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"Three-Dimensional Normal Modes of a Vlasov Plasma With and Without Magnetic Field"

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ABSTRACT: The three-dimensional velocity-space eigenmodes of the linearized Vlasov equation for an electronic plasma, with and without a uniform applied magnetic field, are derived in the electrostatic limit, and are shown to be complete. In general, a continuum of modes exists (except for $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$) and, if the plasma is unstable, an additional set of discrete modes appears. For $\underset{\sim}{B}_0 = 0$ these modes reduce to the Van Kampen-Case spectrum upon integration over all velocities perpendicular to $\underset{\sim}{k}$. The adjoint modes are also derived, and used to establish certain orthogonality properties which are sufficient to enable one to use essentially standard eigenfunction techniques. It is shown for $\underset{\sim}{B}_0 = 0$ that the Balescu-Lenard kinetic equation can readily be derived with the use of the three-dimensional modes. For the case with magnetic field it is found that a particular functional of the undisturbed function plays a role (for $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \neq 0$) analogous to Case's $\eta(v)$ in establishing the properties of the modes. For the case $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$, which must be treated separately, two discrete sets of modes exist, only one of which contributes to the density and field fluctuations in the electrostatic approximation.

I) Introduction

The theory of electrostatic plasma oscillations has progressed along two seemingly widely varying paths. The first, introduced by Landau[1] in the case of disturbances in a uniform (collisionless) electronic plasma without magnetic field, involves solving the initial value problem for the linearized Vlasov equation with the use of Laplace transforms and analytic extension in the complex plane of the transform variable. Bernstein[2] applied the same technique to the more complicated problem of a multi-species collisionless plasma in a uniform magnetic field. Bernstein included transverse (electromagnetic) as well as longitudinal (electrostatic) oscillations.

The second technique, introduced by Van Kampen[3] and extended in a decisive manner by Case[4], applies the theory of distributions to allow the interpretation of one-dimensional (in velocity space) stationary wave solutions of the linearized Vlasov equation (integrated over all velocities perpendicular to the propagation vector $\underset{\sim}{k}$) in a uniform plasma without magnetic field. Van Kampen showed, for a plasma with an isotropic distribution function in the undisturbed state, that this continuum of "stationary wave" solutions forms a complete set of normal modes which can be used through superposition to describe any disturbance of such a plasma which can be adequately described in terms of the one remaining dimension in velocity space. Such a description is sufficient to determine the (electrostatic) electric field perturbations, for example, and Van Kampen demonstrated the equivalence (under certain restrictions) of his electric field solution, obtained by superposition of normal modes, with the solution of Landau. He also indicated certain differences in cases where analytical extension -- essential in Landau's technique -- is not permissible.[5]

Case extended Van Kampen's technique in several important ways. Introducing a set of functions adjoint to the one-dimensional normal modes, he demonstrated orthogonality between the normal modes and their adjoints. Using this property he was able to show that the addition of a set of discrete modes, including modes with discrete complex frequencies (eigenvalues), to the Van Kampen "continuum" modes, provides a complete set of normal modes for a linearly unstable plasma without magnetic field. The condition for the existence of the discrete modes can be shown to correspond to the Penrose criterion for instability.[6] In this sense, Case provided the final proof that the Van Kampen continuum modes (if one includes Case's "1b modes") are complete (in one-dimensional velocity space) for any stable plasma with $\mathbf{B}_0 = 0$. Unfortunately, however, Case did not correctly treat the situation for which marginally stable *discrete* modes exist (the "1c modes", see below). A consequence of this is that his theory does not give a well-defined statement of the contribution of the marginally stable modes to the plasma behavior.

In a later paper[7], Zelazny extended the work of Van Kampen and Case (again for $\mathbf{B}_0 = 0$) to show that the initial value problem for longitudinal plasma oscillations can be solved in three dimensions in velocity space by an analogous superposition of three-dimensional eigenfunctions. He emphasized that the successful solution of the initial value problem by this technique amounts to a proof of the completeness of the normal modes used (by the uniqueness of the solution). An interesting feature of Zelazny's approach, however, is that his "eigenfunctions" actually remain implicit; they contain an arbitrary function $\lambda(v, \mathbf{v}_\perp)$(compare Section II) which is determined differently for each function to be expanded. For the initial value problem, this procedure cannot be criticized; in the following, however, we adopt the point of view that the normal modes are of interest in themselves and that it is useful to have at hand an <u>explicit</u> complete set, independent of the function to be expanded, for use in a variety of problems. Zelazny gives some attention to the marginally stable modes, including the possibility of their being degenerate, but he again gives no explicit prescription fot the determination of their net contribution to the plasma behavior.

Felderhof[8] has also generalized the Van Kampen modes to three dimensions with $\underset{\sim}{B}_0 = 0$. His treatment is limited however, to the stable case (no discrete modes); indeed, he gives an explicit set of modes only if the undisturbed distribution function is Maxwellian. Nevertheless, his procedure, quite different from the following, has a number of points to recommend it, and illustrates the fact that in normal mode theory there is not a unique path to follow.

The purpose of the present paper is two-fold. In the first part an explicit, relatively straightforward generalization to three dimensions in velocity space of the Van Kampen-Case modes (and their adjoints) for electrostatic disturbances of a uniform electronic plasma without magnetic field is given. It is shown (by including discrete modes in the marginally unstable and unstable cases) that these modes form a complete set for three-dimensional disturbances of an electronic plasma, and superposition of the modes provides a solution of the initial value problem which agrees not only with the electric field solution of Landau, but also with his solution for the complete disturbance distribution function in terms of all three dimensions in velocity space. Orthogonality between the normal modes and their adjoints, in an interesting restricted sense, is demonstrated.

Integration of the generalized modes over all velocities perpendicular to the propagation vector reproduces the Van Kampen-Case spectrum. It is also shown, in the proof of completeness, that the difficulties associated with Case's treatment when marginally stable discrete modes exist can be circumvented, and an explicit determination of the contribution of such modes to the solution of any given problem can be provided.

There are important examples for which the integrated ("one-dimensional") modes of Van Kampen and Case are insufficient and for which the three dimensional modes are required if an eigenmode technique is to be employed. [9,10] In this paper we limit ourselves to one interesting example: a simple derivation of the Balescu-Lenard[11,12] equation based on the three-dimensional eigenmodes derived below.

The second part of the paper is devoted to the derivation of a set of normal modes (and their adjoints) for electrostatic perturbations of a plasma in a uniform applied magnetic field . $\underset{\sim}{B}_0$. Here, we shall make essential use of the ideas developed for the zero - $\underset{\sim}{B}_0$ case. In analogy to the zero magnetic field case there exists (for $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \neq 0$) again a continuum, and, if the plasma is unstable, an additional set of discrete set of eigenmodes. A completeness proof is given, and this provides the necessary and sufficient condition for the existence of the discrete modes when $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \neq 0$. It is pointed out that this corresponds to a necessary and sufficient condition for instability of a plasma with uniform magnetic field (for $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \neq 0$).[13] For the special case of propagation perpendicular to the applied magnetic field ($\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$), the normal modes are discrete whether the plasma is stable to such disturbances or not.[2]

II) Three-Dimensional Eigenmodes without Magnetic Field ($\underset{\sim}{B}_0 = 0$)

We seek stationary solutions of the linearized Vlasov equation for a uniform unbounded electronic plasma without magnetic field in the form

$$f(\underset{\sim}{v}, t) = f_0(\underset{\sim}{v}) + \mathcal{E}(\underset{\sim}{v}, \omega, \underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t} \tag{1}$$

We have then, in the electrostatic limit,*

$$(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v})\mathcal{E} - \frac{\omega_p^2}{k^2}\, i\underset{\sim}{k}\cdot\frac{\partial f_0(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \int d^3\underset{\sim}{v}\, \mathcal{E} = 0 \tag{2}$$

where with reference to mks units $\omega_p^2 = n_0 e^2/\varepsilon_0 m$, e is the electronic charge, m the electronic mass, n_0 is the mean electron density, and $\mathcal{E}_0$ is the permittivity of free space. Following Van Kampen and Case we normalize the stationary modes such that**

$$\int d^3\underset{\sim}{v}\, \mathcal{E} = 1 \tag{3}$$

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* See footnote below Equ. (11).

** Zelazny[7] normalized his modes to a function $a(\omega/k)$ which plays the role of our "expansion coefficients" $A_{\parallel}(w)$ introduced later. We shall also find certain modes for which $\int d^3\underset{\sim}{v}\, \mathcal{E} = 0$ (compare Ref. 8).

for any ω and $\mathbf{k}$. Defining $w \equiv \omega/k$, $v = \mathbf{k}\cdot\mathbf{v}/k$, $k \equiv |\mathbf{k}|$, $\mathbf{v}_\perp = \mathbf{v} - \frac{v\mathbf{k}}{k}$, we may then rewrite (1) in the form

$$(w-v)\,\mathcal{E}(v,\mathbf{v}_\perp|w,\mathbf{k}) = -\frac{\omega_p^2}{k^2}\frac{\partial f_0(v,\mathbf{v}_\perp)}{\partial v} \tag{4}$$

where $\partial/\partial v \equiv (\mathbf{k}/k)\cdot\partial/\partial\mathbf{v}$. Within the framework of the theory of distributions[14], the solution of (4) is

$$\mathcal{E}(v,\mathbf{v}_\perp|w) = -\frac{\omega_p^2}{k^2}\frac{\partial f_0(v,\mathbf{v}_\perp)}{\partial v}\frac{\mathcal{P}}{w-v} + \hat{\lambda}(v,\mathbf{v}_\perp)\,\delta(v-w) \tag{5}$$

where $\mathcal{P}$ indicates that the Cauchy principal part is to be taken upon integration over v (or w) and $\hat{\lambda}$ is any function of v and $\mathbf{v}_\perp$. The first term is the "particular" solution of (4), the second term the "homogeneous" solution.

Utilizing (3), we find

$$\lambda(w) \equiv \int d^2\mathbf{v}_\perp\, \lambda(v,\mathbf{v}_\perp) = 1 + \frac{\omega_p^2}{k^2}\,\mathcal{P}\!\int_{-\infty}^{\infty}\frac{F_0'(v)\,dv}{w-v} \tag{6}$$

where the prime denotes differentiation with respect to argument, and

$$F_0(v) \equiv \int d^2\mathbf{v}_\perp\, f_0(v,\mathbf{v}_\perp) \tag{7}$$

For convenience, we define

$$\eta(v) \equiv -\frac{\omega_p^2}{k^2}F_0'(v) \tag{8}$$

and

$$N(v,\mathbf{v}_\perp) \equiv -\frac{\omega_p^2}{k^2}\frac{\partial f_0(v,\mathbf{v}_\perp)}{\partial v} \tag{9}$$

so that (6) reads, for example

$$\lambda(w) = 1 - P\int_{-\infty}^{\infty} \frac{\eta(v)\,dv}{w - v} \tag{10}$$

The dependence of $\hat{\lambda}(v, \mathbf{v}_\perp)$ on $\mathbf{v}_\perp$ is of course not determined by the normalization condition (3); it is left to our choice. Zelazny[7] left $\hat{\lambda}(v, \mathbf{v}_\perp)$ undetermined in the modes themselves, treating it, in effect, as a part of the expansion coefficients, and choosing it therefore differently for each function to be expanded. However, we seek an explicit set of stationary solutions which can provide a complete set of eigenmodes for the description of any disturbance function $f(\mathbf{v})$ in three dimensions. For this purpose, it is apparent that we require two further parameters, in addition to w, to be included in the normal modes $\mathcal{E}$. Let us denote these two parameters by a real two-dimensional vector $\boldsymbol{\gamma}$. We then choose our function $\hat{\lambda}(v, \mathbf{v}_\perp)$ in the form

$$\hat{\lambda}_{\boldsymbol{\gamma}}(v, \mathbf{v}_\perp) = \lambda(v)\,\delta(\mathbf{v}_\perp - \boldsymbol{\gamma}) \tag{11}$$

so that*

$$\mathcal{E}(v, \mathbf{v}_\perp | w, \boldsymbol{\gamma}) = \frac{P N(v, \mathbf{v}_\perp)}{w - v} + \lambda(v)\,\delta(\mathbf{v}_\perp - \boldsymbol{\gamma})\,\delta(v - w) \tag{12}$$

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*One measure of the error inherent in the electrostatic approximation is $(\mathbf{k} \times \mathbf{E})/\mathbf{k}\cdot\mathbf{E} = (w/c)(1 - w^2/c^2)^{-1}(\mathbf{j} \times \mathbf{k})/\rho_e k c$ where ρ_e and $\mathbf{j}$ are the perturbation charge and current. It is apparent that for sufficiently large w and $|\boldsymbol{\gamma}|$, the modes given

(cont'd. p. 9)

If we require the normalization condition (3) to hold for each $\underset{\sim}{\gamma}$ as well as for each w, the expression (10) for still holds, and is identical to that used by Case.[4] It is important to note that $\lambda(v)$ depends only on $\eta(v)$, i.e. the integral of $N(v, \underset{\sim}{v}_\perp)$ over $\underset{\sim}{v}_\perp$. Since

$$\int d^2\underset{\sim}{v}_\perp \; N(v, \underset{\sim}{v}_\perp) = \eta(v) \tag{13}$$

we note further that integration of the modes over $\underset{\sim}{v}_\perp$ reduces the three-dimensional modes (12) to the one-dimensional Van Kampen-Case modes, i.e.,

$$\int d^2\underset{\sim}{v}_\perp \; \mathcal{E}(v, \underset{\sim}{v}_\perp | w, \underset{\sim}{\gamma}) \equiv \mathcal{E}^{1D}(v|w) = \mathcal{P}\frac{\eta(v)}{w-v} + \lambda(v)\,\delta(v-w) \tag{14}$$

In analogy to Case's treatment of the one-dimensional modes, we may distinguish several classes of modes, depending on the properties of η, N and λ. (Recall that $\underset{\sim}{\gamma}$ is always real.)

Class 1a. Here we have w real, $N(w, \underset{\sim}{\gamma}) \neq 0$. These modes form the main continuum of the plasma spectrum. In the present three-dimensional treatment, the special case $\int d^2\underset{\sim}{\gamma}\, N(w, \underset{\sim}{\gamma}) = \eta(w) = 0$, $\lambda(w) \neq 0$, can be included in this class without difficulty (see discussion of adjoints).

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*(cont'd from p. 8) here are not consistent with this approximation. We proceed, however, with the assumption that for many problems of interest the "important" values of w and $|\underset{\sim}{\gamma}|$ in an expansion involving the normal modes will be much less than c.

Class 1b. For these modes, we have w real, $N(w, \underset{\sim}{\gamma}) = 0$, $\lambda(w) \neq 0$. These special modes generally form part of the continuum, and differ from Class 1a most markedly through the nature of their adjoints (Section III). The explicit form for the normal modes themselves is still given by (12). In using these and other continuum modes we must give special attention to the real points of Class 1c (if such points occur).

Class 1c. These modes appear if the undisturbed plasma distribution is such that for real values of $w (\equiv w_i)$ both

$$\eta(w_i) = 0$$
$$\lambda(w_i) = 0 = 1 - \int_{-\infty}^{\infty} \frac{\eta(v)\,dv}{w_i - v} \tag{15}$$

If we assume for simplicity that (15) have only simple roots[7,4], then in the three-dimensional case two independent sets of modes arise at these points:

$$\text{(Class 1c, 1)} \qquad \mathcal{E}^{(1)}(v, \underset{\sim}{v}_\perp | w_i) = \frac{\mathcal{P}\, N(v, \underset{\sim}{v}_\perp)}{w_i - v}$$
$$\text{(Class 1c, 2)} \qquad \mathcal{E}^{(2)}(v, \underset{\sim}{v}_\perp | w_i) = N(v, \underset{\sim}{v}_\perp)\,\delta(v - w_i) \tag{16}$$

Both sets are independent of the parameters $\underset{\sim}{\gamma}$. The Class 1c, 1 modes reduce to Case's one-dimensional modes, upon integration over $\underset{\sim}{v}_\perp$, while the 1c,2 modes vanish after such integration and therefore do not appear in the one-dimensional treatment. The normalization (3) applied to Class 1c, 1 reproduces (15).* Note that even for simple roots of (15) a degeneracy arises at 1c points in the three-dimensional case.

Class 2. Finally we may have eigensolutions for complex w. If such modes occur they form a discrete set

$$\mathcal{E}(v, \underset{\sim}{v}_{\perp} | w_j) = \frac{N(v, \underset{\sim}{v}_{\perp})}{w_j - v} \tag{17}$$

with complex eigenvalues $w = w_j$ such that, with the normalization (3),

$$1 = \int d^3\underset{\sim}{v} \, \frac{N(v, \underset{\sim}{v}_{\perp})}{w_j - v} = \int_{-\infty}^{\infty} \frac{dv \, \eta(v)}{w_j - v} \tag{18}$$

The latter relation defines the w_j. Again, these modes are independent of the parameters $\underset{\sim}{\gamma}$. Note that if w_j is a solution of (18), so is w_j^*.

Before demonstrating completeness of these normal modes, we discuss their adjoints.

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*For further discussion of the 1c modes, see Section III. The reader will note that additional modes of the type 1c,2 can be found at discrete points $\eta(w_k) = 0$, $\lambda(w_k) \neq 0$. However, we shall have no need for such modes, as the modes listed above are already complete for functions of physical interest. (Section IV).

III. Adjoint Modes for $\mathbf{B}_0 = 0$.

The adjoint equation to (2) is

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v})\tilde{\mathcal{E}} - \frac{\omega_p^2}{k^2}\, i\mathbf{k}\cdot\int d^3\mathbf{v}\, \frac{\partial f_0}{\partial \mathbf{v}}\, \tilde{\mathcal{E}} = 0 \tag{19}$$

or

$$(w' - v)\, \tilde{\mathcal{E}}(v,\mathbf{v}_\perp | w', \boldsymbol{\gamma}') = \int d^3\mathbf{v}\; N(v,\mathbf{v}_\perp)\, \tilde{\mathcal{E}}(v,\mathbf{v}_\perp | w', \boldsymbol{\gamma}') \tag{20}$$

Multiplying the complex conjugate of (20) by $\mathcal{E}(v,\mathbf{v}_\perp | w, \boldsymbol{\gamma})$, multiplying (4) by $\tilde{\mathcal{E}}^*(v,\mathbf{v}_\perp | w, \boldsymbol{\gamma}')$, integrating both equations over all $\mathbf{v}$, utilizing (3), and subtracting, we find

$$(w' - w)\int d^3\mathbf{v}\; \tilde{\mathcal{E}}^*(v,\mathbf{v}_\perp | w', \boldsymbol{\gamma}')\, \mathcal{E}(v,\mathbf{v}_\perp | w, \boldsymbol{\gamma}) = 0 \tag{21}$$

or, for the continuum modes (these are purely real, $w^* = w$)

$$\int d^3\mathbf{v}\; \tilde{\mathcal{E}}^*(v,\mathbf{v}_\perp | w', \boldsymbol{\gamma}')\, \mathcal{E}(v,\mathbf{v}_\perp | w, \boldsymbol{\gamma}) = \delta(w - w')\, C(w, \boldsymbol{\gamma}, \boldsymbol{\gamma}') \tag{22}$$

and, for the discrete modes (independent of $\boldsymbol{\gamma}$)

$$\int d^3\mathbf{v}\; \tilde{\mathcal{E}}^*(v,\mathbf{v}_\perp | w_j)\, \mathcal{E}(v,\mathbf{v}_\perp | w_i) = \delta_{w_i w_j^*}\, C_{w_i} \tag{23}$$

We see that equation (4) and its adjoint (20) implies orthogonality between the normal modes and their adjoints with respect to w-space, but not necessarily with respect to $\boldsymbol{\gamma}$-space. For the discrete modes (independent of $\boldsymbol{\gamma}$) this is completely sufficient; for the continuum modes, our choice (11) for $\hat{\lambda}(v,\mathbf{v}_\perp)$ will provide a certain amount of "sharpness" in $\boldsymbol{\gamma}$-space and we shall see that this is sufficient. In what follows we give the explicit adjoint solutions for each class of normal modes, and the corresponding values of the normalization factors $C(w, \underset{\sim}{\gamma}, \underset{\sim}{\gamma}')$ and C_{w_i}.

Class 1a.

For this class of continuum modes we normalize the adjoints such that

$$\int d^3\underset{\sim}{v}\, N(v, \underset{\sim}{v}_\perp)\, \tilde{\mathcal{E}}(v, \underset{\sim}{v}_\perp | w, \underset{\sim}{\gamma}) = 1 \tag{24}$$

and find the solution

$$\tilde{\mathcal{E}}(v, \underset{\sim}{v}_\perp | w, \underset{\sim}{\gamma}) = \frac{\mathcal{P}\,1}{w - v} + \frac{\lambda(w)}{N(w, \underset{\sim}{\gamma})}\,\delta(\underset{\sim}{v}_\perp - \underset{\sim}{\gamma})\,\delta(v - w) \tag{25}$$

where, because of (24), $\lambda(w)$ is again given by (10). The normal modes themselves are given by (12), both $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are real, and we find, utilizing techniques for singular integrals which are related to the proof of the Poincaré-Bertrand formulas[15]

$$\int d^3\underset{\sim}{v}\, \tilde{\mathcal{E}}^*(\underset{\sim}{v} | w', \underset{\sim}{\gamma}')\,\mathcal{E}(\underset{\sim}{v} | w, \underset{\sim}{\gamma}) = \delta(w - w')\left\{\pi^2\eta(w) + \frac{\lambda^2(w)\,\delta(\underset{\sim}{\gamma} - \underset{\sim}{\gamma}')}{N(w, \underset{\sim}{\gamma})}\right\} \tag{26}$$

Again, note that in the three-dimensional treatment the case $\eta(w) = 0$, $\lambda(w) \neq 0$ (the "1b Class" for the <u>one-dimensional</u> modes) can be included in this Class without difficulty.

<u>Class 1b</u> (w real, $N(w, \underset{\sim}{\gamma}) = 0$, $\lambda(w) \neq 0$)

For these modes, we may take

$$\mathcal{E} = \delta(w - v)\, \delta(\underset{\sim}{v}_\perp - \underset{\sim}{\gamma}) \tag{27}$$

since then

$$\int d^3\underset{\sim}{v}\, N(v, \underset{\sim}{v}_\perp)\, \tilde{\mathcal{E}} = N(w, \underset{\sim}{\gamma}) = 0 \tag{28}$$

and (27) indeed solves equation (20) for this class. With $\mathcal{E}$ given by (12) we find

$$\int d^3\underset{\sim}{v}\, \tilde{\mathcal{E}}^*(v, \underset{\sim}{v}_\perp | w', \underset{\sim}{\gamma}')\, \mathcal{E}(v, \underset{\sim}{v}_\perp | w, \underset{\sim}{\gamma}) = \lambda(w)\, \delta(w - w')\, \delta(\underset{\sim}{\gamma} - \underset{\sim}{\gamma}') \tag{29}$$

<u>Class 1c</u> (w_i real, $\eta(w_i) = 0 = \lambda(w_i)$)

In this class we find two independent sets of adjoint modes

$$\begin{array}{ll} \text{(Class 1c, 1)} & \tilde{\mathcal{E}}^{(1)}(v, \underset{\sim}{v}_\perp | w_i) = \dfrac{P\, 1}{w_i - v} \\ \text{(Class 1c, 2)} & \tilde{\mathcal{E}}^{(2)}(v, \underset{\sim}{v}_\perp | w_i) = \delta(v - w_i) \end{array} \tag{30}$$

Clearly, because of the first of (15), the 1c,2 adjoint modes cause both sides of (20) to vanish for $w = w_i$, whereas the normalization (24) applied to the 1c,1 adjoint modes guarantees solution of (20) for $w = w_i$, while at the same time reproducing (15).

In the following we actually have no direct need for the lc adjoints; we include them here for the sake of completeness. Case's prescription[4] for determining the coefficients of the lc modes via orthogonality -- see Section IV -- does not work in the form he gave it; the integrals in his expression for these coefficients do not exist.* We shall see below, however, that a rigorous proof of completeness of the modes of Section II, including proper treatment of any existing lc modes.

On the other hand, we shall be able to make good use of the orthogonality property

$$\int d^3\underset{\sim}{v}\ \mathcal{E}^{(1)}(v, \underset{\sim}{v}_\perp | w_i)\ \tilde{\mathcal{E}}^{*}_{1a}(v, \underset{\sim}{v}_\perp | w, \underset{\sim}{\gamma}) = 0 \tag{31}$$

which can be verified by direct computation.

Zelazny[7] discussed in some detail the situation which arises if equations (15) possess multiple roots at any w_i, in which case further lc,1 modes exist. However, he did not note the need for modes of the type lc,2 (or their multiplicity at multiple roots), nor did he explicitly discuss the problem of finding the "weights" (expansion coefficients) in the solution of his initial value problem.

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*Völk[16] recently pointed out a part of this difficulty. The author is indebted to Dr. Völk for bringing his attention to the problem.

Class 2 (w_j complex)

For this class of discrete modes, which are necessary for the description of the disturbance of an unstable plasma, we again use (24) and find (for simple roots of (18))

$$\tilde{\mathcal{E}}(v, \underset{\sim}{v_\perp} | w_j) = \frac{1}{w_j - v} \tag{32}$$

Again, the normalization reduces to (18), which defines the discrete complex eigenvalues w_j. We again assume for simplicity, that (18) has only simple roots. For Class 2,

$$\int d^3\underset{\sim}{v}\, \tilde{\mathcal{E}}^*(v, \underset{\sim}{v_\perp} | w_j)\, \mathcal{E}(v, \underset{\sim}{v_\perp} | w_i) = \delta_{w_i w_j^*}\, C_{w_i}$$

where (33)

$$C_{w_i} = \int \frac{d^3\underset{\sim}{v}\, N(v, \underset{\sim}{v_\perp})}{(w_i - v)^2} = \int_{-\infty}^{\infty} \frac{dv\, \eta(v)}{(w_i - v)^2}$$

exactly as in the one-dimensional case. As pointed out by Case[4], $C_{w_j} \neq 0$ if w_j is a simple root of (18).

We have been forced to give special attention to the discrete set of points w_i for real w at which both $\eta(w_i) = 0 = \lambda(w_i)$ if such points exist (Class 1c, above). The necessity of doing so can be seen, for example, from the "norm" (26) for the continuum modes. There remains, however, another set of discrete real points which, in the three-dimensional case, could in principle cause difficulty. These are the points w_k such that $\lambda(w_k) = 0$, $\eta(w_k) \neq 0$.

These points exist even for a stable plasma[6], and it would be strange indeed if they caused any real problems. Nevertheless, one sees from (26) that at such points the norm (26) loses its sharpness in $\underset{\sim}{\gamma}$-space. That this is only a formal difficulty will be shown in the following proof of completeness of the normal modes.

IV) Completeness Proof $(\underset{\sim}{B}_0 = 0)$

We wish to show that the normal modes given in Section II are complete in the sense that "any" function* $f(v, \underset{\sim}{v}_\perp)$ can be expanded in the form

$$f(v, \underset{\sim}{v}_\perp) = \sum_{w_j} a_{w_j} \mathcal{E}(v, \underset{\sim}{v}_\perp | w_j) + \sum_{w_i} \left(a_{w_i} \mathcal{E}^{(1)}(v, \underset{\sim}{v}_\perp | w_i) + b_{w_i} \mathcal{E}^{(2)}(v, \underset{\sim}{v}_\perp | w_i) \right) \tag{34}$$
$$+ \int_{-\infty}^{\infty} dw \int d^2\underset{\sim}{\gamma}\, A(w, \underset{\sim}{\gamma})\, \mathcal{E}(v, \underset{\sim}{v}_\perp | w, \underset{\sim}{\gamma})$$

The sums in (34) run over all real (w_i) and complex (w_j) discrete eigenvalues, and we must specify the manner in which the integral over real w in (34) is to be defined at the points w_i of Class 1c.

Utilizing the orthogonality of the complex discrete adjoint modes with the real discrete and continuum modes, we

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The function $f(v, \underset{\sim}{v}_\perp)$ must of course satisfy certain smoothness conditions. Case[4] pointed out that the Hölder condition is sufficient, but probably not necessary, find readily

$$
\begin{aligned}
a_{w_j^*} &= \frac{1}{C_{w_j^*}} \int d^3 v \, f(v, v_\perp) \, \tilde{\mathcal{E}}^*(v, v_\perp | w_j) \\
&= \frac{1}{C_{w_j^*}} \int_{-\infty}^{\infty} dv \, f_{\parallel}(v) \, \tilde{\mathcal{E}}^{1D}(v | w_j^*)
\end{aligned}
\tag{35}
$$

where the C_{w_j} are given by equations (33), and

$$
f_{\parallel}(v) \equiv \int d^2 v_\perp \, f(v, v_\perp)
$$

Thus, the expansion coefficients for the complex discrete modes are precisely the same as their counterparts for the one-dimensional case[4] and depend only on the properties of $f\,(v, v_\perp)$ integrated over $v_\perp$.

Inserting the explicit continuum modes (12) in (34), we now have

$$
\begin{aligned}
f(v, v_\perp) = &\sum_{w_j} a_{w_j} \mathcal{E}(v, v_\perp | w_j) + \sum_{w_i} \left(a_{w_i} \mathcal{E}^{(1)}(v, v_\perp | w_i) + b_{w_i} \mathcal{E}^{(2)}(v, v_\perp | w_i) \right) \\
&+ N(v, v_\perp) \int_{-\infty}^{\infty} dw \, A_{\parallel}(w) \, \frac{\mathcal{P}\,1}{w - v} \quad + A(v, v_\perp) \lambda(v)
\end{aligned}
$$

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*Note from (32) that the discrete Class 2 adjoints are actually independent of $v_\perp$, and equal to their "one-dimensional" counterparts. Case[4] gave an expression similar to (35) for the a_{w_i} of the lc modes; however, as already mentioned, the integrals in his formula are undefined. We shall find the a_{w_i} and b_{w_i} below by a different method.

where

$$A_{\parallel}(w) \equiv \int d^2\underset{\sim}{\gamma}\, A(w,\underset{\sim}{\gamma})$$

We define

$$\hat{f}(v,\underset{\sim}{v}_{\perp}) \equiv f(v,\underset{\sim}{v}_{\perp}) - \sum_{w_j} a_{w_j}\, \mathcal{E}(v,\underset{\sim}{v}_{\perp}|w_j)$$

so that (34) becomes, with the explicit lc modes

$$\begin{aligned} \hat{f}(v,\underset{\sim}{v}_{\perp}) &= \lambda(v) A(v,\underset{\sim}{v}_{\perp}) + N(v,\underset{\sim}{v}_{\perp}) \int_{-\infty}^{\infty} dw\, A_{\parallel}(w)\, \frac{\mathcal{P}\,1}{w-v} \\ &+ \sum_{w_i} a_{w_i} \frac{\mathcal{P}\, N(v,\underset{\sim}{v}_{\perp})}{w_i - v} + \sum_{w_i} b_{w_i} N(v,\underset{\sim}{v}_{\perp})\, \delta(w_i - v) \end{aligned} \tag{36}$$

We shall show below that if simple roots of equations (15) exist, equation (36) can hold only for particular values of the coefficients a_{w_i} and b_{w_i} (which fact determines their choice) and that for these particular values, equation (36) can always be solved for $A_{\parallel}(w)$ in terms of $f_{\parallel}(v)$ provided equations (35) hold. The special choice of the a_{w_i} is required in order that $A_{\parallel}(w)$ be well-defined for all real w, including all lc points.

If no solutions of (15) exist (i.e., no lc modes), (36) can always be solved for $A_{\parallel}(w)$ with the help of (35), and the result is identical to that of Case.[4]

It follows in either situation that the three-dimensional modes derived in Section II are complete, since if $A_{\parallel}(w)$ is known and well-defined for all real w, and if all coefficients of the discrete modes are known, then $\lambda(v)A(v,\underset{\sim}{v}_\perp)$ is determined immediately from (36). Since (36) shows that we actually require only $\lambda(v)A(v,\underset{\sim}{v}_\perp)$ and $A_{\parallel}(v)$ it is then always possible to determine the relevant expansion coefficient such that (34) is valid.

We begin by observing that if the given function $f(v,\underset{\sim}{v}_\perp)$ (and hence $\hat{f}(v,\underset{\sim}{v}_\perp)$) contains no delta-functions*, then we must choose each $b_{w_i} \equiv 0$ except when the integral term in (36) itself gives rise to delta-functions at the points w_i. In the latter case, the b_{w_i} must be chosen to cancel such terms. We shall see that if lc points exist, the b_{w_i} are not in general zero

Consider now the result of integrating (36) over $\underset{\sim}{v}_\perp$. This yields a one-dimensional integral equation for $A_{\parallel}(v)$ of the form

$$\phi(v) = \lambda(v) A_{\parallel}(v) + \eta(v)\int_{-\infty}^{\infty} dw\, A_{\parallel}(w)\,\frac{\mathcal{P}\,1}{w-v} \qquad (37)$$

where
$$\phi(v) \equiv \int d^2\underset{\sim}{v}_\perp\, \hat{f}(v,\underset{\sim}{v}_\perp) - \sum_{w_i} a_{w_i}\frac{\eta(v)}{w_i - v}$$

Equation (37) is independent of the b_{w_i}. This equation is to be solved in conjunction with equation (10). Except for the fact that the a_{w_i} are still undetermined, (37) and (10) are exactly the equations studied by Case.[4]

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*This is obviously the situation of primary physical interest.

Case's solution

$$A_{\parallel}(w) = \frac{\eta(w)}{\lambda^2(w) + \pi^2\eta^2(w)} \left[\mathcal{P} \int \frac{\phi(v)\,dv}{w - v} + \frac{\lambda(w)}{\eta(w)} \phi(w) \right]$$

$$= \frac{\eta(w)}{\lambda^2(w) + \pi^2\eta^2(w)} \left[\mathcal{P} \int \frac{f_{\parallel}(v)\,dv}{w - v} + \frac{\lambda(w)}{\eta(w)} f_{\parallel}(w) \right]$$

in fact holds for both stable and unstable plasmas provided no 1c (marginally stable) modes exist. (In this result one may write either $\phi(v)$ or $f_{\parallel}(v)$ in view of the orthogonality property (31) plus the orthogonality of the Class 2 modes with the 1a adjoints.) It is clear, however, that Case's $A_{\parallel}(w)$ is undefined at any points w_i such that <u>both</u> $\lambda(w_i) = 0 = \eta(w_i)$. If such 1c points exist, Case's treatment must be substantially modified to provide a meaning to the integrals in (34), (36), and (37).

To accomplish this we assume that there exists a set of simple roots w_i of equations (15) and look for solutions of (36) and (37) in the form

$$\lambda(w) A(w, \underset{\sim}{x}) = \lambda(w)\, C(w, \underset{\sim}{x}) \sum_{w_i} \frac{\mathcal{P}\, 1}{w - w_i}$$

$$A_{\parallel}(w) = C_{\parallel}(w) \sum_{w_i} \frac{\mathcal{P}\, 1}{w - w_i} \qquad (38)$$

Inserting this <u>Ansatz</u>* in (36) and (37) we find, respectively,

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* *Footnote on next page*

$$\hat{f}(v, v_\perp) = C(v, v_\perp) \sum_{w_i} \frac{\lambda(v)}{v - w_i} + \pi^2 N(v, v_\perp) \sum_{w_i} C_{\parallel}(w_i)\, \delta(v - w_i)$$

$$+ \sum_{w_i} b_{w_i} N(v, v_\perp)\, \delta(v - w_i) + \sum_{w_i} a_{w_i} \frac{P\, N(v, v_\perp)}{w_i - v} \tag{36a}$$

$$+ N(v, v_\perp) \sum_{w_i} \frac{P\, 1}{w_i - v} \left[P \int \frac{dw\, C_{\parallel}(w)}{w - w_i} - P \int \frac{dw\, C_{\parallel}(w)}{w - v} \right]$$

and

$$\phi(v) - \sum_{w_i} \frac{\eta(v)}{w_i - v} P \int \frac{dw\, C_{\parallel}(w)}{w - v} \equiv \hat{\phi}(v) =$$

$$= \sum_{w_i} \frac{\eta(v)}{w_i - v} \left(P \int \frac{dw\, C_{\parallel}(w)}{w - v} + \frac{\lambda(v)}{\eta(v)} C_{\parallel}(v) \right) \tag{37a}$$

In deriving (36a) we have used (essentially) the Poincaré-Bertrand formula. We see from (36a), according to our previous arguments, that if a solution of (34) or (36) of the type (38) exists, then we must choose

$$b_{w_i} = -\pi^2 C_{\parallel}(w_i) \tag{39}$$

Our task then reduces to finding a self-consistent solution $C_{\parallel}(w)$ of (37a) which is regular over the real line.

Our procedure will be first to solve (37a) implicitly, obtaining $C_{\parallel}(w)$ in terms of $\hat{\phi}(v)$, which itself

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*The reader will note that we could also look for solutions for $\lambda(w) A(w, \gamma)$ including delta-functions at $w = w_i$. The above proof shows that this additional complication is unneccesary.

contains $C_{\parallel}(w)$. We shall then show that a particular choice of the coefficients a_{w_i} is required to render the solution self-consistent and explicit.

We define the Hilbert transforms for complex z

$$N(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{C_{\parallel}(w)\,dw}{w-z}$$

$$Q(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\eta(v)\,dv}{v-z}$$

$$M(z) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\hat{\phi}(v)\,dv}{v-z} \tag{40}$$

$$\hat{Q}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{R(v)\,dv}{v-z} \quad ; \quad R(v) \equiv \sum_{w_i}\frac{\eta(v)}{v-w_i}$$

If a solution of the type we are seeking exists, we must be able to find a function $N(z)$ which is analytic in the cut z-plane cut along the real axis.[3,4,7] Note that $R(v) \equiv \sum_{w_i}\eta(v)(v-w_i)^{-1}$ is regular over the entire real line (or else satisfies a Hölder condition), so Q(z), M(z) and $\hat{Q}(z)$ are analytic in the cut plane.* Now

$$\hat{Q}(z) = \frac{1}{2\pi i}\sum_{w_i}\int\frac{dv\,\eta(v)}{(v-w_i)(v-z)} = \frac{1}{2\pi i}\sum_{w_i}\frac{1}{z-w_i}\left[\int\frac{dv\,\eta(v)}{w_i-v}+\int\frac{dv\,\eta(v)}{v-z}\right]$$

$$= \frac{1}{2\pi i}\left(1+2\pi i\,Q(z)\right)\sum_{w_i}\frac{1}{z-w_i}$$

---

* *This statement is strictly true for $Q(z)$ and $\hat{Q}(z)$. For $M(z)$, which contains $C_{\parallel}(w)$ in its definition, it is true provided we can find a well-defined solution $C_{\parallel}(w)$ by this method.*

where we have used the second of equations (15).

If $Q^{\pm}(w)$ are the limits of $Q(z)$ as z approaches the real axis from above and below respectively, then[3,4] using (10) and the properties of Hilbert transforms, $1+2\pi i\, Q^{\pm}(w) = \lambda(w) \pm \pi i\, \eta(w)$. Thus, both $1+2\pi i\, Q^{\pm}(w)$ have simple zeros at each lc point w_i. We may conclude that the corresponding limits $\hat{Q}^{\pm}(w)$ of $\hat{Q}(z)$ are simply

$$\hat{Q}^{\pm}(w) = \frac{1}{2\pi i}\left(1+2\pi i\, Q^{\pm}(w)\right)\sum_{w_i}\frac{1}{w-w_i}$$

since each of these functions is regular over the entire real line. Moreover, each is non-zero on the real line for $|w| < \infty$ (by our assumption of simple roots of (15)).

Using the properties of Hilbert transforms, with $N^{\pm}(w)$ and $M^{\pm}(w)$ the limits of the corresponding functions (40) as z approaches the real axis from above and below, we find that (37a) and (10) imply

$$M^{+}(w) - M^{-}(w) = N^{+}(w)\cdot 2\pi i\, \hat{Q}^{+}(w) - N^{-}(w)\cdot 2\pi i\, \hat{Q}^{-}(w) \qquad (41)$$

We observe that the complex zeros of $Q(z)$ coincide with the complex zeros (if any) of $1+2\pi i\, Q(z)$. However, any such zeros of $1+2\pi i\, Q(z)$ are just the Class 2 eigenvalues w_j (equation (18)). Therefore, upon utilizing the orthogonality conditions (23), one finds[4] that conditions (35) are precisely what is required to guarantee that $M(z)$

vanish at the complex zeros of $\hat{Q}(z)$.

Hence

$$N(z) = \frac{M(z)}{1 + 2\pi i\,\hat{Q}(z)} \tag{42}$$

is analytic in the complex plane cut along the real axis and satisfies (41); the (implicit) solution we seek is therefore

$$C_{\parallel}(w) = \frac{M^{+}(w)}{2\pi i\,\hat{Q}^{+}(w)} - \frac{M^{-}(w)}{2\pi i\,\hat{Q}^{-}(w)} = \tag{43}$$

$$= \frac{1}{R(w)}\,\frac{\eta^2(w)}{\lambda^2(w) + \pi^2\eta^2(w)}\left(P\!\!\int_{-\infty}^{\infty}\frac{f_{\parallel}(v)\,dv}{w-v} + \frac{\lambda(w)}{\eta(w)}f_{\parallel}(w)\right)$$

which is indeed regular over the real line. In the last equality of (43) we may write either $\hat{\phi}(v)$ or $f_{\parallel}(v)$ in view of (31) and the orthogonality of the Class 2 modes with the 1a adjoints. Thus, the solution $C_{\parallel}(w)$ appears to be explicit. However, again from the properties of the Hilbert transforms

$$P\!\!\int_{-\infty}^{\infty}\frac{dw\,C_{\parallel}(w)}{w-v} = \pi i\left[N^{+}(w) + N^{-}(w)\right] = \frac{\hat{\phi}(v)}{R(v)} - \frac{\lambda(v)}{\eta(v)}C_{\parallel}(v)$$

$$= \frac{1}{\sum\limits_{w_i}\frac{\eta(v)}{v-w_i}}\left(f_{\parallel}(v) - \sum_{w_j} a_{w_j}\frac{\eta(v)}{w_j - v} - \sum_{w_i}\left(a_{w_i} + P\!\!\int_{-\infty}^{\infty}\frac{dw\,C_{\parallel}(w)}{w-w_i}\right)\frac{\eta(v)}{w_i - v}\right) \tag{42a}$$

$$- \frac{\lambda(v)}{\eta(v)}\,C_{\parallel}(v)$$

The latter results checks that (42) indeed provides the (implicit) solution of (37a). However, putting $v = w_\ell$ in the above, where w_ℓ is any one of the points w_i, we find the consistency condition

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{dw\, C_{\parallel}(w)}{w - w_\ell} = \frac{f_{\parallel}(w_\ell)}{\eta'(w_\ell)} + \left(a_{w_\ell} + \mathcal{P}\int_{-\infty}^{\infty} \frac{dw\, C_{\parallel}(w)}{w - w_\ell}\right) - \frac{\lambda'(w_\ell)}{\eta'(w_\ell)}\, C_{\parallel}(w_\ell)$$

which shows that we must choose for each lc point

$$a_{w_i} = \frac{\lambda'(w_i)}{\eta'(w_i)}\, C_{\parallel}(w_i) - \frac{f_{\parallel}(w_i)}{\eta'(w_i)} \tag{44}$$

This yields finally, after some rearrangement

$$a_{w_i} = \frac{\lambda'(w_i)}{\lambda'^2(w_i) + \pi^2 \eta'^2(w_i)} \left(\mathcal{P}\int_{-\infty}^{\infty} \frac{dv\, f_{\parallel}(v)}{w_i - v} - \pi^2 \frac{\eta'(w_i)}{\lambda'(w_i)}\, f_{\parallel}(w_i)\right) \tag{45}$$

Similarly, from (38)

$$b_{w_i} = \frac{-\pi^2 \eta'(w_i)}{\lambda'^2(w_i) + \pi^2 \eta'^2(w_i)} \left(\mathcal{P}\int_{-\infty}^{\infty} \frac{dv\, f_{\parallel}(v)}{w_i - v} + \frac{\lambda'(w_i)}{\eta'(w_i)}\, f_{\parallel}(w_i)\right) \tag{46}$$

Finally, inserting (43) in (38) we find

$$A_{\parallel}(w) = \frac{\mathcal{P}\,\eta(w)}{\lambda^2(w)+\pi^2\eta^2(w)}\left(\mathcal{P}\int_{-\infty}^{\infty}\frac{f_{\parallel}(v)\,dv}{w-v}+\frac{\lambda(v)}{\eta(w)}\,f_{\parallel}(w)\right) \qquad (47)$$

This is Case's result except for the appearance of the additional principal value sign; we see that if lc points exist $A_{\parallel}(w)$ is in fact to be interpreted as a <u>distribution</u> and in that sense the integrals in (34), (36) and (37) are perfectly well-defined. In addition, our derivation of this result has given the further information we require in order to make (34) meaningful, namely the values of the a_{w_i} and b_{w_i}. Moreover, we see explicitly that $A_{\parallel}(w)$ is well-defined at points $\lambda(w) = 0$, $\eta(w) \neq 0$, and such points cause no difficulties. Thus $\lambda(v)\,A(v, \underset{\sim}{v}_{\perp})$ is determined by (36) (we do not need $A(v, \underset{\sim}{v}_{\perp})$ itself) and the eigenmodes are complete.

<u>V) The Expansion Coefficients and the Landau Initial Value Problem</u>

Having demonstrated the completeness of the eigenmodes, we wish now to give a few useful explicit formulas for $\lambda(w)A(w, \underset{\sim}{x}) \equiv \mathcal{B}(w, \underset{\sim}{x})$ appearing in (36). The coefficients for the various discrete modes are given by (35), (45) and (46). We shall then use our results to show the correspondence between the Landau (three-dimensional) solution of the Vlasov initial value problem and the solution obtained by superposition of eigenmodes.

When 1c points (i.e., marginally stable discrete modes) do not exist for the plasma, it is convenient to use the orthogonality property (26). In such cases we obtain from (34) with (26) for the 1a modes

$$\int d^3 v\, f(v, v_\perp)\, \tilde{\mathcal{E}}^*(v, v_\perp | w, \gamma) = \pi^2 \eta(w)\, A_\parallel(w) + \frac{\lambda^2(w)\, A(w, \gamma)}{N(w, \gamma)}$$

or, using (25) and Case's formula for $A_\parallel(w)$

$$B(w,\gamma) \equiv \lambda(w)\, A(w,\gamma) = f(w,\gamma) - \frac{N(w,\gamma)}{\lambda^2(w)+\pi^2\eta^2(w)} \left\{ \pi^2 \eta(w)\, f_\parallel(w) - \lambda(w)\, P\int_{-\infty}^{\infty} \frac{dv\, f_\parallel(v)}{w-v} \right\} \tag{48}$$

When 1c points occur, however, it is more straightforward to use the results of the previous section directly in (36a). We then find, from (47) and (42a)

$$B(w,\gamma) = f(w,\gamma) - N(w,\gamma)\, \frac{P\,\eta(w)}{\lambda^2(w)+\pi^2\eta^2(w)} \left\{ \pi^2 f_\parallel(w) - \frac{\lambda(w)}{\eta(w)}\, P\int_{-\infty}^{\infty} \frac{dv\, f_\parallel(v)}{w-v} \right\} \tag{49}$$

the only change in $B(w,\gamma)$ being the prescription of taking the principal value of the last term. In this form we see immediately that this expression also holds for the Class 1b modes ($N(w,\gamma) = 0$, $\lambda(w) \neq 0$, compare (29)). Note also from (47) that if $\eta(w) = 0$, $\lambda(w) \neq 0$,

$$A_\parallel(w) = \frac{f_\parallel(w)}{\lambda(w)} \tag{50}$$

and $\mathcal{B}(w, \underset{\sim}{\gamma})$ is perfectly well-defined. This justifies our inclusion of such modes in Class 1a, whereas in the one-dimensional treatment they had to be treated separately.[4]

Equation (49) can also be written in the form*

$$\mathcal{B}(w,\underset{\sim}{\gamma}) = f(w,\underset{\sim}{\gamma}) + \frac{\mathcal{P} N(w,\underset{\sim}{\gamma})}{\eta(w)}\left(\lambda(w) A_{\parallel}(w) - f_{\parallel}(w)\right)$$

$$= f(w,\underset{\sim}{\gamma}) - \frac{\mathcal{P} N(w,\underset{\sim}{\gamma})}{\eta(w)}\left[\eta(w)\int dv\, A_{\parallel}(v) \frac{\mathcal{P}\,1}{v-w}\right] - \sum_{w_j} a_{w_j} \frac{N(w,\underset{\sim}{\gamma})}{w_j - w} - \sum_{w_i} a_{w_i} \frac{\mathcal{P} N(w,\underset{\sim}{\gamma})}{w_i - w} \tag{51}$$

where we have used (47) and (37). In this form it is easy to check that $\int d^2\underset{\sim}{\gamma}\, \mathcal{B}(w,\underset{\sim}{\gamma}) = \lambda(w) A_{\parallel}(w)$ as required. (It is fortunate that we were able to show in the previous Section that we need only $\mathcal{B}(w,\underset{\sim}{\gamma})$ and $A_{\parallel}(w)$ but <u>not</u> $A(w,\underset{\sim}{\gamma})$.

The normal modes we have derived are the eigenmodes of the linearized (three-dimensional) Vlasov operator

$$\mathcal{L}(\underset{\sim}{k},\underset{\sim}{v}) \equiv i\underset{\sim}{k}\cdot\underset{\sim}{v} - \frac{\omega_p^2}{k^2}\, i\underset{\sim}{k}\cdot\frac{\partial f_0(v,\underset{\sim}{v}_\perp)}{\partial \underset{\sim}{v}}\int d^3\underset{\sim}{v}' \tag{52}$$

such that

$$\mathcal{L}(\underset{\sim}{k},\underset{\sim}{v})\, \mathcal{E}(\underset{\sim}{v}/w,\underset{\sim}{\gamma}) = ikw\, \mathcal{E}(\underset{\sim}{v}/w,\underset{\sim}{\gamma}) \tag{53}$$

---

*The results of the previous Section show that

$$N(v,\underset{\sim}{v}_\perp)\int dw\, A_{\parallel}(w)\frac{\mathcal{P}\,1}{w-v} = \frac{\mathcal{P}N(v,\underset{\sim}{v}_\perp)}{\eta(v)}\left[\eta(v)\int dw\, A_{\parallel}(w)\frac{\mathcal{P}\,1}{w-v}\right] + \sum_{w_i}\pi^2 C_{\parallel}(w_i) N(v,\underset{\sim}{v}_\perp)\,\delta(w_i - v)$$

so that (51) and (36) agree (using (39)).

The initial value problem for small disturbances of a spatially homogeneous plasma, formulated in terms of the Laplace and Fourier transform of the linearized Vlasov equation takes the form[1] ($\mathcal{R}e\, p > 0$)

$$(p+\mathcal{L}(\underset{\sim}{k},\underset{\sim}{v}))f(p,\underset{\sim}{k},\underset{\sim}{v}) = f(\underset{\sim}{k},\underset{\sim}{v},t=0) \equiv f(\underset{\sim}{k},\underset{\sim}{v},0) \tag{54}$$

We expand both $f(\underset{\sim}{k},\underset{\sim}{v},0)$ and $f(p,\underset{\sim}{k},\underset{\sim}{v})$ according to the prescription (34) -- using an obvious shorthand for the discrete modes:

$$\begin{aligned} f(\underset{\sim}{k},\underset{\sim}{v},0) = \sum_{w_j} a_{w_j}\mathcal{E}_{w_j} &+ \sum_{w_i}\left(a_{w_i}\mathcal{E}^{(1)}_{w_i} + b_{w_i}\mathcal{E}^{(2)}_{w_i}\right) \\ &+ \int_{-\infty}^{\infty} dw \int d^2\underset{\sim}{\gamma}\, \frac{B(w,\underset{\sim}{\gamma})}{\lambda(w)}\,\mathcal{E}(v,\underset{\sim}{v}_\perp|w,\underset{\sim}{\gamma}) \end{aligned} \tag{55}$$

$$\begin{aligned} f(p,\underset{\sim}{k},\underset{\sim}{v}) = \sum_{w_j} c_{w_j}(p)\mathcal{E}_{w_j} &+ \sum_{w_i}\left(c_{w_i}(p)\mathcal{E}^{(1)}_{w_i} + d_{w_i}(p)\mathcal{E}^{(2)}_{w_i}\right) \\ &+ \int_{-\infty}^{\infty} dw \int d^2\underset{\sim}{\gamma}\, \frac{D(p,w,\underset{\sim}{\gamma})}{\lambda(w)}\,\mathcal{E}(v,\underset{\sim}{v}_\perp|w,\underset{\sim}{\gamma}) \end{aligned} \tag{56}$$

The coefficients a_{w_j}, a_{w_i}, b_{w_i}, and $B(w,\underset{\sim}{\gamma})$ are given by (35), (45), (46), and (49), respectively, in terms of the specified initial disturbance $f(\underset{\sim}{k},\underset{\sim}{v},0)$ which also determines their $\underset{\sim}{k}$-dependence. We use (54) and (52) to determine the coefficients in (56). Thus, (54) reads

$$\begin{aligned} &\sum_{w_j}(p+ikw_j)c_{w_j}\mathcal{E}_{w_j} + \sum_{w_i}(p+ikw_i)\left(c_{w_i}\mathcal{E}^{(1)}_{w_i} + d_{w_i}\mathcal{E}^{(2)}_{w_i}\right) + \int_{-\infty}^{\infty} dw\int d^2\underset{\sim}{\gamma}\,\frac{p+ikw}{\lambda(w)}D(p,w,\underset{\sim}{\gamma})\,\mathcal{E}(\underset{\sim}{v}|w,\underset{\sim}{\gamma}) \\ &= \sum_{w_j} a_{w_j}\mathcal{E}_{w_j} + \sum_{w_i}\left(a_{w_i}\mathcal{E}^{(1)}_{w_i} + b_{w_i}\mathcal{E}^{(2)}_{w_i}\right) + \int_{-\infty}^{\infty} dw\int d^2\underset{\sim}{\gamma}\,\frac{B(w,\underset{\sim}{\gamma})}{\lambda(w)}\,\mathcal{E}(\underset{\sim}{v}|w,\underset{\sim}{\gamma}) \end{aligned} \tag{57}$$

We find, therefore, -- either from the orthogonality properties (23), or the uniqueness of the solution of (54) --

$$c_{w_j}(p) = \frac{a_{w_j}}{p+ikw_j} \quad ; \qquad c_{w_i}(p) = \frac{a_{w_i}}{p+ikw_i} \tag{58}$$

$$d_{w_i}(p) = \frac{b_{w_i}}{p+ikw_i} \tag{59}$$

and

$$D(p, w, \underline{x}) = \frac{B(w, \underline{x})}{p+ikw} \tag{60}$$

Thus,

$$\begin{aligned} f(p, \underline{k}, \underline{v}) &= \sum_{w_j} \frac{a_{w_j}}{p+ikw_j} \mathcal{E}_{w_j} + \sum_{w_i} \frac{a_{w_i}\mathcal{E}^{(1)}_{w_i} + b_{w_i}\mathcal{E}^{(2)}_{w_i}}{p+ikw_i} \\ &+ \frac{B(v, \underline{v}_\perp)}{p+ikv} + N(v, \underline{v}_\perp)\int_{-\infty}^{\infty} dw\, A_{\parallel}(w) \frac{1}{p+ikw} \cdot \frac{P\,1}{w-v} \end{aligned} \tag{61}$$

Adding and subtracting (55) divided by ($p+ikv$) -- this is equivalent to using (51) -- we find

$$\begin{aligned} f(p, \underline{k}, \underline{v}) = {} & \frac{f(\underline{k}, \underline{v}, 0)}{p+ikv} + N(v, \underline{v}_\perp)\left[\int_{-\infty}^{\infty} dw \frac{A_{\parallel}(w)}{p+ikw} \cdot \frac{P\,1}{w-v} - \int_{-\infty}^{\infty} dw \frac{A_{\parallel}(w)}{p+ikv} \cdot \frac{P\,1}{w-v}\right] \\ & + \sum_{w_j} a_{w_j} \mathcal{E}_{w_j}\left[\frac{1}{p+ikw_j} - \frac{1}{p+ikv}\right] + \sum_{w_i} a_{w_i} \mathcal{E}^{(1)}_{w_i}\left[\frac{1}{p+ikw_i} - \frac{1}{p+ikv}\right] \\ = {} & \frac{f(\underline{k}, \underline{v}, 0)}{p+ikv} - \frac{ikN(v, \underline{v}_\perp)}{p+ikv}\left\{\int_{-\infty}^{\infty} \frac{dw\, A_{\parallel}(w)}{p+ikw} + \sum_{w_j} \frac{a_{w_j}}{p+ikw_j} + \sum_{w_i} \frac{a_{w_i}}{p+ikw_i}\right\} \end{aligned} \tag{62}$$

From (61) and (3), with the help of the Poincare-Bertrand formula, we find

$$n(p,\underset{\sim}{k}) \equiv \int d^3\underset{\sim}{v}\, f(p,\underset{\sim}{k},\underset{\sim}{v}) = \sum_{w_j} \frac{a_{w_j}}{p+ikw_j} + \sum_{w_i} \frac{a_{w_i}}{p+ikw_i} + \int_{-\infty}^{\infty} \frac{dw\, A_{11}(w)}{p+ikw} \qquad (63)$$

To show that this result leads to the correct time dependence[1] for the density (or electric field) fluctuations, one need only complete the proof given by Case[4] based on two-sided transforms by explicitly computing the a_{w_i} arising from the partial residues at the imaginary poles (1c points) mentioned by Case. One finds that the contributions at such points are exactly those given by equations (45). Thus, (62) is precisely the correct Laplace transform of the three-dimensional solution of the Vlasov initial value problem as given by Landau.

In the next Section we shall give an example in which the use of the three-dimensional eigenmodes of the linearized Vlasov operator appears to be quite powerful.

VI) A Derivation of the Balescu-Lenard Equation

We consider a spatially uniform, stable electronic plasma without magnetic field far from thermodynamic equilibrium. The single-particle distribution function evolves on the slow ("collisional") time scale ϵt according to the relation [19,11,12,17]

$$\frac{\partial f_0}{\partial(\epsilon t)} = -\frac{\partial}{\partial \underset{\sim}{v}_1}\cdot\int d^3\underset{\sim}{k}\, i\underset{\sim}{k}\,\frac{\omega_p^2}{k^2}\int d^3\underset{\sim}{v}_2\left\{\lim_{t\to\infty} g_{12}(\underset{\sim}{v}_1,\underset{\sim}{v}_2,\underset{\sim}{k},t|\epsilon t)\right\} \tag{64}$$

where $g_{12}(\underset{\sim}{v}_1,\underset{\sim}{v}_2,\underset{\sim}{k},t|\epsilon t)$ is the Fourier transform of the two-particle correlation function $g_{12}(\underset{\sim}{v}_1,\underset{\sim}{v}_2,\underset{\sim}{x}_1-\underset{\sim}{x}_2,t|\epsilon t)$, and ϵ is the "plasma discreteness" parameter, $\epsilon^{-1}\equiv n_0 r_D^3$ with r_D the Debye length, $r_D^2=\varepsilon_0 KT/n_0e^2$.

The fast-time behavior of the two-particle correlation function is determined to first order in ϵ by an equation made up of two linearized Vlasov operators as defined by (51) [20, 17]:

$$\frac{\partial g_{12}}{\partial t}+\left[\mathcal{L}(\underset{\sim}{k},\underset{\sim}{v}_1)+\mathcal{L}(-\underset{\sim}{k},\underset{\sim}{v}_2)\right]g_{12}=\frac{i\phi(k)}{m}\underset{\sim}{k}\cdot\left(\frac{\partial}{\partial\underset{\sim}{v}_1}-\frac{\partial}{\partial\underset{\sim}{v}_2}\right)f_0(\underset{\sim}{v}_1)f_0(\underset{\sim}{v}_2) \tag{65}$$

$$\equiv S_{12}(\underset{\sim}{k},\underset{\sim}{v}_1,\underset{\sim}{v}_2)$$

where $\phi(k)=\varepsilon_0^{-1}(2\pi)^{-3}e^2/k^2$ is the Fourier transform of the Coulomb interaction potential, and equation (65) defines $S_{12}(\underset{\sim}{k},\underset{\sim}{v}_1,\underset{\sim}{v}_2)$. Utilizing the fact that $\mathcal{L}(\underset{\sim}{k},\underset{\sim}{v}_1)$ and $\mathcal{L}(-\underset{\sim}{k},\underset{\sim}{v}_2)$ commute[20] and the hypothesis that $f_0(\underset{\sim}{v})$ does not change appreciably on the fast time scale (adiabatic hypothesis[19,20] or "multiple-time scales"[21]) one may formally write the solution to (65) in the form [21,17]

$$g_{12}(\underset{\sim}{v}_1,\underset{\sim}{v}_2,\underset{\sim}{k},t)=\iint_{L_1L_2}\frac{dp_1}{2\pi i}\frac{dp_2}{2\pi i}\frac{e^{(p_1+p_2)t}}{p_1+p_2}\left(p_2+\mathcal{L}(-\underset{\sim}{k},\underset{\sim}{v}_2)\right)^{-1}\left(p_1+\mathcal{L}(\underset{\sim}{k},\underset{\sim}{v}_1)\right)^{-1}S_{12} \tag{66}$$

where the Laplace contours L_1 and L_2 run as usual from $-i\infty+\sigma_{1,2}$ to $+i\infty+\sigma_{1,2}$ to the right of any singularities of the integrand in the p_1- and p_2- planes, respectively. The quantities in (66) are defined for $\mathrm{Re}\, p_1>0$, $\mathrm{Re}\, p_2>0$; they may be analytically extended to negative real values. In (66) we have neglected the initial value of $g_{12}(t=0)$ because one can prove readily for a stable plasma that the initial value will not contribute to

$$\lim_{t\to\infty}\int d^3 v_2\, g_{12}(k, v_1, v_2, t) \tag{67}$$

because of phase mixing, and this is the quantity actually required in (64).

We shall start from (66) and utilize the three-dimensional eigenmodes of the operators $\mathcal{L}(k, v_1)$ and $\mathcal{L}(-k, v_2)$. We put (for the stable case assumed in this Section, there are no discrete modes)

$$S_{12}(k, v_1, v_2) = \int_{-\infty}^{\infty} dw_1 \int d^2\gamma_1 \frac{B(w_1, \gamma_1 | v_2, k)}{\lambda(w_1)} \mathcal{E}_1(v_1, v_{1\perp} | w_1, \gamma_1) \tag{68}$$

and in turn

$$B(w_1, \gamma_1 | v_2, k) = \int_{-\infty}^{\infty} dw_2 \int d^2\gamma_2 \frac{B(w_1, \gamma_1, w_2, \gamma_2, k)}{\lambda(w_2)} \mathcal{E}_2(v_2, v_{2\perp} | w_2, \gamma_2) \tag{69}$$

so that finally

$$S_{12}(k, v_1, v_2) = \int \frac{dw_1\, dw_2\, d^2\gamma_1\, d^2\gamma_2}{\lambda(w_1)\lambda(w_2)} B(w_1, \gamma_1, w_2, \gamma_2, k)\, \mathcal{E}_1 \mathcal{E}_2 \tag{70}$$

Applying the same arguments as those used under equation (59), we find immediately

$$g_{12}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{k}, t) = \iint_{L_1 L_2} \frac{dp_1\, dp_2}{(2\pi i)^2} \frac{e^{(p_1+p_2)t}}{p_1+p_2} \int \frac{dw_1\, dw_2\, d^2\boldsymbol{\gamma}_1\, d^2\boldsymbol{\gamma}_2\, B(w_1, \boldsymbol{\gamma}_1, w_2, \boldsymbol{\gamma}_2, \mathbf{k})\, \mathcal{E}_1 \mathcal{E}_2}{\lambda(w_1)\lambda(w_2)(p_1+ikw_1)(p_2-ikw_2)} \tag{71}$$

where we have used the fact that $w(-\mathbf{k}) = -w(\mathbf{k})$ (see equations (52) and (53)). We may perform the p_2-integration over L_2 to find

$$g_{12}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{k}, t) = \int \frac{dw_1\, dw_2\, d^2\boldsymbol{\gamma}_1\, d^2\boldsymbol{\gamma}_2}{\lambda(w_1)\lambda(w_2)} \int \frac{dp_1}{2\pi i} \left[\frac{-B\mathcal{E}_1\mathcal{E}_2 + e^{(p_1+ikw_1)t} B\mathcal{E}_1\mathcal{E}_2}{(p_1+ikw_1)(p_1+ikw_2)} \right] \tag{72}$$

The first term in brackets in (72) vanishes upon integration over L_1 since we may close the L_1 contour to the right for that term, and there are no singularities in the right-half p_1-plane. The second term yields simply

$$\begin{aligned} g_{12}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{k}, t) &= \int \frac{dw_1\, dw_2\, d^2\boldsymbol{\gamma}_1\, d^2\boldsymbol{\gamma}_2}{\lambda(w_1)\lambda(w_2)} B\mathcal{E}_1\mathcal{E}_2 \frac{1-e^{-ik(w_1-w_2)t}}{ik(w_1-w_2)} \\ &= \int \frac{dw_1\, dw_2\, d^2\boldsymbol{\gamma}_1\, d^2\boldsymbol{\gamma}_2}{\lambda(w_1)\lambda(w_2)} B\mathcal{E}_1\mathcal{E}_2 \int_0^t e^{-ik(w_1-w_2)\tau}\, d\tau \end{aligned} \tag{73}$$

Thus, for large times

$$\lim_{t\to\infty} g_{12}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{k}, t) = \int \frac{dw_1\, dw_2\, d^2\boldsymbol{\gamma}_1\, d^2\boldsymbol{\gamma}_2}{\lambda(w_1)\lambda(w_2)} B\mathcal{E}_1\mathcal{E}_2 \left[\frac{P}{ik(w_1-w_2)} + \pi\delta(kw_1 - kw_2) \right] . \tag{74}$$

Now $S_{12}(\mathbf{k}, \mathbf{v}_1, \mathbf{v}_2)$ is pure imaginary and odd in $\mathbf{k}$; therefore, from (70) and the known properties of $\mathcal{E}_1$ and $\mathcal{E}_2$ (real, even in $\mathbf{k}$), B is also imaginary and odd in $\mathbf{k}$. Since the eigenvalues w are odd in $\mathbf{k}$, as already pointed out, we see that the first term in square brackets in (74) is real and even in $\mathbf{k}$; it therefore contributes nothing to the $\mathbf{k}$-integral in (64). We are left only with the second term, which is pure imaginary and odd in $\mathbf{k}$.

Moreover, we require only the integral over $\mathbf{v}_2$ of g_{12} , and $\int d^3\mathbf{v}_2\, \mathcal{E}_2 = 1$. Thus (64) with (74) becomes

$$\frac{\partial f_0(\mathbf{v}_1, \epsilon t)}{\partial(\epsilon t)} = -\pi\omega_p^2 \frac{\partial}{\partial \mathbf{v}_1}\cdot \int d^3\mathbf{k}\, \frac{i\mathbf{k}}{k^3} \int \frac{dw_1\, d^2\boldsymbol{\gamma}_1\, d^2\boldsymbol{\gamma}_2\, B(w_1, w_1, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)\, \mathcal{E}_1}{\lambda^2(w_1)} \tag{75}$$

We make immediate use of the fact that B in (75) is integrated over $\boldsymbol{\gamma}_2$. From the defining expansion (69) and from (47)

$$\int d^2\boldsymbol{\gamma}_2 \frac{B(w_1, w_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)}{\lambda(w_2)} = \frac{\eta(w_2)}{\lambda^2(w_2)+\pi^2\eta^2(w_2)} \int_{-\infty}^{\infty} dv_2\, \tilde{\mathcal{E}}^{1D}_{w_2}(v_2) \int d^2\mathbf{v}_{2\perp}\, B(w_1, \boldsymbol{\gamma}_1 | \mathbf{v}_2, \mathbf{k}) \tag{76}$$

where $\tilde{\mathcal{E}}^{1D}_{w_2}(v_2) = \mathcal{P}\, 1/(w_2 - v_2) + (\lambda(w_2)/\eta(w_2))\,\delta(w_2 - v_2)$. On the other hand, from the defining expansion (68) and from (51)

$$B(w_1, \boldsymbol{\gamma}_1 | \mathbf{v}_2, \mathbf{k}) = S_{12}(\mathbf{k}, w_1, \boldsymbol{\gamma}_1 | v_2, \mathbf{v}_{2\perp}) - \frac{N(w_1, \boldsymbol{\gamma}_1)}{\eta(w_1)} S_{12}^{\parallel(1)}(\mathbf{k}, w_1 | v_2, \mathbf{v}_{2\perp})$$

$$+ \frac{N(w_1, \boldsymbol{\gamma}_1)\, \lambda(w_1)}{\lambda^2(w_1) + \pi^2\eta^2(w_1)} \int_{-\infty}^{\infty} dv_1\, S_{12}^{\parallel(1)}(\mathbf{k}, v_1 | v_2, \mathbf{v}_{2\perp})\, \tilde{\mathcal{E}}^{1D}_{w_1}(v_1) \tag{77}$$

where we have introduced the notation

$$S_{12}^{\parallel(\alpha)} \equiv \int d^2 v_{\alpha\perp}\; S_{12}\left(\underset{\sim}{k}, v_1, \underset{\sim}{v}_{1\perp} \middle| v_2, \underset{\sim}{v}_{2\perp}\right)\;;\quad \alpha = 1,2 \tag{78}$$

i.e.

$$S_{12}^{\parallel(1)} = -ik^3 \frac{\phi(k)}{m}\left[\eta(v_1) f_0(\underset{\sim}{v}_2) - N(\underset{\sim}{v}_2) F_0(v_1)\right]\omega_p^{-2} \tag{78a}$$

$$S_{12}^{\parallel(2)} = -ik^3 \frac{\phi(k)}{m}\left[N(\underset{\sim}{v}_1) F_0(v_2) - \eta(v_2) f_0(\underset{\sim}{v}_1)\right]\omega_p^{-2} \tag{78b}$$

Introducing further

$$S_{12}^{\parallel(1,2)} \equiv \int d^2 v_{1\perp} \int d^2 v_{2\perp}\; S_{12}\left(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2\right)$$

$$= -ik^3 \frac{\phi(k)}{m}\left[\eta(v_1) F_0(v_2) - \eta(v_2) F_0(v_1)\right]\omega_p^{-2} \tag{79}$$

we have

$$\int d^2 v_{2\perp}\, B(w_1, \underset{\sim}{\gamma}_1 | v_2, \underset{\sim}{v}_{2\perp}, \underset{\sim}{k}) = S^{\parallel(2)}(\underset{\sim}{k}, w_1, \underset{\sim}{\gamma}_1 | v_2) - \frac{N(w_1, \underset{\sim}{\gamma}_1)}{\eta(w_1)} S^{\parallel(1,2)}(\underset{\sim}{k}, w_1 | v_2)$$

$$+ \frac{N(w_1, \underset{\sim}{\gamma}_1)\lambda(w_1)}{\lambda^2(w_1) + \pi^2\eta^2(w_1)} \int_{-\infty}^{\infty} dv_1\; S_{12}^{\parallel(1,2)}(\underset{\sim}{k}, v_1 | v_2)\, \tilde{\mathcal{E}}_{w_1}^{1D}(v_1) \tag{80}$$

Thus, setting $w_2 = w_1$ in (76), as required by (75), we find

$$\int d^2 \underset{\sim}{\gamma}_2 \frac{B(w_1, w_1, \underset{\sim}{\gamma}_1, \underset{\sim}{\gamma}_2)}{\lambda(w_1)} = \frac{\eta(w_1)}{\lambda^2(w_1) + \pi^2\eta^2(w_1)}\left[\frac{N(w_1, \underset{\sim}{\gamma}_1)\lambda(w_1)}{\lambda^2(w_1) + \pi^2\eta^2(w_1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_1 dv_2\; S_{12}^{\parallel(1,2)}(\underset{\sim}{k}, v_1 | v_2)\, \tilde{\mathcal{E}}_{w_1}^{1D}(v_1)\, \tilde{\mathcal{E}}_{w_2}^{1D}(v_2)\right.$$

$$\left. + \int_{-\infty}^{\infty} dv_2\; \tilde{\mathcal{E}}_{w_1}^{1D}(v_2)\left\{S_{12}^{\parallel(2)}(\underset{\sim}{k}, w_1, \underset{\sim}{\gamma}_1 | v_2) - \frac{N(w_1, \underset{\sim}{\gamma}_1)}{\eta(w_1)} S_{12}^{\parallel(1,2)}(\underset{\sim}{k}, w_1 | v_2)\right\}\right] \tag{81}$$

Since $S_{12}^{\parallel(1,2)}(\underset{\sim}{k}, v_1 | v_2)$ is odd under interchange of v_1 and v_2 (see (79)) while the product of $\tilde{\mathcal{E}}_{w_1}^{1D}(v_2)\, \tilde{\mathcal{E}}_{w_1}^{1D}(v_1)$ is even, the double integral in (81) vanishes by symmetry. This corresponds to the fact that there is no kinetic equation to this order in a one-dimensional plasma; only the "three-dimensional effects" exemplified by the next two terms contribute.

Using the expressions (78) and (79) for $S_{12}^{\parallel(2)}$ and $S_{12}^{\parallel(1,2)}$ and the explicit form of the one-dimensional (integrated) adjoint modes, one may reduce the remaining terms in (81) after some minor algebra to

$$\int d^2\underset{\sim}{\gamma}_2 \frac{B(w_1, w_1, \underset{\sim}{\gamma}_1, \underset{\sim}{\gamma}_2)}{\lambda(w_1)} = \frac{S_{12}^{\parallel(2)}(\underset{\sim}{k}, w_1, \underset{\sim}{\gamma}_1 | w_1)}{\lambda^2(w_1) + \pi^2\eta^2(w_1)} \quad . \tag{82}$$

In deriving this relation, we have used (10). This expression is to be inserted in (75), divided by $\lambda(w_1)$ and integrated over $\underset{\sim}{\gamma}_1$ and w_1. We recall, however, that part of $\mathcal{E}_1$ (the principal value part) is independent of $\underset{\sim}{\gamma}_1$. Thus, for this part of $\mathcal{E}_1$ the integration over $\underset{\sim}{\gamma}_1$ effects only $S_{12}^{\parallel(2)}$ in (82) and leads to a term proportional to

$$S_{12}^{\parallel(1,2)}(\underset{\sim}{k}, w_1 | w_1) = 0$$

Thus, this term vanishes, and we are left only with the delta-function term. We may integrate this immediately, with the result

$$\frac{\partial f_0(\underset{\sim}{v}_1, \epsilon t)}{\partial(\epsilon t)} = -\pi\omega_p^2 \frac{\partial}{\partial \underset{\sim}{v}_1} \cdot \int d^3\underset{\sim}{k} \frac{i\underset{\sim}{k}}{k^3} \frac{S_{12}^{\parallel(2)}(\underset{\sim}{k}, v_1, \underset{\sim}{v}_{1\perp} | v_1)}{\lambda^2(v_1) + \pi^2\eta^2(v_1)} \tag{83}$$

If we recall that

$$\begin{aligned} \lambda(v) + \pi i \eta(v) &= 1 - \frac{\omega_p^2}{k^2} P\int_{-\infty}^{\infty} \frac{F_0'(u)\,du}{u - v} - \pi i \frac{\omega_p^2}{k^2} F_0'(v) \\ &\equiv D(-i\underset{\sim}{k}\cdot\underset{\sim}{v}, \underset{\sim}{k}) \end{aligned} \tag{84}$$

is exactly the "dielectric constant" of the Landau theory evaluated at $p = -i\underset{\sim}{k} \cdot \underset{\sim}{v}$, we recognize in (83) the familiar denominator obtained by Balescu[11] and Lenard[12] .

$$\lambda^2(v_1) + \pi^2 \eta^2(v_1) = \left| \overset{\circ}{D}(-i\underset{\sim}{k} \cdot \underset{\sim}{v}_1, \underset{\sim}{k}) \right|^2 \tag{85}$$

Equation (83) is one form of the Balescu-Lenard kinetic equation. Its derivation via the three-dimensional eigenmodes illustrates the power of the normal mode approach for some problems involving the linearized Vlasov operator.

VII) Normal Modes with Uniform Magnetic Field

We turn now to a derivation of a set of normal modes for an electronic plasma in a uniform applied magnetic field $\underset{\sim}{B}_0$. Bernstein[2] studied the general* initial value problem in a multi-species plasma using Laplace transform techniques and included electromagnetic (transverse) disturbances. We shall limit ourselves here to electrostatic disturbances. Harris[22] has given a very helpful discussion of the conditions under which this restriction is a realistic one.

We use the usual cylindrical coordinate system with the z-axis aligned with the applied magnetic field $\underset{\sim}{B}_0$ (Figure 1). As is well-known the equilibrium (undisturbed) distribution function is necessarily independent of the azimuthal angle about the $\underset{\sim}{B}_0$ -field:

$$\underset{\sim}{v} = (v_z, v_\perp, \phi) \quad ; \quad v_\perp = \sqrt{v_x^2 + v_y^2} \tag{86}$$

$$\frac{\partial f_0}{\partial \phi} = 0 \quad ; \quad f_0 = f_0(v_z, v_\perp) \tag{87}$$

We seek stationary solutions of the linearized Vlasov equation for this case in the form

$$f(\underset{\sim}{v}, t) = f_0(v_z, v_\perp) + h(v_z, v_\perp, \phi, \omega, \underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t} \tag{88}$$

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* Although non-relativistic. See Ref. 17, Chap. 10 for a discussion of the relativistic equations.

We make no restriction, for the moment, on the direction of $\mathbf{k}$ with respect to $\mathbf{B}_0$. We have then for the disturbance function h [17,22]

$$(-\omega + k_z v_z + k_\perp v_\perp \cos\theta)\, h - i\omega_c \frac{\partial h}{\partial \theta} = \frac{\omega_p^2}{k^2}\left(k_z \frac{\partial f_0}{\partial v_z} + k_\perp \cos\theta \, \frac{\partial f_0}{\partial v_\perp}\right) \int d^3\mathbf{v}\, h \tag{89}$$

where $\omega_c \equiv e|\mathbf{B}_0|/m$, $\mathbf{k} = (k_z, k_\perp, \alpha)$, $k_\perp = \sqrt{k_x^2 + k_y^2}$ and $\theta = \phi - \alpha$ is the angle between $\mathbf{v}_\perp$ and $\mathbf{k}_\perp$ ** (Figure 1). We see that the entire equation is invariant to rotation about the $\mathbf{B}_0$ - axis, i.e. $h(\theta)$ is independent of α. Equation (89) is restricted to electrostatic disturbances, since we have used $\mathbf{k} \times \mathbf{E} = 0$ and Poisson's equation instead of the full Maxwell equations.[22]

We normalize the stationary modes for $\mathbf{B}_0 \neq 0$ such that

$$\int d^3\mathbf{v}\, h_\nu(v_z, v_\perp, \theta | \omega, \mu, \mathbf{k}) = \mathcal{M}_\nu(\mu | \mathbf{k}) \tag{90}$$

and look for solutions of (89) periodic in θ with period 2π. (The meaning of the two real parameter ν and μ will be made clear shortly (see discussion above equ. (95).) They are related to the fact that we seek modes "complete" in three-dimensional velocity space and thus require (in analogy to our previous work) two parameters in addition to ω (or ω/k). We shall choose $\mathcal{M}_\nu(\mu)$ below in a convenient

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** In this Section and all that follows the symbol $\perp$ refers to "perpendicular to magnetic field", and is not to be confused with its use in earlier sections with reference to the direction of $\mathbf{k}$.

way). Defining

$$\beta \equiv \frac{k_z v_z - \omega}{\omega_c} \; ; \; \gamma \equiv \frac{k_\perp v_\perp}{\omega_c} \; ; \; \sigma(\theta) \equiv \frac{i\omega_p^2}{\omega_c k^2}\left(k_z \frac{\partial f_0}{\partial v_z} + k_\perp \cos\theta \frac{\partial f_0}{\partial v_\perp}\right) \quad (91)$$

we find, if $\mathcal{M}_\nu(\mu) \neq 0$*

$$h_\nu(v_z, v_\perp, \theta | \omega, \mu) = \mathcal{M}_\nu(\mu) e^{-i\beta\theta - i\gamma\sin\theta} \left\{ \frac{e^{-2\pi i\beta}}{1 - e^{-2\pi i\beta}} \int_0^{2\pi} d\theta' e^{i\beta\theta' + i\gamma\sin\theta'} \sigma(\theta') + \int_0^\theta d\theta' e^{i\beta\theta' + i\gamma\sin\theta'} \sigma(\theta') \right\} \quad (92)$$

This expression is readily seen to satisfy $h(\theta + 2\pi) = h(\theta)$, since $\sigma(\theta+2\pi) = \sigma(\theta)$, and to be the "solution" of (89) with (90). However, we are left with the problem of interpreting the singular denominator in (92), in analogy to the similar problem of interpreting $(\underset{\sim}{k} \cdot \underset{\sim}{v} - \omega)^{-1}$ in the case without magnetic field, and of guaranteeing that (90) is satisfied (see below).

To facilitate this interpretation, we first rewrite

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*We shall see below that it is again possible to find non-trivial solutions of (89) for which $\mathcal{M}_\nu(\mu) = 0$; i.e., modes whose integral over velocity space vanishes. Such modes do not contribute to the electric field fluctuations in our electrostatic approximation, although they may contribute to the disturbance distribution function. Such "zero field" modes are especially important for the special case $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$.

(92) using the Bessel identity[23]

$$e^{\pm i\gamma \sin\theta} = \sum_{n=-\infty}^{\infty} J_n(\gamma)\, e^{\pm in\theta} \qquad (93)$$

to enable us to evaluate the integrals. Performing the θ'-integrals, and combining terms, we arrive finally at the expression

$$h_\nu(v_z, v_\perp, \theta|\omega, \mu) = m_\nu(\mu)\, \frac{\omega_p^2}{k^2}\, \frac{e^{-i\gamma\sin\theta}}{\omega_c} \sum_{n=-\infty}^{\infty} J_n(\gamma)\, e^{in\theta} \left\{ \frac{k_z \frac{\partial f_0}{\partial v_z} + \frac{nk_\perp}{\gamma}\frac{\partial f_0}{\partial v_\perp}}{\beta + n} \right\} \qquad (94)$$

The singularities at $\beta = -n$, of course, are the same as those of the singular denominator $(1 - e^{-2\pi i\beta})$ in (92).

At this point we restrict our attention to disturbances for which $k_z \neq 0$ $(\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \neq 0)$. We shall disucss this limiting case subsequently. So long as $k_z \neq 0$, all characteristic frequencies are Doppler shifted by the particle motion along z; this is the physical reason for the continuum of characteristic frequencies we shall find. We now note, from the point of view of the theory of distributions, that taking the Cauchy principal value (with respect to integration over v_z) of each $(\beta + n)^{-1}$ in the sum in (94) provides us with a "particular solution" to the inhomogeneous equation (89), provided the normalization condition (90) holds. As before, we must then seek a "homogeneous solution", again within the theory of distributions, which provides us with the freedom to satisfy (90). At the same time, as we learned from our earlier work, we must require of our homogeneous solution that it provides the maximum amount of "completeness" in θ- and $v_\perp$- space. In analogy to our procedure for zero $\underset{\sim}{B}_0$ -field, we use two parameters: the discrete parameter ν (all real integer values) corresponding to the θ -variation, and the continuous (positive, real) parameter μ corresponding to the $v_\perp$ -variation. We are then led from the above considerations to the choice

$$h_\nu^{homo}(\mu) = e^{-i\gamma\sin\theta} e^{i\nu\theta} \lambda_\nu''(v_z,\mu)\, \delta(v_\perp - \mu)\, \delta(\beta+\nu)\, m_\nu(\mu) \tag{95}$$

The reader will readily verify that the "generalized function" $h_\nu^{homo}(\mu)$ is indeed a homogeneous solution of (89) for any ν, i.e. it satisfies left-hand-side (89) = 0. Moreover, the dependence on $v_\perp$ is arbitrary. Our final form for the continuum modes will thus involve for each ν the sum of (95) and (94) with each $(\beta + n)^{-1}$ in the sum over n in (94) replaced by its Cauchy principal value. (Our restriction to $k_z \neq 0$ must be kept in mind; the solutions (95) do not apply for $k_z = 0$. As we shall see, all modes for $k_z = 0$ require discrete frequencies ω .)

Defining

$$W \equiv \frac{\omega}{k} \quad ; \quad W_c \equiv \frac{\omega_c}{k}$$

$$k = \sqrt{k_z^2 + k_\perp^2} \tag{96}$$

and

$$N_z(v_z, v_\perp) \equiv -\frac{\omega_p^2}{k^2}\frac{\partial f_0}{\partial v_z}(v_z, v_\perp)$$

$$N_\perp(v_z, v_\perp) \equiv -\frac{\omega_p^2}{k^2}\frac{\partial f_0}{\partial v_\perp}(v_z, v_\perp) \qquad (97)$$

we may finally write our continuum modes in the form ($k_z \neq 0$)

$$h_\nu(v_z, v_\perp, \theta | w, \mu) = 2\pi\mu J_\nu(\gamma_\mu) e^{-i\gamma\sin\theta}\left\{\sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta}\frac{\mathcal{P}\left[\frac{k_z}{k}N_z + \frac{nw_c}{v_\perp}N_\perp\right]}{w - w_c n - k_z v_z/k}\right.$$

$$\left. + \frac{\lambda_\nu(v_z)e^{i\nu\theta}}{2\pi\mu J_\nu(\gamma_\mu)}\delta(v_\perp - \mu)\delta(w - w_c\nu - k_z v_z/k)\right\} \qquad (98)$$

where $\gamma_\mu \equiv k_\perp\mu/\omega_c$, and $\lambda_\nu''(v_z, \mu)$ has been replaced by $(k/\omega_c)\lambda_\nu(v_z)(2\pi\mu J_\nu(\gamma_\mu))^{-1}$ in anticipation of application of (90); the μ-dependence of this term is then explicit. In (98) we have chosen our normalization (90) in the form

$$\mathcal{M}_\nu(\mu) = 2\pi\mu J_\nu(\gamma_\mu) \qquad (90a)$$

so that the modes are well-defined at the zeros of $2\pi\mu J_\nu(\gamma_\mu)$ in μ-space. This is especially helpful in interpreting the limit $k_\perp = 0$ (disturbances propagating along $\underset{\sim}{B}_0$, see below). That $\lambda_\nu(v_z)$ defined by (98) is in fact independent of μ will be seen immediately below. Thus, we require*

$$\int_0^{2\pi} d\theta \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_z\, h_\nu(v_z, v_\perp, \theta | w, \mu) = 2\pi\mu J_\nu(\gamma_\mu) \qquad (90)$$

and again using the identity (93), we may carry out the θ -

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*We shall find it convenient throughout the present discussion to replace the integral over ϕ in $\int d^3\underset{\sim}{v}$ by the integral over $\theta = \phi - \alpha$. For the normal modes this is trivial since their angular dependence involves only θ . For a more general function, $f(\phi, \alpha)$ we transform to $\tilde{f}(\theta, \alpha)$ before integrating.

integrals explicitly to find, with the help of the delta-functions:

$$\frac{k}{|k_z|}\lambda_\nu\left(\frac{k}{k_z}(w-w_c\nu)\right) = 1 - \int_{-\infty}^{\infty} dv_z \sum_{n=-\infty}^{\infty} \frac{\mathcal{P}}{w-w_c n - k_z v_z/k} \int_0^\infty 2\pi v_\perp dv_\perp J_n^2(\gamma)\, \eta_n(v_z, v_\perp)$$

$$\eta_n(v_z, v_\perp) \equiv \frac{k_z}{k} N_z(v_z, v_\perp) + \frac{n w_c}{v_\perp} N_\perp(v_z, v_\perp) \tag{99}$$

We may immediately draw an important conclusion from (99). We see explicitly that $\lambda_\nu\left(\frac{k}{k_z}(w-w_c\nu)\right)$ is in fact also independent of ν, i.e., it is the <u>same function of w for any ν.</u> We therefore define

$$\hat{\lambda}(w) \equiv \frac{k}{|k_z|}\lambda_\nu\left(\frac{k}{k_z}(w-w_c\nu)\right); \quad \eta_n(v_z) \equiv \int_0^\infty 2\pi v_\perp dv_\perp J_n^2(\gamma)\, \eta_n(v_z, v_\perp) \tag{100}$$

and find that

$$\begin{aligned}\hat{\lambda}(w) &= 1 - \int_{-\infty}^{\infty} dv_z \sum_{n=-\infty}^{\infty} \eta_n(v_z) \frac{\mathcal{P}}{w-w_c n - k_z v_z/k} \\ &= 1 - \mathcal{P}\int_{-\infty}^{\infty} \frac{du \sum_{n=-\infty}^{\infty} \operatorname{sgn}(k_z)\frac{k}{k_z}\eta_n\left(\frac{k}{k_z}(u-w_c n)\right)}{w-u} \\ &= 1 - \mathcal{P}\int_{-\infty}^{\infty} \frac{du\, P(u)}{w-u}\end{aligned} \tag{101}$$

where

$$P(u) \equiv \sum_{n=-\infty}^{\infty} \frac{k}{k_z} \operatorname{sgn}(k_z)\, \eta_n\left(\frac{k}{k_z}(u-w_c n)\right) \tag{102}$$

$P(u)$ is a reasonably smooth function of u if $k_z \neq 0$.

These results will be very useful in our later discussion of completeness of the modes; they are also important in determining the classification of the modes. It is apparent that $P(u)$ plays a role for $\underset{\sim}{B}_0 \neq 0$ analogous to the role played by $\eta(v)$ in the case of zero magnetic field. This fact has been exploited elsewhere[13] to study the stability properties of a plasma in a uniform magnetic field.

Before proceeding it is useful to discuss a few special points where difficulties or ambiguities could arise.

<u>The zeros of $2\pi\mu J_\nu(\gamma_\mu)$, $(k_\perp \neq 0, k_z \neq 0)$</u>

The reader will note that at the zeros of $2\pi\mu J_\nu(\gamma_\mu)$ the normal modes are normalized <u>locally</u> to zero, and at these points only the "homogeneous solution" (second part of (98)) survives. Since $\lambda_\nu(v_z)$ is independent of μ it can be regarded as being determined from (99) by virtue of a <u>limiting process</u> as γ_μ approaches one of the zeros of $J_\nu(\gamma_\mu)$.

<u>The special case $k_\perp = 0$</u>

For disturbances propagating along $\underset{\sim}{B}_0$, $(k_\perp = 0)$, the sum in (98) reduces to a single term (n = 0) and $J_\nu(k_\perp\mu/\omega_c) \to 0$ for $\nu \neq 0$ while $J_0(k_\perp\mu/\omega_c) \to 1$. The reader will see, however, that "homogeneous solutions", normalized to zero, are preserved for $\nu \neq 0$. It is easy to check that these are indeed solutions of (89) for $k_\perp = 0$. Again, we regard the $\lambda_\nu(v_z)$ $(\nu \neq 0)$ as being determined by (99) (where now the sum contains only one term) by virtue of a limiting process.

Final justification of these remarks, however, will depend on the "completeness", i.e. usefulness, of the modes so chosen.

The special case $k_z = 0$ ($\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$).

For disturbances propagating perpendicular to $\underset{\sim}{B}_0$, ($k_z = 0$), we note that our particular use of generalized functions in v_z -space as a means of obtaining normalizable eigensolutions of (89) no longer applies, and, because of the lack of Doppler shifting, modes with a continuous spectrum of frequencies do not exist for this case. Instead, two distinct types of modes with discrete frequencies occur for this propagation direction. The first set of modes, all with non-vanishing normalization integral, arise for the discrete frequencies ω_ℓ (or phase speeds $\omega_\ell / k \equiv W_\ell$) such that for fixed $k_\perp$ (= k)

$$1 = \sum_{n=-\infty}^{\infty} \frac{n W_c}{W_\ell - n W_c} \int_0^\infty 2\pi\, dv_\perp \int_{-\infty}^{\infty} dv_z\, J_n^2\left(\frac{k v_\perp}{\omega_c}\right) N_\perp(v_z, v_\perp) \tag{103a}$$

(we note in passing that (103a) corresponds formally to putting $\hat{\lambda}(W_\ell) = 0$ with $k_z = 0$.) Apart from notation, and our neglect here of the ion motion, this is the dispersion relation studied by Bernstein[2] (he discussed this relation, however, only for a Maxwellian distribution f_0). If f_0 is Maxwellian, only real W_ℓ can satisfy (103a). Equation (103a) has in any case a denumerably infinite number of discrete solutions $W_\ell \neq W_c n$.

If we normalize these discrete modes for $k_z = 0$ such that $\int d^3\underset{\sim}{v}\, h_\nu(v_z, v_\perp, \theta | W_\ell, \mu) = 1$, they are independent of ν and μ and may be written explicitly:

$$h\left(v_z, v_\perp, \theta \middle| w_\ell, k_z=0\right) = e^{-i\frac{v_\perp}{w_c}\sin\theta} \sum_{n=-\infty}^{\infty} \frac{J_n\left(\frac{v_\perp}{w_c}\right) e^{in\theta} \frac{n w_c}{v_\perp} N_\perp(v_z, v_\perp)}{w_\ell - w_c n} \tag{104a}$$

It is easy to check that these modes are solutions of (89) for $k_z = 0$ and $\omega = k w_\ell$ and that their normalization to 1 reproduces (103a). Note that these modes contain no free parameters corresponding to the θ - and $v_\perp$-variations; they can scarcely be expected to be complete for the expansion of a function in three-dimensional velocity space. (See Section X).

This dilemma is resolved by the recognition of the existence of an independent set of modes h_m with discrete frequencies given by the cyclotron frequency or one of its harmonics

$$\omega_m = m\omega_c \;;\; |m| = 1, 2, 3, \ldots \tag{103b}$$

plus a "d.c." or zero-frequency mode h_0 . For $m \neq 0$ these modes are solutions of (89) for $k_z = 0$ having the property $\int d^3\underset{\sim}{v}\, h_m(v_z, v_\perp, \theta | \omega_m, \sigma, K) = 0$, i.e. they are solutions which do not contribute to the electric field or density fluctuations. They have the general form, for each integer $m \neq 0$

$$h_m\left(v_z, v_\perp, \theta \middle| \omega_m, \sigma, K\right) = e^{-i\frac{v_\perp}{w_c}\sin\theta} e^{im\theta} H_m\left(v_z, v_\perp \middle| \sigma, K\right) \tag{104b}$$

Their "zero-field" property requires for each $m \neq 0$, σ, and K (using (93)):

$$\int_0^\infty dv_\perp\, v_\perp \int_{-\infty}^{\infty} dv_z\, J_m\left(\frac{v_\perp}{w_c}\right) H_m\left(v_z, v_\perp \middle| \sigma, K\right) = 0$$

We shall discuss in Section X a possible special choice of the functions $H_{m1}(v_z, v_\perp | \sigma, K)$. We shall show there that continuous parameters σ and K can be chosen to correspond, respectively, to the v_z- and $v_\perp$-variations in velocity space (the θ-variation being associated with the parameter m). We remark here that such modes do not contribute to the dispersion relation of Bernstein since they do not contribute to the electric field fluctuations. One can readily verify, by studying the initial value problem of the linearized Vlasov equation in the electrostatic limit for $k_z = 0$, that modes of the general type (104b) are included in the solution for $f(\underset{\sim}{v}, t)$, but vanish upon integration over $\underset{\sim}{v}$. In contrast, the "d.c. component" (m = 0 mode) does contribute to the electric field perturbation and represents an interesting static solution of the linearized Vlasov equation for $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$ (Section X).

VIII) The Adjoint Modes ($\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \neq 0$) and Mode Classification

The adjoint modes $\tilde{h}$ satisfy the equation

$$\left(-w + \frac{k_z v_z}{k} + \frac{k_\perp v_\perp}{k} \cos\theta\right)\tilde{h} - i w_c \frac{\partial \tilde{h}}{\partial \theta} = -\int d^3\underset{\sim}{v} \left(\frac{k_z}{k} N_z + \frac{k_\perp}{k} N_\perp \cos\theta\right)\tilde{h} \tag{105}$$

Utilizing the same (standard) procedure as that described under equation (20) we may prove immediately from (105) and (89) that

$$(w'^* - w)\int d^3\underset{\sim}{v}\, h_\mu(v_z, v_\perp, \theta | w, \mu)\, \tilde{h}^*_{\mu'}(v_z, v_\perp, \theta | w', \mu') = 0 \tag{106}$$

Once again, the basic equations imply orthogonality in w-space but not necessarily with respect to μ or ν.

The main continuum (Class 1a)

In this class of modes we have ω real, $k_z \neq 0$, the eigenmodes are given explicitly by (98), and for each ν, μ and ω in this class

$$\frac{k_z}{k} N_z\left(\frac{k}{k_z}(\omega-\omega_c\nu), \mu\right) + \frac{\nu\omega_c}{\mu} N_\perp\left(\frac{k}{k_z}(\omega-\omega_c\nu), \mu\right)$$

$$\equiv \eta_\nu\left(\frac{k}{k_z}(\omega-\omega_c\nu), \mu\right) \neq 0 \tag{107}$$

For the modes so defined we normalize the adjoints such that

$$\int d^3v \left(\frac{k_z N_z}{k} + \frac{k_\perp \cos\theta}{k} N_\perp\right) \tilde{h}_\nu(v_z, v_\perp, \theta|\omega,\mu) = 2\pi\mu J_\nu(\gamma_\mu) \tag{108}$$

The Class 1a adjoint modes then take the form ($k_z \neq 0$)

$$\tilde{h}_\nu(v_z, v_\perp, \theta|\omega,\mu) = 2\pi\mu J_\nu(\gamma_\mu) e^{-i\gamma\sin\theta}\Bigg\{\sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta} \frac{P}{\omega-\omega_c n - k_z v_z/k}$$

$$+ \frac{\lambda_\nu(v_z) e^{i\nu\theta}}{2\pi\mu J_\nu(\gamma_\mu)\eta_\nu(v_z, v_\perp)} \delta(v_\perp-\mu)\delta(\omega-\omega_c\nu-k_z v_z/k)\Bigg\} \tag{109}$$

since with (108) this solves the adjoint equation (105), and application of (108) leads again to expression (99) for $\lambda_\nu(v_z)$. With the restriction (107), the adjoint modes given by (109) are well-defined within this class. They also hold in the limit $k_\perp \to 0$.

We may also determine explicitly the orthogonality properties; again, careful application of techniques related to the Poincaré-Bertrand formula yields, in analogy with (26):

$$\int d^3\underline{v}\, \tilde{h}^{*}_{\nu'}(v_z, v_\perp, \theta | w', \mu')\, h_\nu(v_z, v_\perp, \theta | w, \mu) = \delta(w-w') \left\{ \pi^2 P(w) + \frac{\frac{|k_z|}{k} \hat{\lambda}^2(w)\, \delta_{\nu\nu'}\, \delta(\mu-\mu')}{2\pi\mu J_\nu^2(\gamma\mu)\, \eta_\nu\left(\frac{k}{k_z}(w-w_c\nu), \mu\right)} \right\} 2\pi\mu J_\nu(\gamma\mu) \cdot 2\pi\mu' J_{\nu'}(\gamma\mu') \tag{110}$$

where $P(w)$ is defined by (102), $\hat{\lambda}(w)$ by (100), and $\eta_\nu(v_z, v_\perp)$ in (99). We see that with the restriction (107) the "norm" (110) is also well-defined for Class 1a, and holds in the limit $k_\perp \to 0$.

Class 1b.

In this class we have w real, $k_z \neq 0$, but $\eta_\nu\left(\frac{k}{k_z}(w-w_c\nu), \mu\right) = 0$. These modes are the analog of Class 1b for the zero-magnetic field case (Sections II and III). The normal modes themselves are still given by (98); however, for the adjoint modes we may now take

$$\tilde{h}_\nu(v_z, v_\perp, \theta | w, \mu) = e^{-i\gamma\sin\theta}\, e^{i\nu\theta}\, \delta(v_\perp - \mu)\, \delta\left(w - w_c\nu - \frac{k_z v_z}{k}\right) \tag{111}$$

since this expression causes the left hand side of (105) to vanish, and

$$\int d^3\underline{v} \left(\frac{k_z N_z}{k} + \frac{k_\perp\cos\theta}{k} N_\perp\right) e^{-i\gamma\sin\theta}\, e^{i\nu\theta}\, \delta(v_\perp - \mu)\, \delta(w - w_c\nu - k_z v_z/k)$$

$$= 2\pi\mu J_\nu(\gamma\mu) \frac{k}{|k_z|}\, \eta_\nu\left(\frac{k}{k_z}(w-w_c\nu), \mu\right) = 0 \tag{112}$$

Thus, (111) indeed solves (105) for this class. Moreover,

$$\int d^3\underline{v}\, \tilde{h}^{*}_{\nu'}(v_z, v_\perp, \theta | w', \mu')\, h_\nu(v_z, v_\perp, \theta | w, \mu) = \frac{k}{|k_z|} \lambda_\nu\left(\frac{k}{k_z}(w - w_c \nu)\right) \delta_{\nu\nu'}\, \delta(\mu - \mu')\, \delta(w - w')$$

$$= \hat{\lambda}(w)\, \delta_{\nu\nu'}\, \delta(\mu - \mu')\, \delta(w - w') \qquad (113)$$

In using the continuum modes 1a and 1b, we are to omit the points of 1c.

<u>The discrete modes</u> (classes 1c and 2).

<u>Class 1c.</u> This class of modes is defined for the set of real points w_i such that <u>both</u> (see Equ. (101))

$$\hat{\lambda}(w_i) = 0 \qquad (114)$$

and

$$P(w_i) = 0 \qquad (115)$$

It is useful to note that this definition of the discrete modes <u>includes as special case those discrete modes (104a) of the case $k_z = 0$ for which w_ℓ is real.</u> That this is so can be seen as follows: as already noted the condition (114) is the same as (103a) if $k_z = 0$, and the solutions of (103a) are such that $w_\ell \neq w_c n$ for any n. But inspection of definition (102) shows that $P(w) \rightarrow 0$ for $k_z \rightarrow 0$ <u>except</u> where $w = w_c n$. (In fact, $P(w)$ approaches a sum of delta-functions in the limit $k_z \rightarrow 0$ with singular points at

$w = w_c n$). Hence, at the points w_i defined by $\hat{\lambda}(w_i) = 0$, $P(w_i) = 0$ if $k_z \to 0$, and the (real) discrete modes (104a) are a simple limiting case of the general Class 1c modes.

In Class 1c we find two sets of solutions such that $\mathcal{M}_\nu(\mu) = 1$ (Class 1c,1) and $\mathcal{M}_\nu(\mu) = 0$ (Class 1c,2):

$$\text{Class 1c,1} \qquad h^{(1)}(v_z, v_\perp, \theta / w_i) = e^{-i\gamma \sin\theta} \sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta} \frac{\mathcal{P}\, \eta_n(v_z, v_\perp)}{w_i - w_c n - k_z v_z / k}$$

$$\text{Class 1c,2} \qquad h^{(2)}(v_z, v_\perp, \theta / w_i) = e^{-i\gamma \sin\theta} \sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta} \eta_n(v_z, v_\perp)\, \delta(w_i - w_c n - k_z v_z / k) \tag{116}$$

Note that (104a) is the limit of the 1c,1 modes as $k_z \to 0$ (the principal value sign $\mathcal{P}$ is then irrelevant). These discrete modes are <u>independent of the parameters ν and μ</u>. For the <u>adjoint modes</u>, we normalize according to

$$\int d^3\underline{v}\, \tilde{h}^{(1)}(v_z, v_\perp, \theta / w_i) \left(\frac{k_z}{k} N_z + \frac{k \cos\theta}{k} N_\perp \right) = 1$$

or

$$\int d^3\underline{v}\, \tilde{h}^{(2)}(v_z, v_\perp, \theta / w_i) \left(\frac{k_z}{k} N_z + \frac{k \cos\theta}{k} N_\perp \right) = 0 \tag{117}$$

and find again two sets of adjoint modes

$$\tilde{h}^{(1)}(v_z, v_\perp, \theta / w_i) = e^{-i\gamma \sin\theta} \sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta} \frac{\mathcal{P}}{w_i - w_c n - k_z v_z / k}$$

$$\tilde{h}^{(2)}(v_z, v_\perp, \theta / w_i) = e^{-i\gamma \sin\theta} \sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta} \delta(w_i - w_c n - k_z v_z / k) \tag{118}$$

Application of (117) to (118) or (90) with $\mathcal{M}_\nu(\mu) = 1$ or 0 to (116) leads simply to $\hat{\lambda}(w_i) = 0$, and $P(w_i) = 0$, which defines the Class 1c modes.

Class 2. (w discrete and complex). This class of modes is defined for the set of discrete complex points w_j, if they exist, which satisfy

$$1 = \int_{-\infty}^{\infty} dv_z \sum_{n=-\infty}^{\infty} \frac{1}{w_j - w_c n - k_z v_z/k} \int_0^{\infty} 2\pi v_\perp dv_\perp \, J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right) \eta_n(v_z, v_\perp)$$

$$= \int_{-\infty}^{\infty} dv_z \sum_{n=-\infty}^{\infty} \frac{\eta_n(v_z)}{w_j - w_c n - k_z v_z/k} = \int_{-\infty}^{\infty} \frac{du\, P(u)}{w_j - u} \tag{119}$$

(Note that (119) reduces to (103a) for $k_z = 0$ if (103a) has complex roots.) Normalizing these modes such that

$$\int d^3\underset{\sim}{v}\, h(v_z, v_\perp, \theta | w_j) = \int d^3\underset{\sim}{v}\, \hat{h}(v_z, v_\perp, \theta | w_j)\left(\frac{k_z}{k} N_z + \frac{k_\perp \cos\theta}{k} N_\perp\right) = 1 \tag{120}$$

we have explicitly

$$h(v_z, v_\perp, \theta | w_j) = e^{-i\gamma \sin\theta} \sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta} \frac{\eta_n(v_z, v_\perp)}{w_j - w_c n - k_z v_z/k} \tag{121}$$

and

$$\tilde{h}(v_z, v_\perp, \theta | w_j) = e^{-i\gamma \sin\theta} \sum_{n=-\infty}^{\infty} J_n(\gamma) e^{in\theta} \frac{1}{w_j - w_c n - k_z v_z/k} \tag{122}$$

Again, the normalization conditions simply reproduce the defining relation (119). All the discrete modes arising in the limit $k_z = 0$ are included in Classes 1c and 2.

For Class lc' and Class 2, we find*

$$\int d^3v\, \tilde{h}^*(v_z, v_\perp, \theta | w_j)\, h(v_z, v_\perp, \theta | w_i) = \delta_{w_j^* w_i} C_{w_i} \tag{123}$$

where

$$C_{w_i} = \mathcal{P}\int_{-\infty}^{\infty} du \frac{P(u)}{(w_i - u)^2}$$

For the lc' modes this integral exists, since $P(w_i) = 0$. For the Class 2 modes (w_i complex), of course, the principal value sign is not needed. (See also discussion in Section III relative to the lc modes for $B_0 = 0$). For simplicity we assume that only simple roots of (119) and (114) exist, which guarantees $C_{w_i} \neq 0$. In the limit $k_z \to 0$ (123) reduces to

$$C_{w_i}(k_z = 0) = \sum_{n=-\infty}^{\infty} \frac{\int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi\, dv_\perp\, J_n^2\left(\frac{v_\perp}{w_c}\right) n w_c\, H_\perp(v_z, v_\perp)}{(w_i - w_c n)^2} \tag{124}$$

for w_i both real and complex.

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*Some care must be exercised here in the case of the lc' modes. Upon forming the indicated product in (123) and integrating over θ and $v_\perp$, one finds that the only surviving contribution is the integral of a sum:

$$\int_{-\infty}^{\infty} dv_z \left(\sum_{n=-\infty}^{\infty} \eta_n(v_z) \frac{\mathcal{P}}{w_i - w_c n - k_z v_z/k} - \frac{\mathcal{P}}{w_i - w_c n - k_z v_z/k} \right)$$

For $k_z \neq 0$ the integrand has meaning only as a sum, i.e. it cannot be integrated term by term. However, use of the same transformation as in (101) shows that the integral of the sum is well-defined because $P(w_i) = 0$, and (123) results.

IX) Completeness of the Normal Modes ($\underset{\sim}{k} \cdot \underset{\sim}{B}_0 \neq 0$)

We have already noted the analogous role played by $P(w)$ in the construction of the normal modes with magnetic field to that of $\eta(w)$ in the Van Kampen-Case theory and its three-dimensional extension given in this paper. This analogy holds for all propagation directions except the limiting case $k_z = 0$ ($\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$). We have also seen that in this limiting case ($k_z = 0$) we always obtain a denumerably infinite set of discrete normal modes belonging to the general Class 1c and/or 2 (whether the plasma is stable or not) plus a second discrete set of "zero-field" modes. This is an essential new feature of the plasma spectrum introduced by the presence of the magnetic field. In a plasma without magnetic field, the treatment of Case[4], combined with the Penrose criterion[6], shows that discrete modes exist if and only if the plasma is unstable or marginally stable at the given value of $\underset{\sim}{k}$. We shall see below that this is also true for a plasma with magnetic field for any propagation direction of electrostatic disturbances other than $k_z = 0$; i.e., for $k_z \neq 0$, the discrete modes exist for the given value of $\underset{\sim}{k}$ and given $\underset{\sim}{B}_0$ if and only if the plasma is unstable or marginally stable to such disturbances for the specified conditions.

In discussing completeness of the normal modes, we shall first limit our attention to disturbances for which $k_z \neq 0$. For the sake of brevity we shall also restrict our proof to plasmas for which no 1c points exist. (It will become apparent below that the generalization to include 1c points proceeds exactly as in Section IV.) We wish to show for such cases that it is always possible to expand any* function $\hat{f}(\underset{\sim}{v}, \underset{\sim}{k}) = f(v_z, v_\perp, \theta, k_z, k_\perp, \alpha)$ (see Fig. 1) according to the scheme

$$f(v_z, v_\perp, \theta, k_z, k_\perp, \alpha) = \sum_{w_i} a_{w_i} h\,(v_z, v_\perp, \theta/w_i)$$
$$+ \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} dw \int_0^{\infty} d\mu\, A_\nu(w,\mu) h_\nu(v_z, v_\perp, \theta/w,\mu) \tag{125}$$

The sum over w_i runs over all complex discrete modes (Class 2). We shall have no difficulty here in defining the integral because of our assumption of no lc points. Although in the interest of simplicity we have suppressed the dependence on $\underset{\sim}{k}$ in the right hand side of (125), we recall here explicitly that the normal modes are functions of $k_\perp$ and k_z. Moreover, the expansion coefficients a_{w_i} and $A_\nu(w,\mu)$ will depend parametrically on k_z, $k_\perp$ and α.

We use the orthogonality of the continuum and discrete modes with the adjoint modes to find (suppressing the $\underset{\sim}{k}$ dependence)

$$a_{w_i^*} = \frac{1}{C_{w_i^*}} \int d^3\underset{\sim}{v}\, f(v_z, v_\perp, \theta)\, \tilde{h}^*(v_z, v_\perp, \theta/w_i) \tag{126}$$

where the C_{w_i} are given by (123).

We define

$$f_1(v_z, v_\perp, \theta) = f(v_z, v_\perp, \theta) - \sum_{w_i} a_{w_i} h\,(v_z, v_\perp, \theta/w_i) \tag{127}$$

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* See footnote under first sentence, Section IV.

and using the explicit continuum modes (98) in (125), find

$$f_1(v_z, v_\perp, \theta) = e^{-i\gamma\sin\theta}\sum_{\nu=-\infty}^{\infty}\int_{-\infty}^{\infty}dw\int_0^{\infty}d\mu\, A_\nu(w,\mu)\Big\{\lambda_\nu(v_z)e^{i\nu\theta}\delta(v_\perp-\mu)\delta(w-w_c\nu-k_z v_z/k)$$

$$+2\pi\mu\, J_\nu(\gamma_\mu)\sum_{n=-\infty}^{\infty}J_n(\gamma)e^{in\theta}\frac{\mathcal{P}\,\eta_n(v_z,v_\perp)}{w-w_c n-k_z v_z/k}\Big\}$$

$$= e^{-i\gamma\sin\theta}\sum_{\nu=-\infty}^{\infty}\Big\{e^{i\nu\theta}A_\nu\Big(w_c\nu+\frac{k_z v_z}{k}, v_\perp\Big)\lambda_\nu(v_z)$$

$$+\sum_{n=-\infty}^{\infty}J_n(\gamma)e^{in\theta}\eta_n(v_z,v_\perp)\,\mathcal{P}\!\!\int_{-\infty}^{\infty}\frac{dw}{w-w_c n-k_z v_z/k}\int_0^{\infty}2\pi\mu\, J_\nu(\gamma_\mu)A_\nu(w,\mu)\,d\mu\Big\} \tag{128}$$

Since $e^{-i\gamma\sin\theta}\neq 0$, we may define

$$g(v_z, v_\perp, \theta) \equiv e^{i\gamma\sin\theta} f_1(v_z, v_\perp, \theta) \tag{129}$$

and in turn the Fourier coefficient

$$g_{\nu'}(v_z, v_\perp) \equiv \frac{1}{2\pi}\int_0^{2\pi}d\theta\, e^{-i\nu'\theta}g(v_z, v_\perp, \theta) \tag{130}$$

(With the coefficients a_{w_i} given by (126), $g_{\nu'}(v_z, v_\perp)$ is completely determined—through definitions (127), (129) and (130)— in terms of the given function f(v_z , $v_\perp$, θ).) Performing in (128) the operations indicated in (129) and (130) we find

$$g_{\nu'}(v_z, v_\perp) = A_{\nu'}\Big(w_c\nu'+\frac{k_z v_z}{k}, v_\perp\Big)\lambda_{\nu'}(v_z)$$

$$+J_{\nu'}(\gamma)\eta_{\nu'}(v_z,v_\perp)\,\mathcal{P}\!\!\int_{-\infty}^{\infty}\frac{dw}{w-w_c\nu'-k_z v_z/k}\sum_{\nu=-\infty}^{\infty}\int_0^{\infty}2\pi\mu\, J_\nu(\gamma_\mu)A_\nu(w,\mu)\,d\mu \tag{131}$$

Thus, in analogy to our previous work, to prove the validity of (125) it is sufficient to show that (131) can always be solved for the quantity

$$\mathcal{Q}(w) \equiv \sum_{\nu=-\infty}^{\infty} \int_0^{\infty} 2\pi v_\perp dv_\perp \, J_\nu(\gamma_1) A_\nu(w, v_\perp) \tag{132}$$

since if $\mathcal{Q}(w)$ is known then the product $\lambda_{\nu'}(v_z) A_{\nu'}(w_c\nu' + \frac{k_z v_z}{k}, v_\perp)$ is determined directly from (131). (This product is analogous to the B(w, $\underset{\sim}{\gamma}$) used in Section IV, and we see from (128) that it is just this product, in addition to $\mathcal{Q}(w)$, which we need to determine the expansion (125). We do not require $A_{\nu'}(w_c\nu' + \frac{k_z v_z}{k}, v_\perp)$ alone.)

To show that $\mathcal{Q}(w)$ can always be determined, we write (131) <u>for each ν'</u> in terms of the variable $u = k_z v_z/k + w_c\nu'$ *

$$\begin{aligned} g_{\nu'}\left(\tfrac{k}{k_z}(u-w_c\nu'), v_\perp\right) = \; & A_{\nu'}(u, v_\perp)\,\lambda_{\nu'}\left(\tfrac{k}{k_z}(u-w_c\nu')\right) \\ & + J_{\nu'}(\gamma)\,\eta_{\nu'}\left(\tfrac{k}{k_z}(u-w_c\nu'), v_\perp\right) \mathcal{P}\int_{-\infty}^{\infty} \frac{dw\, \mathcal{Q}(w)}{w-u} \end{aligned} \tag{133}$$

Multiplying by $2\pi v_\perp J_{\nu'}(\gamma) k/|k_z|$, integrating over $v_\perp$ and summing* over all ν', we arrive finally at the integral equation

$$G(u) = \mathcal{Q}(u)\,\hat{\lambda}(u) + P(u)\,\mathcal{P}\int_{-\infty}^{\infty} \frac{dw\, \mathcal{Q}(w)}{w-u} \tag{134}$$

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*We use no index on u in (133) by virtue of the following argument. If (131) has a solution for each ν', then it must hold at the points $v_z = \frac{k}{k_z}(u - w_c\nu')$ for each ν'. We then perform the operations leading to (134), including the sum over ν', at fixed u rather than fixed v_z. The result is (134).

with

$$\hat{\lambda}(u) = 1 - \mathcal{P}\int_{-\infty}^{\infty} \frac{du' \, P(u')}{u - u'} \tag{101}$$

where

$$G(u) \equiv \sum_{\nu=-\infty}^{\infty} \int_0^{\infty} 2\pi v_\perp dv_\perp \, J_\nu(\gamma) \frac{k}{|k_z|} g_\nu\left(\frac{k}{k_z}(u - w_c \nu), v_\perp\right) \tag{135}$$

P(u) is defined by (102). In deriving (134) we have made use of the fact, proven earlier, that $\lambda_{\nu'}\left(\frac{k}{k_z}(u - w_c \nu')\right)$ is independent of ν' .

Equations (134) and (101) are once again the integral equations studied by Case and Van Kampen. Thus, we know immediately that if there are no points at which $\hat{\lambda}(u)$ and P(u) vanish simultaneously (i.e., no Class 1c or Class 2 modes), then (134) with (101) always has a solution $\mathcal{Q}(u)$. This is true, for example, for all isotropic distributions f_0, as one can readily prove from the definitions of P(u) and $\hat{\lambda}$(u).[13,3]

We wish now to show that (134) with (101) can always be solved for $\mathcal{Q}(u)$, even when complex discrete modes exist. We define for complex z

$$\begin{aligned} R(z) &\equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathcal{Q}(w)\, dw}{w - z} \\ S(z) &\equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{P(w)\, dw}{w - z} \\ T(z) &\equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G(w)\, dw}{w - z} \end{aligned} \tag{136}$$

Then, by precisely the same arguments used by Case[4] (and Van Kampen[3]), if a well-behaved solution $a(\nu)$ of (134) exists, then

$$R(z) = \frac{T(z)}{1 + 2\pi i\, S(z)} \tag{137}$$

and R(z) must be analytic in the complex z-plane cut along the real axis. Moreover, if R(z) has these properties, and if $R^{\pm}(u)$, $S^{\pm}(u)$, $T^{\pm}(u)$ are the limits of R, S and T as z approaches the real axis from above and below, then

$$a(u) = R^{+}(u) - R^{-}(u) = \frac{T^{+}(u)}{1 + 2\pi i\, S^{+}(u)} - \frac{T^{-}(u)}{1 + 2\pi i\, S^{-}(u)} \tag{138}$$

To show that R(z) has the requisite properties, we note that (for $k_z \neq 0$) T(z) and S(z) are analytic in the cut plane. Hence the only poles of R(z) that can arise are at those points z_i (if they exist[13]), such that

$$1 + 2\pi i\, S(z_i) = 0 \tag{139}$$

But these are just our points w_j defined by (119). Thus, R(z) can be analytic in the cut plane only if

$$T(w_j) = 0 \tag{140}$$

This requires

$$\int_{-\infty}^{\infty} \frac{dw\, G(w)}{w - w_j} = 0$$

or

$$\int_{-\infty}^{\infty} \frac{dw \displaystyle\sum_{\nu=-\infty}^{\infty} \int_0^{\infty} 2\pi v_\perp\, dv_\perp\, J_\nu(\gamma) \frac{k}{|k_z|}\, g_\nu\!\left(\frac{k}{k_z}(w - w_c \nu), v_\perp\right)}{w_j - w} = 0$$

Transforming back to v_z - space, we see that the above reduces to (with (129) and (130))

$$0 = \int_{-\infty}^{\infty} dv_z \sum_{\nu=-\infty}^{\infty} \frac{\int_0^{\infty} v_\perp\, dv_\perp \int_0^{2\pi} e^{-i\nu\theta} e^{i\gamma \sin\theta} f_1(v_z, v_\perp, \theta)\, d\theta}{w_j - w_c \nu - k_z v_z / k}$$

$$= \int_{-\infty}^{\infty} dv_z \int_0^{\infty} v_\perp\, dv_\perp \int_0^{2\pi} d\theta \sum_{\nu=-\infty}^{\infty} \frac{e^{i\gamma \sin\theta} e^{-i\nu\theta} J_\nu(\gamma)}{w_j - w_c \nu - k_z v_z / k} \left[f(v_z, v_\perp, \theta) - \sum_{w_i} a_{w_i} h(v_z, v_\perp, \theta | w_i) \right] \quad (141)$$

$$= \int d^3\underset{\sim}{v}\, \tilde{h}^*(v_z, v_\perp, \theta | w_j^*) \left[f(v_z, v_\perp, \theta) - \sum_{w_i} a_{w_i} h(v_z, v_\perp, \theta | w_i) \right]$$

Thus, from the orthogonality of the discrete modes, the condition $T(w_j) = 0$ requires

$$C_{w_j^*}\, a_{w_j^*} = \int d^3\underset{\sim}{v}\, f(v_z, v_\perp, \theta)\, \tilde{h}^*(v_z, v_\perp, \theta | w_j) \quad (142)$$

Hence, the requirement that $T(w_j)$ vanish is indeed satisfied, since the condition (142) is precisely the same as our formula (126) for the expansion coefficients of the discrete modes. (As already noted, by assuming simple roots of (119),

we guarantee $C_{w_j^*} \neq 0$). We conclude that a solution $\hat{a}(w)$ of (134) always exists for $k_z \neq 0$ and hence the product $\lambda_{p'}(v_z) A_{p'}\left(\frac{k_z v_z}{\omega_c} + m_e p', v_\perp\right)$ is determined from (131).*
Therefore, for $k_z \neq 0$, the normal modes derived in Section VII are complete, i.e., the expansion (125) is valid.

It has been shown elsewhere[13], by studying the properties of the dispersion relation for electrostatic disturbances in the form given by Harris[22] that a Vlasov plasma with applied uniform magnetic field is unstable to such disturbances over a range of values of $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$ (for $k_z \neq 0$) <u>if and only if</u> the <u>Penrose criterion</u>[6] is satisfied by the function P(u) in place of $\eta(u)$. We may conclude from this (compare the Penrose criterion and the definitions (114) and (115) of the lc modes) that discrete modes exist for $k_z \neq 0$ if and only if the plasma is unstable or marginally stable at the <u>given</u> values of $\underset{\sim}{B}_0$ and $\underset{\sim}{k}$ we are considering.

In contrast, for $k_z = 0$, the continuum vanishes altogether, and we always have two independent denumerably infinite sets of modes with discrete frequencies, whether the plasma is stable or not. Our above treatment says nothing concerning the completeness of these modes for arbitrary disturbances with $k_z = 0$. In the following section we discuss the normal modes for this special case.

*As already stated, the proof given here is restricted to the case for which there are no lc points. When such points exist, the proof can be generalized exactly as in Section IV. The results for $a(u)$ and the corresponding coefficients of the lc modes are then obtained by replacing $\eta(w)$ by $P(u)$ and $f_\parallel(v)$ by $G(v)$ in formulas (10), (45), (46), and (47).

x) The normal modes for $k_z = 0$.

For the special case of propagation perpendicular to $\underset{\sim}{B}_0$ there exist two independent sets of normal modes corresponding to discrete excitation frequencies, the general properties of which (except for the zero-frequency component) are described in equations (103a)-(104b). The first of these sets, with frequencies given by the Bernstein dispersion relation, contributes to the electric field oscillations and is a simple limiting case (equation (104a)) of the general Class 1c or 2 modes (equations (116) and (121)). The adjoints of the modes (104a) are given by (118) or (122) for $k_z = 0$. The second of these two sets is made up of modes excited at the cyclotron frequency and its harmonics, plus a zero-frequency component which we shall derive below. A particular choice of the functions $H_m(v_z, v_\perp/\sigma, K)$ $(m \neq 0)$ in (104b) is apparently left to our discretion.

We first note a special property of the modes (104a): if we multiply any one of them by $e^{i\frac{v_\perp}{w_c}\sin\theta}$ and integrate over θ from 0 to 2π, the resulting integral vanishes. Thus, any function $f(v_z, v_\perp, \theta)$ which is expanded in terms of the (104a) modes alone would have to have the property

$$g_0(v_z, v_\perp) \equiv \frac{1}{2\pi}\int_0^{2\pi} g(v_z, v_\perp, \theta)\,d\theta \equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{i\frac{v_\perp}{w_c}\sin\theta} f(v_z, v_\perp, \theta) = 0 \tag{143}$$

However, an arbitrary function cannot have this property. This provides a direct proof that the modes (104a) alone are not complete in velocity space.

The quantity $g_o(v_z, v_\perp)$ plays an interesting role in the theory for the special case $k_z = 0$. Let us consider the linearized Vlasov equation for $k_z = 0$ in the electrostatic approximation:

$$\frac{\partial f(v_z, v_\perp, \theta, t)}{\partial t} + (ikv_\perp \cos\theta) f + \omega_c \frac{\partial f}{\partial \theta} = i\frac{\omega_p^2}{k^2}\frac{\partial f_o(v_z,v_\perp)}{\partial v_\perp} k\cos\theta \int d^3\underline{v}\, f$$

Noting that for $g(v_z, v_\perp, \theta, t) \equiv e^{i\frac{v_\perp}{\omega_c}\sin\theta} f(v_z, v_\perp, \theta, t)$

$$\frac{\partial g}{\partial t} = e^{i\frac{v_\perp}{\omega_c}\sin\theta}\frac{\partial f}{\partial t} \quad ; \quad \frac{\partial g}{\partial \theta} = e^{i\frac{v_\perp}{\omega_c}\sin\theta}\left(\frac{\partial f}{\partial \theta} + i\frac{kv_\perp}{\omega_c}\cos\theta \cdot f\right)$$

we find

$$\frac{\partial g}{\partial t} + \omega_c \frac{\partial g}{\partial \theta} = i\frac{\omega_p^2}{k^2}\frac{\partial f_o(v_z,v_\perp)}{\partial v_\perp} k\cos\theta\, e^{i\frac{v_\perp}{\omega_c}\sin\theta} \int d^3\underline{v}\, f$$

We note that g, as well as f, must be periodic in θ . Hence, upon integration over θ from 0 to 2π, there results

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi} d\theta\, g(v_z, v_\perp, \theta, t) \equiv \frac{\partial}{\partial t} g_o(v_z, v_\perp, t) = 0 \qquad (144)$$

i.e., this particular weighted integral of the solution is <u>constant in time</u>. We are thus led to expect that those initial disturbances in velocity space which do not satisfy $g_o = 0$, and hence are not expandable in terms of the modes (104a) alone, will excite a static d.c. mode. (Note that

(144) and the conclusions we have drawn from it apply only for $k_z = 0$.)

We have noted at the end of Section VII that the fact that the additional modes (104b)($m \neq 0$) do not contribute to the electric field fluctuations leads to the property

$$\int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_z \, J_m\left(\frac{v_\perp}{\omega_c}\right) H_m(v_z, v_\perp / \sigma, K) = 0 \qquad (145)$$

For reasons similar to those discussed above, we cannot choose $H_m(v_z, v_\perp/\sigma, K)$ so that the integral over v_z alone vanishes (for example), since this would again lead to a special property of the functions expanded in terms of such modes.

To arrive at a particular choice of the functions $H_m(v_z, v_\perp/\sigma, K)$ we have chosen the somewhat unconventional procedure of studying the initial value problem of the linearized Vlasov equation in the electrostatic limit for $k_z = 0$. For the sake of brevity, we omit this derivation since it involves standard techniques[1,2], and the result, which will be given below, is readily checked. The only part which may not be completely familiar is that contribution to the disturbance distribution function which vanishes upon integration over $\underset{\sim}{v}$ plus that part representing a static mode (pole at $p = 0$). This study leads us to choose (for $\omega = m\omega_c$, all integer $m \neq 0$)

$$h_m(v_z, v_\perp, \theta/\sigma, K) \equiv e^{-i\frac{v_\perp}{w_c}\sin\theta} e^{im\theta} H_m(v_z, v_\perp/\sigma, K) =$$

$$= e^{-i\frac{v_\perp}{w_c}\sin\theta} e^{im\theta}\left(\frac{\delta(v_z-\sigma)\delta(v_\perp-K)}{2\pi K J_m(K/w_c)} - \frac{N_\perp(v_z,v_\perp)\frac{m w_c}{v_\perp} J_m\left(\frac{v_\perp}{w_c}\right)}{\int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^{\infty} dv_z N_\perp(v_z,v_\perp)\frac{m w_c}{v_\perp} J_m^2\left(\frac{v_\perp}{w_c}\right)}\right) \quad (146)$$

The $H_m(v_z, v_\perp/\sigma, K)$ so defined clearly satisfy (145). Moreover, the adjoints to these modes are simply given by

$$\tilde{h}_m(v_z, v_\perp, \theta/\sigma, K) = e^{-i\frac{v_\perp}{w_c}\sin\theta} e^{im\theta} \frac{H_m(v_z, v_\perp/\sigma, K)}{\frac{m w_c}{v_\perp} N_\perp(v_z, v_\perp)} \quad (147)$$

The adjoint modes (147) are orthogonal to the Bernstein modes (104a) as well as to the d.c. mode h_0 discussed below (eq. (152)).

The orthogonality properties of the "zero-field" modes and their adjoints are in turn given by $(\ell, m \neq 0)$

$$\int d^3\underset{\sim}{v}\, \tilde{h}_\ell^*(v_z, v_\perp, \theta/\sigma', K') h_m(v_z, v_\perp, \theta/\sigma, K) =$$

$$= \delta_{\ell m}\left\{\frac{\delta(\sigma-\sigma')\delta(K-K')}{2\pi K J_\ell^2\left(\frac{K}{w_c}\right)\frac{\ell w_c}{K} N_\perp(\sigma, K)} - \frac{1}{\int_{-\infty}^{\infty} dv_z \int_0^\infty 2\pi v_\perp dv_\perp N_\perp(v_z, v_\perp)\frac{\ell w_c}{v_\perp} J_\ell^2\left(\frac{v_\perp}{w_c}\right)}\right\} \quad (148)$$

We turn now to the d.c. mode mentioned previously. If a zero-frequency eigensolution of (89) for $k_z = 0$ exists, it must satisfy

$$(k v_\perp \cos\theta) h_0 - i\omega_c \frac{\partial h_0}{\partial\theta} = -(k\cos\theta) N_\perp \int d^3\underset{\sim}{v}\, h_0 \quad (149)$$

We seek a homogeneous and a particular solution of (149) to enable us to satisfy $\int d^3\underset{\sim}{v}\, h_0 = 1$. A homogeneous solution of (149) is seen to be

$$h_0^{homo} = e^{-i\frac{v_\perp}{w_c}\sin\theta}\, \hat{H}_0(v_z, v_\perp | \sigma, K) \tag{150}$$

where $\hat{H}_0(v_z, v_\perp | \sigma, K)$ is any function of v_z and $v_\perp$. If the assumed normalization can be satisfied, a particular solution of (149) is then simply (independent of θ)

$$h_0^{part} = -\frac{1}{v_\perp} N_\perp \tag{151}$$

Thus, we may choose

$$h_0(v_z, v_\perp | \sigma, K) = \frac{\delta(v_z-\sigma)\,\delta(v_\perp-K)}{2\pi K J_0(K/w_c)}\, e^{-i\frac{v_\perp}{w_c}\sin\theta}\left(1+\int_0^\infty 2\pi\, dv_\perp \int_{-\infty}^{\infty} dv_z\, N_\perp\right) - \frac{N_\perp(v_z, v_\perp)}{v_\perp} \tag{152}$$

With the help of (93), we find $\int d^3\underset{\sim}{v}\, h_0 = 1$ as required.

The adjoint $\hat{\tilde{h}}_0$ of this mode must satisfy (105) with $k_z = 0 = w$. Here we may choose

$$\tilde{h}_0(v_z, v_\perp | \sigma, K) = \frac{\delta(v_z-\sigma)\,\delta(v_\perp-K)}{2\pi J_0\left(\frac{K}{w_c}\right) N_\perp(\sigma, K)}\, e^{-i\frac{v_\perp}{w_c}\sin\theta} \tag{153}$$

since this causes both sides of (105) to vanish if $k_z = 0 = w$. Finally, we have (using (93))

$$\int d^3\underset{\sim}{v}\, \tilde{h}_0^{*}(v_z, v_\perp, \theta | \sigma', K')\, h_0(v_z, v_\perp, \theta | \sigma, K) =$$

$$= \frac{\delta(\sigma-\sigma')\,\delta(K-K')}{2\pi J_0^2\left(\frac{K}{w_c}\right) N_\perp(\sigma, K)}\left(1+\int_0^\infty 2\pi\, dv_\perp \int_{-\infty}^{\infty} dv_z\, N_\perp\right) - 1 \tag{154}$$

One may readily verify that the adjoint mode $\tilde{h}_0$ is orthogonal to the Bernstein modes (104a) and the remaining h_{ym} modes

(Eq. (146), m $\neq$ 0).

With the choices (146) and (152) of the h_m modes, let us seek to expand an arbitrary function* in velocity space in the form

$$f(v_z, v_\perp, \theta) = \sum_{w_r} a_{w_r} h(v_z, v_\perp, \theta | w_r, k_z = 0)$$

$$+ \sum_{m=-\infty}^{\infty} \int_0^\infty dK \int_{-\infty}^{\infty} d\sigma \, b_m(\sigma, K) h_m(v_z, v_\perp, \theta | \sigma, K) \qquad (155)$$

where the sum over W_r runs over all the solutions of (103a). (We recall that the coefficients a_{w_r} and $b_m(\sigma, K)$ depend parametrically on k and α .)

An immediate consequence of (155), if it is valid, (recalling the normalization of the modes) is

$$\int d^3\underset{\sim}{v} \, f(v_z, v_\perp, \theta | k, \alpha) = \sum_{w_r} a_{w_r}(k, \alpha) + \int_0^\infty dK \int_{-\infty}^{\infty} d\sigma \, b_0(\sigma, K | k, \alpha) \qquad (156)$$

Moreover, utilizing the explicit form of the modes, we find $(\ell \neq 0)$

$$g_\ell(v_z, v_\perp) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta \, g(v_z, v_\perp, \theta) e^{-i\ell\theta} \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta \, f(v_z, v_\perp, \theta) e^{i \frac{v_\perp}{w_c} \sin\theta} e^{-i\ell\theta}$$

$$= \sum_{w_r} a_{w_r} \frac{J_\ell\left(\frac{v_\perp}{w_c}\right) \frac{\ell w_c}{v_\perp} N_\perp(v_z, v_\perp)}{w_r - \ell w_c} + \int_{-\infty}^{\infty} d\sigma \int_0^\infty dK \, b_\ell(\sigma, K) H_\ell(v_z, v_\perp | \sigma, K)$$

$$- J_\ell\left(\frac{v_\perp}{w_c}\right) \frac{N_\perp(v_z, v_\perp)}{v_\perp} \int_{-\infty}^{\infty} d\sigma \int_0^\infty dK \, b_0(\sigma, K) \qquad (157)$$

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*As before, limited smoothness conditions on the function may be required for complete rigor.

For $\ell = 0$, we find

$$g_0(v_z, v_\perp) = \left(1 + \int_0^\infty 2\pi dv_\perp \int_{-\infty}^\infty dv_z N_\perp\right) \frac{b_0(v_z, v_\perp)}{2\pi J_0(v_\perp/w_c)} - J_0\left(\frac{v_\perp}{w_c}\right) \int_0^\infty dK \int_{-\infty}^\infty d\sigma \, b_0(\sigma, K) \tag{158}$$

Thus, the "constant of the motion" given by (144) is indeed associated with the d.c. ($m = 0$) component of the h_m modes; we may in fact determine the expansion coefficient $b_0(\sigma, K)$ from (158). Hence, if (155) is valid, the Bernstein modes alone contribute the time-varying density and field fluctuations, while the h_0 mode provides the required static part of the solution.

The expansion coefficients a_{w_r} are determined, from the orthogonality of the corresponding adjoints, by equation (126) for $k_z = 0$:

$$a_{w_r^*} = \frac{1}{C_{w_r^*}} \int d^3\underline{v} \, f(v_z, v_\perp, \theta) \tilde{h}^*(v_z, v_\perp, \theta \,|\, w_r, k_z = 0) \tag{126}$$

where the adjoints $\tilde{h}_{w_r}$ are obtained from (118) and/or (122) with $k_z = 0$, and the C_{w_r} are given by (124). Similarly, for $m \neq 0$, the coefficients $b_m(\sigma, K)$ must each satisfy, from the orthogonality relations (148)

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$$b_m(\sigma,K) = 2\pi K J_m\left(\frac{K}{w_c}\right) g_m(\sigma,K) - \frac{2\pi K J_m^2\left(\frac{K}{w_c}\right)\frac{mw_c}{K} N_\perp(\sigma,K)\int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z J_m\left(\frac{v_\perp}{w_c}\right) g_m}{\int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z N_\perp(v_z,v_\perp)\frac{mw_c}{v_\perp} J_m^2\left(\frac{v_\perp}{w_c}\right)}$$

$$+ \frac{2\pi K J_m^2\left(\frac{K}{w_c}\right)\frac{mw_c}{v_\perp} N_\perp(\sigma,K)}{\int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z N_\perp(v_z,v_\perp)\frac{mw_c}{v_\perp} J_m^2\left(\frac{v_\perp}{w_c}\right)} \int_{-\infty}^\infty d\sigma \int_0^\infty dK\, b_m(\sigma,K) \tag{159}$$

This set of integral equations for the $b_m(\sigma,K)$ $(m \neq 0)$ has the interesting property that if we integrate over σ and K, we obtain a set of identities, i.e., each equation has solutions for <u>any</u> value (A_m, say) of the integral $\int_{-\infty}^\infty d\sigma \int_0^\infty dK\, b_m(\sigma,K)$. However, as it turns out, we do not need unique solutions of equations (159); the equations themselves are all we need to determine the expansion (155). Put another way, the determination of the expansion (155) is independent of the choice of A_m.

To see this, we use the explicit form (146) of the modes h_m $(m \neq 0)$ in (155) to find

$$f(v_z,v_\perp,\theta) - \sum a_{w_r} h(v_z,v_\perp,\theta / w_r, k_z=0) - \int_0^\infty dK \int_{-\infty}^\infty d\sigma\, b_0(\sigma,K) h_0(v_z,v_\perp,\theta|\sigma,K) =$$

$$= e^{-i\frac{v_\perp}{w_c}\sin\theta} \sum_{m=-\infty}^{\infty}{}' e^{im\theta}\left(\frac{b_m(v_z,v_\perp)}{2\pi v_\perp J_m\left(\frac{v_\perp}{w_c}\right)} - \frac{N_\perp(v_z,v_\perp)\frac{mw_c}{v_\perp} J_m\left(\frac{v_\perp}{w_c}\right) A_m}{\int_{-\infty}^\infty dv_z \int_0^\infty 2\pi v_\perp dv_\perp N_\perp \frac{mw_c}{v_\perp} J_m^2\left(\frac{v_\perp}{w_c}\right)}\right) \tag{155a}$$

where the prime on the sum over m indicates that we are to omit m = 0. We see that the quantity in parentheses on the right hand side is in fact determined for each m $\neq$ 0 from (159).

Use of the orthogonality relation (154) for the h_0 mode and its adjoint simply reproduces Eq. (158). Solving (158), we find*

$$\int_{-\infty}^{\infty} d\sigma \int_0^{\infty} dK\, b_0(\sigma,K) = \frac{\int_0^{\infty} 2\pi v_\perp\, dv_\perp \int_{-\infty}^{\infty} dv_z\, J_0\left(\frac{v_\perp}{w_c}\right) g_0(v_z,v_\perp)}{1+\int_0^{\infty} 2\pi v_\perp\, dv_\perp \int_{-\infty}^{\infty} dv_z \left(1-J_0^2\left(\frac{v_\perp}{w_c}\right)\right) \frac{N_\perp(v_z,v_\perp)}{v_\perp}}$$

$$\frac{b_0(v_z,v_\perp)}{2\pi J_0\left(\frac{v_\perp}{w_c}\right)}\left(1+\int_0^{\infty} 2\pi\, dv_\perp \int_{-\infty}^{\infty} dv_z\, N_\perp(v_z,v_\perp)\right) = g_0(v_z,v_\perp) + \frac{J_0\left(\frac{v_\perp}{w_c}\right)\frac{N_\perp(v_\perp,v_z)}{v_\perp}\int_0^{\infty} 2\pi v_\perp\, dv_\perp \int_{-\infty}^{\infty} dv_z\, J_0\left(\frac{v_\perp}{w_c}\right) g_0(v_z,v_\perp)}{1+\int_0^{\infty} 2\pi v_\perp\, dv_\perp \int_{-\infty}^{\infty} dv_z \left(1-J_0^2\left(\frac{v_\perp}{w_c}\right)\right)\frac{N_\perp(v_z,v_\perp)}{v_\perp}} \tag{160}$$

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* We assume, for simplicity, that $f_0(v_z,v_\perp)$ is such that the denominator in (160) does not vanish. This is the case, for example, if f_0 is Maxwellian. If the denominator were to vanish, (158) would have no solution; i.e. the modes discussed here would not be complete for such cases. We observe, however, that the vanishing of the denominator in (160) corresponds to a solution of the Bernstein dispersion relation (103a) at $w_r = 0$. If such a solution exists an <u>additional</u> (d.c.) mode appears which then completes the set of modes for this case. This leads to a <u>degeneracy</u>, i.e. to two eigensolutions with zero eigenvalue. We do not discuss this case further here.

However, if we insert the explicit form (152) in (155a), we see that we actually require

$$\frac{b_0(v_z, v_\perp)\, e^{-i\frac{v_\perp}{w_c}\sin\theta}}{2\pi v_\perp J_0\left(\frac{v_\perp}{w_c}\right)}\left(1+\int_0^\infty 2\pi\, dv_\perp \int_{-\infty}^\infty dv_z\, N_\perp(v_z, v_\perp)\right) - \frac{N_\perp(v_z, v_\perp)}{v_\perp}\int_{-\infty}^\infty d\sigma \int_0^\infty dK\, b_0(\sigma, K) =$$

$$= g_0(v_z, v_\perp)\, e^{-\frac{i v_\perp}{w_c}\sin\theta} + \frac{\frac{N_\perp(v_z, v_\perp)}{v_\perp}\left(J_0\left(\frac{v_\perp}{w_c}\right) e^{-i\frac{v_\perp}{w_c}\sin\theta} - 1\right)\int_0^\infty 2\pi v_\perp\, dv_\perp \int_{-\infty}^\infty dv_z\, J_0\left(\frac{v_\perp}{w_c}\right) g_0(v_z, v_\perp)}{1+\int_0^\infty 2\pi v_\perp\, dv_\perp \int_{-\infty}^\infty dv_z \left(1 - J_0^2\left(\frac{v_\perp}{w_c}\right)\right)\frac{N_\perp(v_z, v_\perp)}{v_\perp}} \tag{161}$$

where the right-hand-side follows from (160).

Our final result, upon use of (159) and (161) and the explicit form of the Bernstein modes in (155), is then

$$f(v_z, v_\perp, \theta) = \sum a_{w_r}\, e^{-i\frac{v_\perp}{w_c}\sin\theta} \sum_{n=-\infty}^{\infty} \frac{J_n\left(\frac{v_\perp}{w_c}\right)\frac{m w_c}{v_\perp} N_\perp(v_z, v_\perp)\, e^{in\theta}}{w_r - n w_c}$$

$$+ e^{-i\frac{v_\perp}{w_c}\sin\theta} \sum_{\substack{m=-\infty \\ (m\neq 0)}}^{\infty}{}' e^{im\theta}\left(g_m(v_z, v_\perp) - \frac{J_m\left(\frac{v_\perp}{w_c}\right)\frac{m w_c}{v_\perp} N_\perp \int_0^\infty 2\pi v_\perp\, dv_\perp \int_{-\infty}^\infty dv_z\, J_m\left(\frac{v_\perp}{w_c}\right) g_m(v_z, v_\perp)}{\int_0^\infty 2\pi v_\perp\, dv_\perp \int_{-\infty}^\infty dv_z\, N_\perp(v_z, v_\perp)\frac{m w_c}{v_\perp} J_m^2\left(\frac{v_\perp}{w_c}\right)}\right) \tag{162}$$

$$+ \ (\text{r.h.s. } (161)).$$

where the $g_m(v_z, v_\perp)$ are defined in (157) for all m.

If $f(v_z, v_\perp, \theta)$ is taken to be the initial disturbance of the plasma (with $k_z = 0$) then $f(v_z, v_\perp, \theta, t)$ is obtained by multiplying each of the Bernstein modes in (162) by the factor $e^{-ikw_r t}$, where w_r is the corresponding root of the Bernstein dispersion relation (103a), and by multiplying each of the h_m modes in (162) by $e^{-ikmw_c t}$. If the expansion (155) is valid, i.e., if the modes are complete, this is the solution, according to the normal mode approach, of the initial value problem for the three-dimensional linearized Vlasov equation with $k_z = 0$, the a_{w_r} being given by (126) and (124). Precisely the same result is obtained by direct solution in terms of Laplace transforms and subsequent inversion of the transforms, provided $\int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z \left(J_0^2\left(\frac{v_\perp}{w_c}\right)-1\right)\frac{N_\perp}{v_\perp} \neq 1$.

We turn now to the question of completeness for the $k_z = 0$ modes (with the restriction already noted under Eq. (160)). We begin by observing the following: equation (155a) can be interpreted as an equation for the $b_m(\sigma, K)$ $(m \neq 0)$ for given $f(v_z, v_\perp, \theta)$ since the coefficients a_{w_r} and $b_0(\sigma, K)$ are known in terms of $f(v_z, v_\perp, \theta)$ and are independent of the $b_m(\sigma, K)$ $(m \neq 0)$. Multiplying both sides by $e^{i\frac{v_\perp}{w_c}\sin\theta}$ and computing the Fourier coefficient (with respect to θ-space) of the result, we obtain an equivalent set of integral equations for each of the $b_m(\sigma, K)$ $(m \neq 0)$

$$g_m(v_z, v_\perp)\, 2\pi v_\perp J_m\left(\frac{v_\perp}{w_c}\right) - \sum_{w_r} a_{w_r} \frac{2\pi J_m^2\left(\frac{v_\perp}{w_c}\right)\frac{mw_c}{v_\perp} N_\perp(v_z, v_\perp)}{w_r - mw_c}$$

$$+ 2\pi J_m^2\left(\frac{v_\perp}{w_c}\right)\frac{N_\perp(v_z,v_\perp)}{v_\perp}\int_0^\infty dK \int_{-\infty}^\infty d\sigma\, b_0(\sigma, K) = \tag{163}$$

$$= b_m(v_z, v_\perp) - \frac{2\pi v_\perp N_\perp(v_z,v_\perp)\frac{mw_c}{v_\perp}J_m^2\left(\frac{v_\perp}{w_c}\right)\int_0^\infty dK \int_{-\infty}^\infty d\sigma\, b_m(\sigma, K)}{\int_{-\infty}^\infty dv_z \int_0^\infty 2\pi v_\perp dv_\perp\, N_\perp(v_z, v_\perp)\frac{mw_c}{v_\perp}J_m^2\left(\frac{v_\perp}{w_c}\right)}$$

Integrating over $v_\perp$ and v_z, we find that the right-hand-side vanishes; therefore, (163) has solutions for each $m \neq 0$ if and only if

$$G_m \equiv \int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z J_m\left(\frac{v_\perp}{w_c}\right) g_m(v_z, v_\perp) = \sum_{w_r} \frac{a_{w_r}}{w_r - m w_c} \int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z J_m^2\left(\frac{v_\perp}{w_c}\right) \frac{m w_c}{v_\perp} N_\perp(v_z, v_\perp)$$

$$- \left(\int_0^\infty dK \int_{-\infty}^\infty d\sigma\, b_0(\sigma, K)\right)\left(\int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z J_m^2\left(\frac{v_\perp}{w_c}\right) \frac{N_\perp(v_z, v_\perp)}{v_\perp}\right) \qquad (164)$$

with the a_{w_r} given by (126) and (124) and b_0 by (160). Thus, if (164) holds, there is always a solution of (163) which guarantees the validity of expansion (155). Moreover, (164) is precisely the condition for compatibility of (163) with (159), which was obtained by invoking orthogonality. (We have already seen that the fact that (159) has an infinity of solutions causes no difficulty.) If we sum (164) over all $m \neq 0$, use (103a), and add the corresponding G_0 computed from (158), the result reduces to (156); however, (164) is the stronger condition.

The coefficients a_{w_r} given by (126) can be written

$$a_{w_r} = \frac{1}{C_{w_r}} \sum_{\ell=-\infty}^{\infty} \int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z\, g_\ell(v_z, v_\perp)\, J_\ell\left(\frac{v_\perp}{w_c}\right) \frac{1}{w_r - \ell w_c}$$

$$\equiv \frac{1}{C_{w_r}} \sum_{\ell=-\infty}^{\infty} \frac{G_\ell}{w_r - \ell w_c} \qquad (\text{all } \ell) \qquad (165)$$

where we have used our previous definitions of $g_m(v_z, v_\perp)$ and the integrals G_m. If we now define

$$B_m \equiv \int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z \, J_m^2\left(\frac{v_\perp}{w_c}\right) \frac{m w_c}{v_\perp} N_\perp(v_z, v_\perp) \tag{166}$$

use the expression (124) for the C_{w_r}, and (160) for $\int_{-\infty}^\infty d\sigma \int_0^\infty dK \, b_c(\sigma, K)$, the requirement (164) reduces to ($m \neq 0$)

$$G_m = \sum_{w_r} \sum_{l=-\infty}^\infty \frac{B_m G_l}{(w_r - m w_c)(w_r - l w_c) \sum_{n=-\infty}^\infty B_n (w_r - n w_c)^{-2}}$$

$$- \frac{B_m G_o}{m w_c} \frac{1}{1 + \int_0^\infty 2\pi v_\perp dv_\perp \int_{-\infty}^\infty dv_z \left(1 - J_o^2\left(\frac{v_\perp}{w_c}\right)\right) \frac{N_\perp(v_z, v_\perp)}{v_\perp}} \tag{164a}$$

where the sum over w_r runs over all the roots of (103a):

$$0 = 1 - \sum_{n=-\infty}^\infty \frac{B_n}{w_r - n w_c} \equiv D_B(p)\Big]_{p = -ikw_r} \tag{167}$$

If (164a) is to hold it must be a property of the Bernstein dispersion relation ((103a) or (167)), or more precisely a property of the function $D_B(p)$ *

$$D_B(p) \equiv 1 + ik \sum_{n=-\infty}^{\infty}{}' \frac{B_n}{p + ik n w_c} \tag{168}$$

---

*As already pointed out, the denominator in (160) is just $D_B(p=0)$. We are restricting our present discussion to cases for which $D_B(p=0) \neq 0$.

defined in the complex p-plane for $\mathcal{Re}\, p > 0$. (Note that $B_0 = 0$; it is useful nevertheless to note explicitly that the sum in (168) excludes n = 0. This is indicated by the prime on the sum in (168).) We observe, for example, that

$$\sum_{n=-\infty}^{\infty}{}' B_n (w_r - n w_c)^{-2} = -ik\, D_B'(p)\Big]_{p=-ikw_r} \tag{169}$$

This suggests that a study of the initial value problem for the linearized Vlasov equation would be helpful in establishing (164a).

Following up this idea, we are led to consider a set of functions $G_m(t)$, whose Laplace transforms $G_m(p) \equiv \int_0^\infty e^{-pt} G_m(t)\,dt$ satisfy the equations $(\mathcal{Re}\, p > 0)$

$$\begin{aligned} G_m(p) &= \frac{G_m(0)}{p + ikmw_c} - \frac{ikB_m}{p + ikmw_c} \sum_{l=-\infty}^{\infty} G_l(p) \\ G_0(p) &= \frac{G_0(0)}{p} \end{aligned} \tag{170}$$

so that

$$\sum_{m=-\infty}^{\infty} G_m(p) = \frac{\displaystyle\sum_{l=-\infty}^{\infty} \frac{G_l(0)}{p + iklw_c}}{D_B(p)} \tag{171}$$

We now observe that the right-hand-side of (164a) (provided $D_B(p=0) \neq 0$) is simply the evaluation via the residue theorem of an integral involving $D_B(p)$ in the complex p-plane; i.e., for each $m \neq 0$ with $G_m = G_m(0)$

$$rhs\ (164a) = \int_L \frac{dp}{2\pi i}\left(\frac{-ikB_m}{D_B(p)(p+ikmw_c)}\sum_{l=-\infty}^{\infty}\frac{G_l(0)}{p+iklw_c} + \frac{G_m(0)}{p+ikmw_c}\right) \tag{172}$$

where L is the Laplace contour, running to the right of the zeros of $D_B(p)$. Thus, from (171) and (170)

$$rhs\ (164a) = \int_L \frac{dp}{2\pi i}\left(\frac{-ikB_m\sum_{l=-\infty}^{\infty}G_l(p)}{p+ikmw_c} + \frac{G_m(0)}{p+ikmw_c}\right)$$

$$= \int_L \frac{dp}{2\pi i} G_m(p) = G_m(0) \tag{173}$$

where the last equality follows from the Laplace inversion theorem at $t = 0$. This proves (164). We conclude, therefore, that (155) is valid, and the modes described in this section are complete provided $D_B(p=0) \neq 0$. (In effect, in this case, the fact that the normal mode treatment gives the correct solution of the initial value problem guarantees that the modes are complete.)

As we have seen, the Bernstein modes (104a) can be obtained from the more general discrete modes $1c,1$ and/or 2 in the limit $k_z = 0$. There seems to be no simple way, however, of interpreting the "zero-field" modes as the limit as $k_z \to 0$ of the $1c,2$ modes (116), although there is a certain resemblance of the expansion (128) for $k_z \neq 0$ with the expansion (155a) for $k_z = 0$. The implications of this are that if we consider, over the whole of $\underset{\sim}{k}$-space, the Fourier transform $f(\underset{\sim}{v}, \underset{\sim}{k}, t=0)$ of a general initial disturbance, $f(\underset{\sim}{x}, \underset{\sim}{v}, t=0)$, of a homogeneous plasma with magnetic field, we must treat, at least in a normal mode expansion, the plane defined by $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$ separately from the rest of $\underset{\sim}{k}$-space. This appears to be associated with the lack of Doppler shifting of the characteristic frequencies for disturbances in this plane.

One might well ask what the physical meaning of the "zero-field" modes discussed in this section might be. We remark that their somewhat unphysical character is certainly associated in part with the electrostatic approximation we have used (they would be macroscopically undetectable in the electrostatic approximation). For example, all of the modes discussed here satisfy $\underset{\sim}{k} \cdot \underset{\sim}{j} = -\omega n_0 |e| \int d^3\underset{\sim}{v}\, h$, as they must, to be consistent with the Vlasov equation as we have used it. However, as in the case $\underset{\sim}{B}_0 = 0$ (see footnote under Eq. (11)), the eigenmodes possess various non-vanishing values of $\underset{\sim}{k} \times \underset{\sim}{j}$ and hence must always violate the electrostatic approximation to some extent. As is well known, in any problem in which the very high-frequency components (for example) become important, the electrostatic approximation cannot be used.[22]

The static mode we have found for $k_z = 0$ appears to have been overlooked in the literature. We note here only that it is consistent, for $k_z = 0$, with the translational invariance of the Vlasov equation along $\underset{\sim}{B}_0$.

XI) Conclusions

The normal modes for a collisionless electronic (Vlasov) plasma in three dimensions in velocity space, with and without a uniform applied magnetic field, have been derived in the electrostatic approximation. In general, there exists a continuum of modes (except for $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$) and, if the plasma is unstable, an additional set of discrete modes. With a modest extension of the techniques introduced by Van Kampen and Case, we have been able to show that these three-dimensional normal modes are complete. By introducing the appropriate three-dimensional adjoints in the manner suggested by Case, we have demonstrated certain orthogonality properties which are sufficient to enable us to use essentially standard eigenfunction techniques. Certain examples of the use of these eigenmodes for problems involving the three-dimensional linearized Vlasov operator have been given for $\underset{\sim}{B}_0 = 0$. For a plasma with magnetic field, the special case $\underset{\sim}{k} \cdot \underset{\sim}{B}_0 = 0$ has to be treated separately; however, in this case a complete set of modes can also be found.

As pointed out by Case[4], the use of an eigenfunction method rather than other standard techniques[1,2] is largely a matter of taste. However, there seem to be definite advantages to knowing the explicit form of the plasma normal modes, and for certain problems the eigenfunction method is encouragingly powerful.

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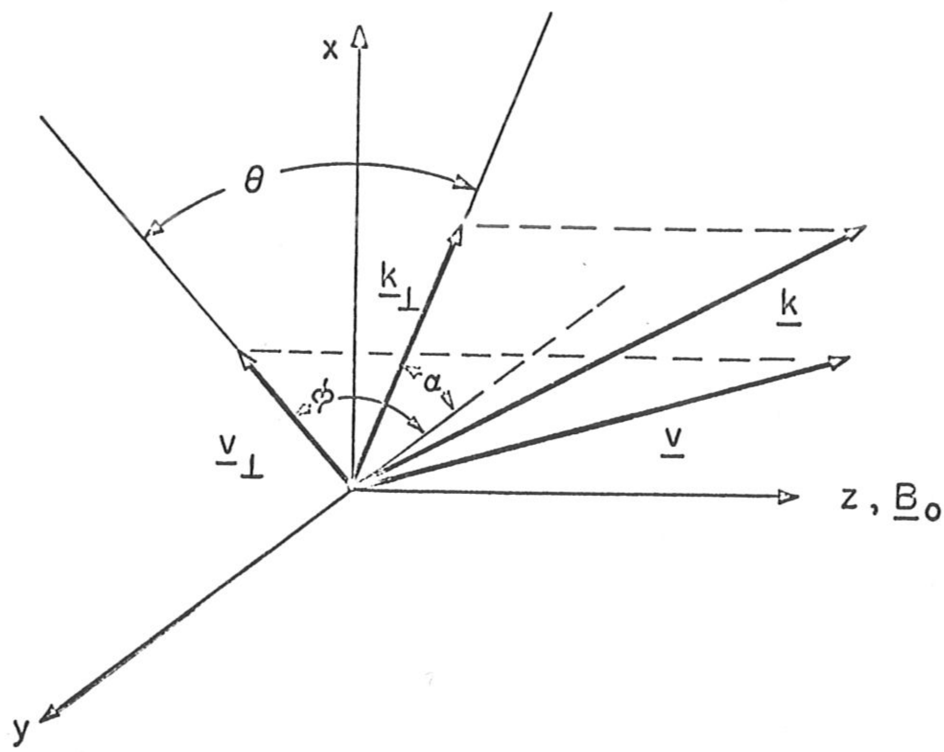

FIG. 1

References

1) Landau, L., J. Phys. U.S.S.R.10, 25 (1946).

2) Bernstein, I.B., Phys. Rev. 109, 1, 10 (1958).

3) Van Kampen, N.G., Physica 21, 949 (1955).

4) Case, K.M., Ann. Phys. 7, 349 (1959). See also, several subsequent papers by Case and co-workers; a bibliography is available in: Aamodt, R.E., and Case, K.M., Ann. Phys. 21, 284 (1963). These papers emphasize the use of the eigenfunction method for boundary value problems. To the author's knowledge, all problems treated are one-dimensional.

5) See also: Weitzner, H., Phys. Flu., 6, 1123 (1963).

6) Penrose, O., Phys. Flu., 3, 258 (1960).

7) Zelazny, R.S., Ann. Phys. 19, 177-185 (1962).

8) Felderhof, B.U., Physica 30, 1771-1792 (1964).

9) Drummond, W.E., and Pines, D., Nucl. Fusion, 1962 Suppl., p. 1049; Velikhov, E.P., Vedenov, A.A. and Sagdeev, R.Z., Nucl. Fusion 1, 82 (1961); 1962 Suppl., p. 465; Frieman, E.A., and Rutherford, P.H., Ann. Phys. 28, 134 (1964).

10) Baldwin, D.E., Phys. Flu. 7, 782 (1964). Baldwin considers the "quasi-linear theory" (Refs. 9) within the framework of the one-dimensional Van Kampen-Case modes. His discussion of the non-linear (mode-coupling) terms is also limited to the strictly one-dimensional description.

11) Balescu, R., Phys. Flu. 3, 52 (1960).

12) Lenard, A., Ann.Phys. 3,390 (1960).

13) McCune, J.E., Phys. Rev. Letters 15, 9, 398 (1965); also Phys. Rev. Letters 15, 16, 684 (1965).

14) Lighthill, M.J., Fourier Analysis and Generalised Functions, Cambridge University Press, Cambridge, England (1959).

15) Muskhelishvili, N.I., Singular Integral Equations, Noordhoff, Groningen (1953).

16) Völk, H., Doctoral Dissertation, University of Munich, to be published (1965).

17) Montgomery, D.C., and Tidman, D.A., Plasma Kinetic Theory, McGraw-Hill, New York, N.Y. (1964).

18) Balescu, R., Statistical Mechanics of Charged Particles, Monographs in Statistical Physics, Vol. 4, Interscience Publishers, New York, N.Y. (1963).

19) Bogoliubov, N.N., Problems of a Dynamical Theory in Statistical Physics (1946, in Russian). [English translation by E.K. Gora, in Studies in Statistical Mechanics, edited by J. de Boer and G.E. Uhlenbeck (North-Holland Publishing Co., Amsterdam, 1962), Vol. 1.]

20) Dupree, T.H., Phys. Flu. 4, 696 (1961).

21) Rutherford, P.H. and Frieman, E.A., Phys. Flu. 6, 1139 (1963).

22) Harris, E.G., J. Nucl. Energy, C, 2, 138 (1961).

23) Watson, G.N., Theory of Bessel Functions, Cambridge University Press, Cambridge, England (1945).