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General Interchange Instability

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ABSTRACT: Magnetohydrostatic equilibria are considered whose lines of force are interchangeable. The energy change is computed which follows from the interchange of a pair of flux tubes. The separation between the members of the pair and the flux of either tube are kept unspecified. Necessary conditions for stability against interchanges over finite distances are derived and expressed by $p, q := \int dl/B$, and $I := (1/\mu) \int B\, dl$. In the case of infinitesimal interchanges, a necessary and sufficient condition can be derived and considerably simplified by using the equilibrium condition $dI = -q\, dp$. This stability condition is shown to be equivalent to the simultaneous stability against interchange at constant volume and at constant flux.

1) We consider a magnetohydrostatic configuration which permits the interchange of magnetic lines of force of finite length. The lines of force may end in an insulating end plate which intersects the lines of force perpendicularly and takes up the gas pressure. We neglect the effects produced beyond the end plates by the interchange. Alternatively, the lines of force may be closed upon themselves. Finally, the configuration may possess a periodicity along the magnetic field; we then consider a single interval. In all cases, each line of force has a defined length, and we may define for each of them the integrals

$$q := \int dl / B \quad \text{and} \quad I := \frac{1}{\mu_0} \int B \, dl . \tag{1}$$

(dl = line element along $\underset{\sim}{B}$; $B = |\underset{\sim}{B}|$). We may associate with a line of force a flux tube of the (infinitesimal) flux Φ and the (infinitesimal) cross section A which are related by

$$A \cdot B = \Phi \tag{2}$$

A, of course, varies in general along any given field line while

$$\partial \Phi / \partial l = 0 \quad \text{similar to} \quad \partial p / \partial l = 0 . \tag{3}$$

By these definitions we may ascribe to a flux tube a certain (infinitesimal) volume

$$V := \int A \, dl = \Phi q . \tag{4}$$

2) The most general interchange of lines of force can be thought of as being composed of the superposition of interchanges of pairs of infinitesimal flux tubes of (in general) different flux and (in general) finite separation. Considering one such

pair, we denote these tubes in their original positions by the indices 1 and 2 respectively. In the interchange tube 1 takes the original position of 2; in this position its physical state will be denoted by index 1 and a prime. In the same interchange tube 2 takes the original position of 1 and will then be denoted by 2 and a prime. Given the geometry of the flux tubes, there is a unique way in which matter must be distributed along the field lines after the interchange such that the pressure is constant along each field line. We assume that this condition obtains. If we further assume that the matter behaves adiabatically during the interchange, then the conservation of entropy yields

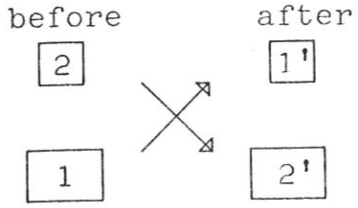

Fig. 1
Notation for the interchange (cross sections through the infinitesimal flux tubes)

$$p_1 V_1^{\gamma} = p_1' V_1'^{\gamma} ; \quad p_2 V_2^{\gamma} = p_2' V_2'^{\gamma}$$
$$(V_1 = V_2' ; \quad V_2 = V_1') \tag{5}$$

while the conservation of flux leads to

$$\Phi_1 = \Phi_1' ; \quad \Phi_2 = \Phi_2' . \tag{6}$$

This allows us to relate the values of the integrals of eq. (1) before and after the interchange with the ratio V_1 / V_2 as the only parameter

$$q_1' = \frac{V_1'}{\Phi_1'} = \frac{V_2}{V_1} q_1 \quad ; \quad 1 \leftrightarrow 2 \tag{7}$$

$$I_1' = \frac{1}{\mu_o} \Phi_1' \int \frac{dl_1'}{A_1'} = \frac{1}{\mu_o} \Phi_1 \int \frac{dl_2}{A_2}$$
$$= \frac{\Phi_1}{\Phi_2} I_2 = \frac{V_1 q_2}{V_2 q_1} I_2 \quad ; \quad 1 \leftrightarrow 2 \tag{8}$$

3) The energy contained in the flux tube 1 is:

$$\begin{aligned} W_1 &= \frac{1}{2\mu_0} \int B_1^2 A_1 \, dl_1 + \int \frac{p_1}{\gamma - 1} A_1 \, dl_1 \\ &= \frac{1}{2} \Phi_2 I_1 + \frac{p_1}{\gamma-1} V_1 \\ &= \frac{1}{2} \frac{I_1 V_1}{q_1} + \frac{p_1 V_1}{\gamma - 1} \end{aligned} \tag{9}$$

With the aid of eqs. (5), (7) and (8) the energy after the interchange can be expressed by the unprimed quantities

$$W_1' = \frac{1}{2} \frac{q_2}{q_1^2} I_2 \cdot \frac{V_1^2}{V_2} + \frac{p_1}{\gamma-1} \left(\frac{V_1}{V_2}\right)^{\gamma} \cdot V_2 \, . \tag{10}$$

This, together with the obvious relations for W_2 and W_2', allows us to express the energy difference between the state after and before interchange

$$\Delta W := (W_1' - W_1) + (W_2' - W_2) \tag{11}$$

by quantities of the original configuration and the parameters V_1, V_2 . To do this in as concise a form as possible we introduce the following notation:

$$a_1 := q_1^{-2} \, ; \quad b_1 = q_1 \cdot I_1 \; ; \quad 1 \leftrightarrow 2 \tag{12}$$

and for any quantity x which may bear the indices 1 and 2:

$$\delta x := x_2 - x_1 \; ; \; \overline{x} = \frac{1}{2} (x_1 + x_2) \, . \tag{13}$$

This then gives our fundamental relation:

$$\begin{aligned} \Delta W &= - \frac{1}{2} \delta (a V^2) \cdot \delta (b V^{-1}) \\ &\quad - \frac{1}{\gamma - 1} \delta (p V^{\gamma}) \cdot \delta (V^{2-\gamma}) \end{aligned} \tag{14}$$

$$\begin{aligned} &= - \frac{1}{2} (\overline{a} \, \delta V^2 + \delta a \cdot \overline{V^2}) \cdot (\overline{b} \, \delta V^{-1} + \delta b \cdot \overline{V^{-1}}) \\ &\quad - \frac{1}{\gamma - 1} (\overline{p} \, \delta V^{\gamma} + \delta p \, \overline{V^{\gamma}}) \cdot \delta V^{1-\gamma} \, . \end{aligned} \tag{15}$$

4) A necessary condition for stability is that there exists no pair of interchangeable field lines that with any choice of V_2/V_1 ΔW becomes negative. If we consider a specific pair (1,2) and choose V_2/V_1 to take in succession the following values:

$$\frac{V_2}{V_1} = \left(\frac{a_2}{a_1}\right)^{-\frac{1}{2}} ; \left(\frac{b_2}{b_1}\right) ; \left(\frac{p_2}{p_1}\right)^{-\frac{1}{\gamma}} ; 1 \qquad (16)$$

then the four δ-terms in eq. (15) become successively zero, such that only one product of δ-terms remains. Thereby four necessary conditions of stability result:

$$\begin{aligned}
&(a) \quad \delta(p\,a^{-\gamma/2}) \cdot \delta(a^{(\gamma-1)/2}) \leq 0 \\
&(b) \quad \delta(p\,b^{\gamma}) \cdot \delta(b^{1-\gamma}) \leq 0 \\
&(c) \quad \delta(a\,p^{-2/\gamma}) \cdot \delta(b\,p^{1/\gamma}) \leq 0 \\
&(d) \quad \delta a \cdot \delta b \leq 0
\end{aligned} \qquad (17)$$

all of which must be fulfilled simultaneously. The first two of these special interchanges leave the magnetic energy unchanged ((a) on both field lines individually, (b) by exactly interchanging it), and the two remaining ones leave the thermal energy constant. Inspection of eq. (17) shows that only three of these conditions are independent. Using $\gamma > 1$ and the definition $a = q^{-2}$ (and recalling that $b = q \cdot I$) we may reformulate eq. (17) as:

$$q_2 \geq q_1 \; ; \; b_2 \geq b_1 \; ; \; \left(\frac{p_2}{p_1}\right)^{1/\gamma} \geq \max\left\{\frac{q_1}{q_2}, \frac{b_1}{b_2}\right\} \qquad (18)$$

or <u>all</u> inequality signs inverted must hold for all pairs (1,2) as necessary condition for stability.

The third condition is certainly fulfilled if $p_2 \geq p_1$. If, however, $p_2 < p_1$, then $I_2 > I_1$ (as shall be proved) and $b_1/b_2 < q_1/q_2$ so that in all cases the third condition may be replaced by

$$p_2 q_2^{\gamma} \geq p_1 q_1^{\gamma} . \tag{19}$$

Without further knowledge about the relation between the quantities b, p, q, it does not follow that no value V_2/V_1 exists which renders ΔW negative, even if the conditions (18) are satisfied.

The question of the sufficiency of the criteria for stability against interchanges can, however, be answered if only interchanges between neighbouring lines of force are considered.

5) We restrict our attention to such values of V_2/V_1, that $\delta V/\overline{V} =: \varepsilon$ is a small quantity into powers of which we may expand the right hand side of eq. (15)

$$\begin{aligned} \Delta W/\overline{V} = \; & \varepsilon^0 (-1/2\ \delta a\ \delta b) \\ & + \varepsilon^1 (-\overline{a}\delta b + 1/2\ \overline{b}\delta a + \delta p) \\ & + \varepsilon^2 (\overline{a}\overline{b} + \gamma\overline{p}) \\ & + \varepsilon^2 (-1/4\ \delta a\ \delta b) \\ & + \varepsilon^3 \left(-1/4\ \overline{a}\delta b + 1/4\ \overline{b}\delta a + \frac{\gamma(2\gamma-1)}{12}\delta p\right) \\ & + \varepsilon^4 \left(1/4\ \overline{a}\overline{b} + \frac{\gamma(\gamma^2-\gamma+1)}{12}\overline{p}\right) \\ & + \varepsilon^4 (-1/16\ \delta a\ \delta b) \\ & + \varepsilon^5 \ldots\ldots \\ & \ldots\ldots \end{aligned} \tag{20}$$

Consistent with our later application, we now assume that all δ are small quantities of the same order as ε ("infinitesimal interchange"). Then the first three lines of the right hand side of eq. (20) are of the same order (namely ε^2), the next three lines are of order ε^4, and so on. We retain only the first three lines, replace δ by d to indicate the assumption of smallness, and omit the bar. This shortened expression for ΔW has as function of ε just one minimum which corresponds to the most unstable possible infinitesimal interchange:

$$dw := \text{Min}\,(\Delta W / V) = -\frac{1}{2}\,da\,db - \frac{1}{4}\,\frac{(-a\,db + 1/2\,b\,da + dp)^2}{ab + \gamma p} \geq 0 \quad \text{for stability} \tag{21}$$

This condition is not only necessary but also sufficient for stability against infinitesimal interchanges if fulfilled for all infinitesimally separated pairs of field lines. A surprising simplification is possible if we use the fact that the state before the interchange was one of magneto-hydrostatic equilibrium.

6)

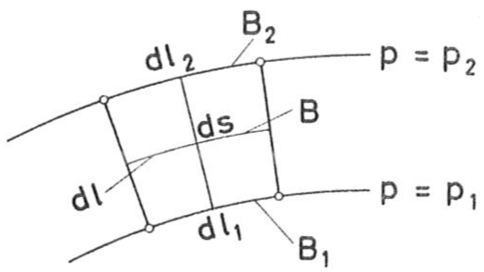

Fig. 2
Notation for equilibrium condition

To derive an equilibrium condition, we consider two neighbouring field lines 1 and 2 on the isobars p_1 and p_2. On these, we take line elements cut out by two neighbouring orthogonals to $\underline{B}$. The two line elements dl_1 and dl_2 (separated by the distance ds) define a surface element. In its plane we evaluate the equilibrium condition

$$\frac{1}{\mu}\,\text{curl}\ \underline{B} \times \underline{B} = \text{grad}\ p \qquad \text{as} \tag{22}$$

$$\frac{1}{\mu}\,\frac{B_1\,dl_1 - B_2\,dl_2}{dl \cdot ds} \cdot B = \frac{p_2 - p_1}{ds} \tag{23}$$

Multiplication by $dl \cdot ds / B$ yields

$$\frac{1}{\mu}\,(B_1\,dl_1 - B_2\,dl_2) = \frac{dl}{B} \cdot (p_2 - p_1)\,. \tag{24}$$

Integration along the lines of force is then possible and results in

$$dI = -q\,dp\,. \tag{25}$$

Since eq. (25) has to hold for all pairs of neighbouring field lines I, q, p must be functions of each other. These need not be single-valued though;for instance, if there exist two disconnected isobaric surfaces to the same pressure, then I may have different values on each of them. Except for such situations our previous assertion (in deriving eq. (19)) follows directly.

7) Using (25) in eq. (21) yields

$$dw = \frac{q^{-\gamma}}{b+\gamma q^2 p}\; db \cdot d(q^\gamma p) \tag{26}$$

with the ensuing criterion:

$$db \cdot d(q^\gamma p) \geq 0 \text{ for stability } (b = q \cdot I)\,. \tag{27}$$

The advantage gained by using the equilibrium condition goes far beyond the gain in formal simplicity. Since $q^\gamma p$ is a function of b, we need only know the sign of the slope

$$d(q^\gamma p)/db \tag{28}$$

of that function of one variable for all values of b occuring (if it is not single-valued, then on all branches of it) to determine whether or not the equilibrium is stable against infinitesimal exchanges. If we had not had information on the equilibrium we would have had to apply (21) on the whole infinitesimal neighbourhood of each line of force separately.

We obtain a useful transformation of condition (27) by introducing q as independent variable:

$$\frac{dqI}{dq} \cdot \frac{dq^{\gamma}p}{dq} = q^{\gamma-2}\left(-q^2\frac{dp}{dq} + I\right)\cdot\left(q^2\frac{dp}{dq} + \gamma q\right) \tag{29}$$

[$\geq$ 0 for stability].

Since both I and γq are non-negative, one of the brackets at most can be negative. For stability both have, therefore, to be positive, giving the equivalent criteria:

$$\left.\begin{matrix} dq \cdot dq\, I \\ dq \cdot dq^{\gamma} p \end{matrix}\right\} \text{ both } \geq 0 \text{ for stability} \tag{30}$$

$$-\gamma p \leq q\frac{dp}{dq} \leq \frac{I}{q} \qquad \text{for stability} \tag{31}$$

$$\gamma q p \geq q\frac{dI}{dq} \geq -I \qquad \text{for stability} \tag{32}$$

The first line of (30) and the right hand inequalities of (31) and (32) correspond to the interchange of flux tubes of equal volume, the other inequalities to the interchange of flux tubes of equal flux. We therefore find that stability under these two simple exchanges is not only necessary but also sufficient for stability under all infinitesimal interchanges+). Furthermore, it is evident that the attainment of the condition in the small (particularly in the form of the criterion (30)) is sufficient for the fulfilment of the necessary conditions (18) and (19) in the large, as long as the functions q, b, p are uniquely related.

The author wishes to express his thanks for many helpful discussions to Dr. K.U. v. Hagenow.

+) That this should be so, was suggested by Dr. v. Hagenow. He also drew my attention to the fact, that a condition equivalent to eq. (31) has been derived from the general energy principle by B.B. Kadomtsev (in Plasma Physics, ed. Leontovich, Pergamon 1960).