln n + \ln \frac{v_{we}}{2\sqrt{\pi}\, C_R U_{th}^2}\right] \tag{79}$$

and

$$E_r = - \frac{\partial U_s}{\partial n} \nabla_r n - \frac{\partial U_s}{\partial U_{th}} \nabla_r U_{th} \tag{80}$$

$$= - \frac{U_{th}}{n} \nabla_r n - \left[\ln \frac{n v_{we}}{2\sqrt{\pi} C_R U_{th}^2} - \frac{3}{2} \right] \nabla_r U_{th} \tag{81}$$

For an estimation U_{th} can be put to 0.2 V in the logarithmic term of eq.(81). This yields for E_r

$$E_r \approx - \frac{U_{th}}{n} \nabla_r n + \left[48.1 - 2.3 \log_{10} n \right] \nabla_r U_{th} \tag{82}$$

If the plate temperature is decreasing towards the edge of the plate the two contributions to E_r are of opposite sign and the second term is for non of the devices known large enough to be responsible for the experimental observations. That means that also a temperature variation across the plate surface must be excludet as the reason for the experimental findings of largely increased ion energies.

It should be noted, finally, that the increased loss rates of ions as well as the increase of the mean ion energy can be avoided by compensating the influence of the density distribution on the radial electric field by a suitably chosen distribution of the plate temperature, i.e. by satisfying the local condition

$$\left[\frac{3}{2} - \ln \frac{n v_{we}}{2\sqrt{\pi} C_R U_{th}^2} \right] \nabla_r U_{th} = + 2 \frac{U_{th}}{n} \nabla_r n \tag{83}$$

but such an adjustment might be difficult especially, since it depends on the particle density itself and its distribution.

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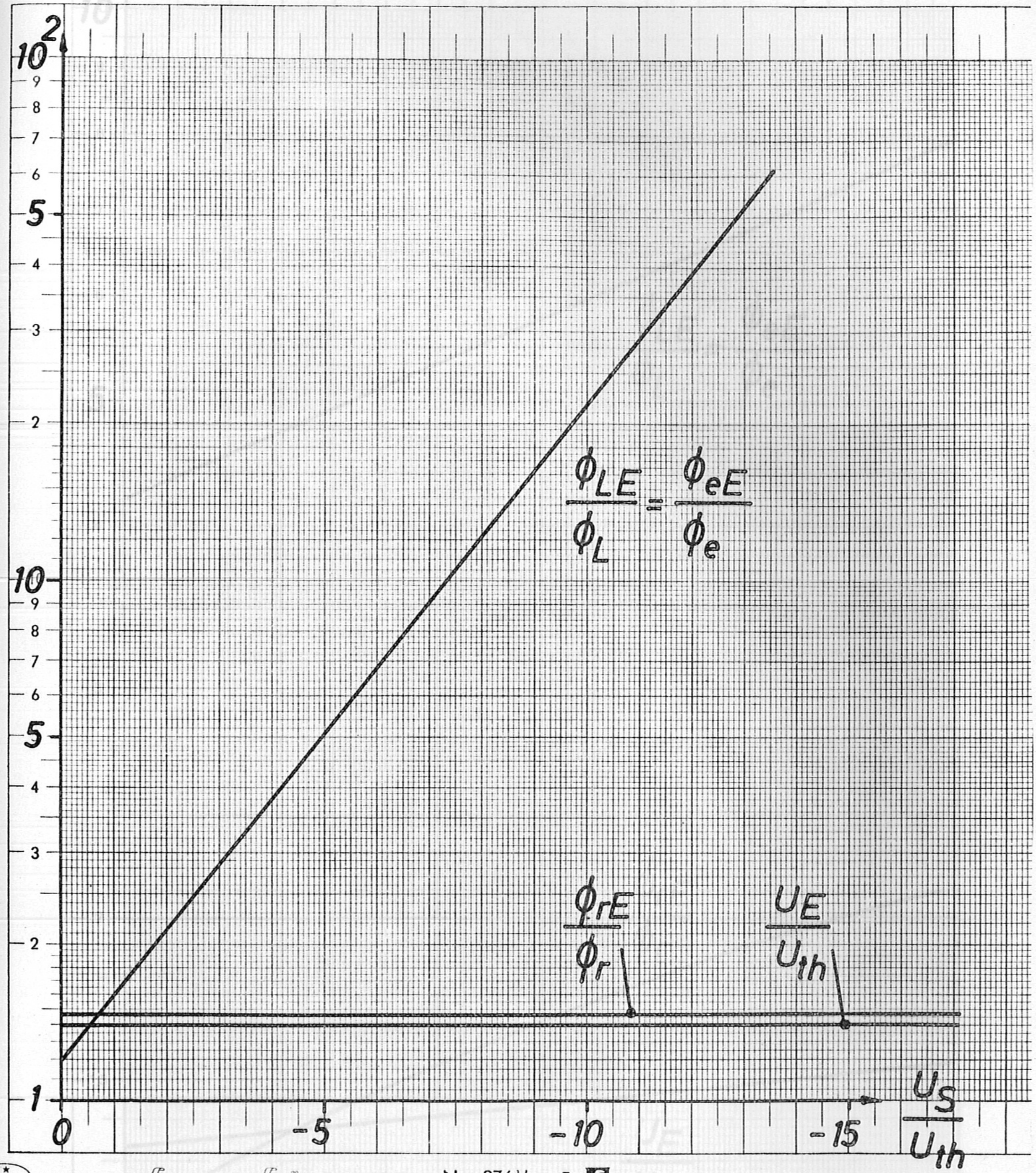
Figure Captions

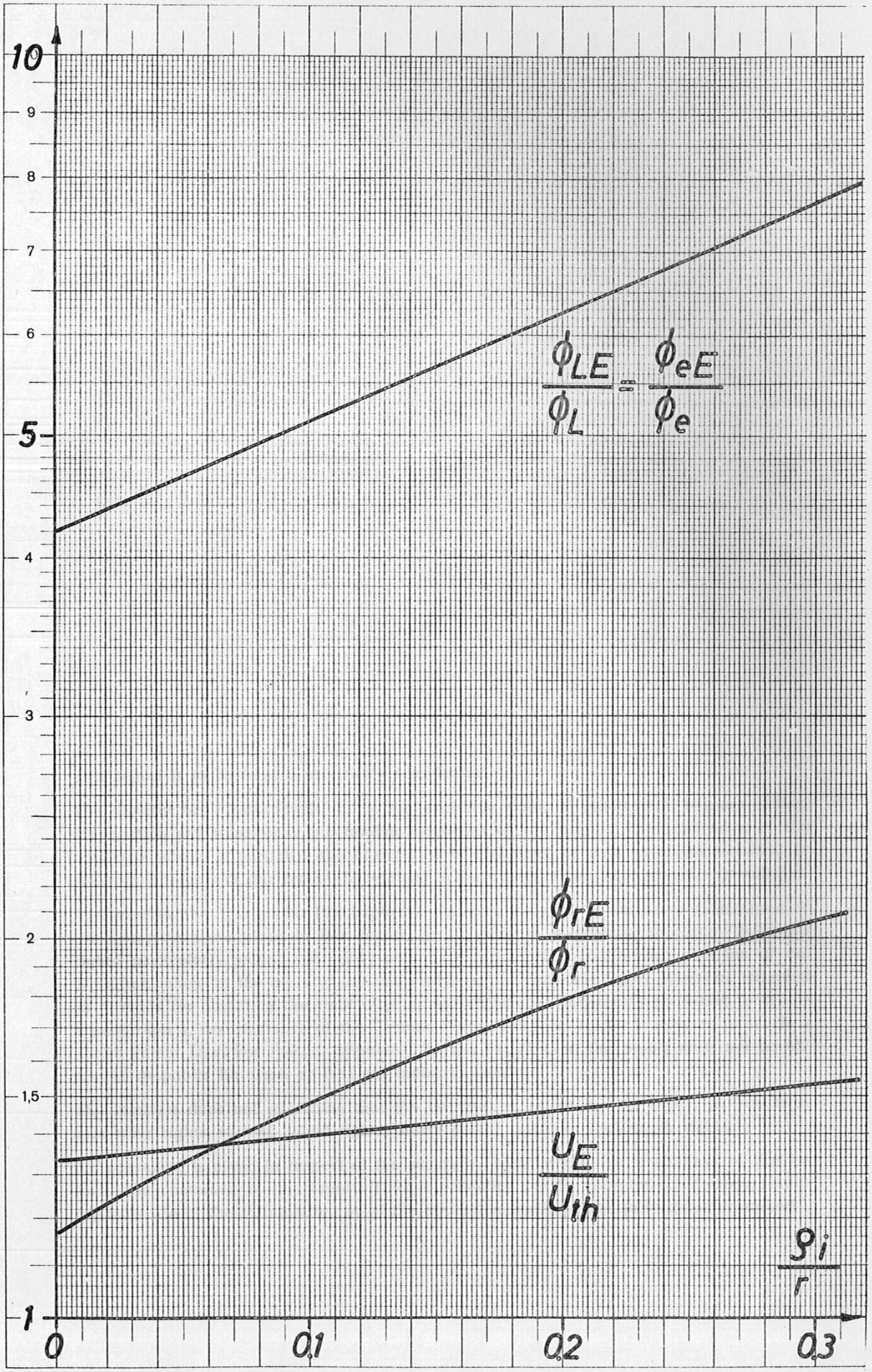
fig.1 Relative increase of mean particle energy, U_E (eq.(53)), of resistive diffusion, ϕ_{rE} (eq.(48)), of the loss rates to the end plates, ϕ_{LE}, and end plate diffusion, ϕ_{eE} (eq.(65)) vs. the ratio of sheath voltage, U_s, to the thermal voltage, U_{th}, for $a = 2/3$, $\lambda = \varrho_i$ and $r = 10\ \varrho_i$.

fig.2 Relative increase of the same quantities as shown in fig.1 vs. the ratio of the ion radius of gyration to the radius under consideration for $a = 2/3$, $\lambda = \varrho_i$ and $U_s/U_{th} = -5$.



10^2
5
10
5
1
0
-5
-10
-15
$\frac{U_S}{U_{th}}$
$\frac{\phi_{LE}}{\phi_L} = \frac{\phi_{eE}}{\phi_e}$
$\frac{\phi_{rE}}{\phi_r}$
$\frac{U_E}{U_{th}}$

fig.1



10
9
8
7
6
5
4
3
2
1,5
1
0
0,1
0,2
0,3
$\frac{\phi_{LE}}{\phi_L} = \frac{\phi_{eE}}{\phi_e}$
$\frac{\phi_{rE}}{\phi_r}$
$\frac{U_E}{U_{th}}$
$\frac{\rho_i}{r}$

Eine /

fig. 2