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CONNECTION BETWEEN CORRELATIONS AND FLUCTUATIONS IN A PLASMA

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ABSTRACT:

The space time fluctuations of the density in a plasma is obtained by introducing in the Vlasov equation an initial distribution describing the microscopic fluctuations at a given time. The statistical description of this initial condition is entirely provided by the two particles correlation function H (k,u) of the Bogoliubov-Lenard theory. The result can be considered as a generalisation of the Nyquist theorem: A static description of a fluctuating microscopic quantity plus a macroscopic equation describing a time evolution - gives the frequency spectrum of the microscopic quantity.

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INTRODUCTION

At thermal equilibrium the simplest way to derive the ω (frequency) k (wavenumber) spectrum of a fluctuating quantity - the electron density for example, is to use the Nyquist theorem. This theorem connects a "collective property" of the plasma - the transconductivity $\sigma(\omega,k)$ a quantity obtained through the Vlasov equation - with the microscopic fluctuations. If we introduce [1] the graininess parameter $g = 1/nD^3$ (n density, D Debeye length) σ is a quantity of order g^{1} while the fluctuations are of order g^{0}. The connecting coefficient is the temperature Θ , a quantity of order g^{-1} in the theory. We have

EQUATIONS

(1) $$\langle |J(k,\omega)|^2\rangle = \frac{\Theta}{\pi}\,\sigma(\omega,k)$$

$$\sigma(\omega,k) + i\,\phi(\omega,k) = \frac{\omega}{4\pi i}\,\frac{1-\epsilon}{\epsilon}\,.$$

The form of ϵ , the well known dielectric constant, will be defined later. Two questions can be considered. The first is to extend the fluctuation-dissipation theorem (as the Nyquist theorem is sometimes called) to non equilibrium situation - more exactly to situations where we can consider a metaequilibrium. The second is to try to understand (1) and see why and how we need, essentially, to solve the Vlasov equation to obtain the frequency spectrum of the fluctuations.

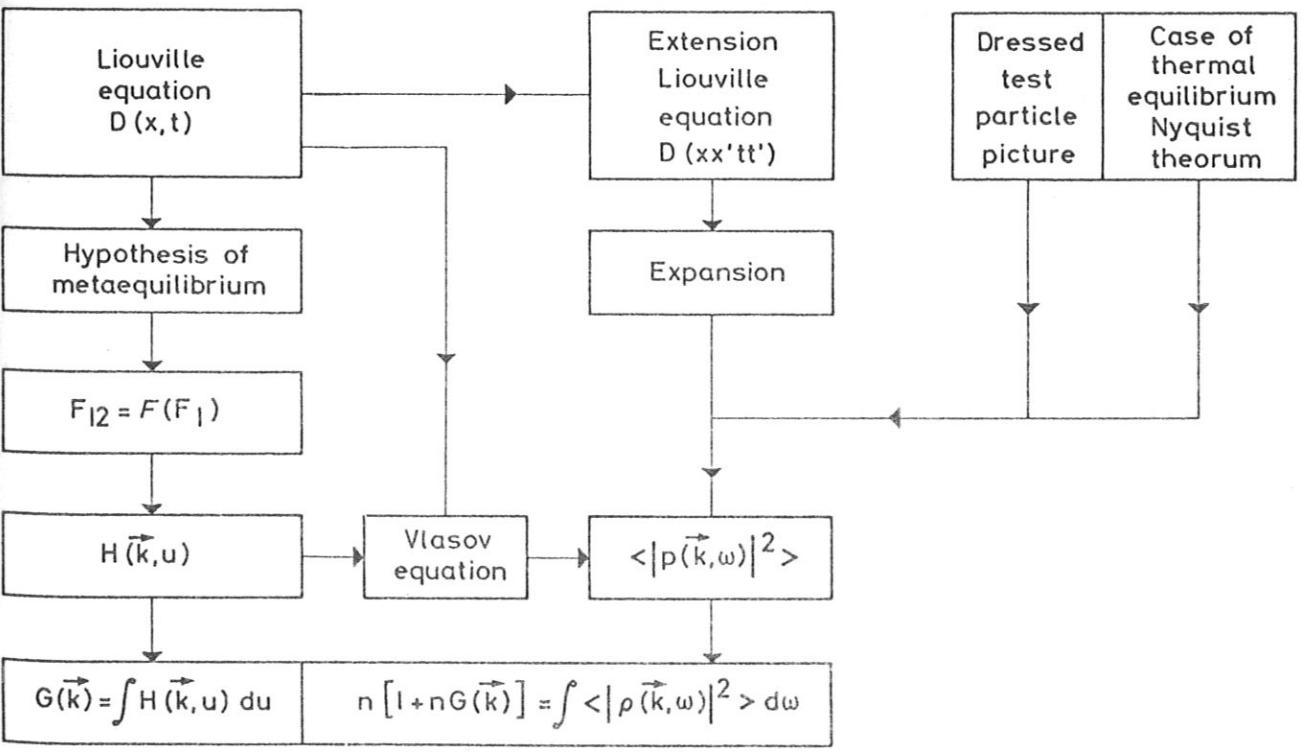
Hagfors [2] deals with the second question. Considering the plasma at thermal equilibrium, to get $\langle|\rho(k,\omega)|^2\rangle$ he introduced as initial conditions in the Vlasov equation a quantity describing statistically the fluctuations at a given time. He takes as "average" initial conditions $f^0(k,u)$

$$f^0(k,u) = F(u)\,\rho(k)$$

i.e. he assumes that the spatial density fluctuations are functions of the velocities of the individual particles. Finally the statistical average $\langle|\rho(k)|^2\rangle$, for a one species plasma - is taken equal to $n k^2 D^2/(1+k^2 D^2)$, a result given by Bohm and Pines [3]. Rostoker [4] deals with the two questions: by an extension of the Bogoliubov technique applied to a "Superliouville" equation he gets $\langle |\rho(k,\omega)|^2 \rangle$ outside thermal equilibrium. Also he gives a picture of the phenomena: Dressing a test particle with its asymptotic cloud and treating the test particle plus its cloud as a free quasiparticle he gets

$\langle |\rho(k,\omega)|^2 \rangle$. See also [5].

Here we deal with a generalization of the Hagfors method outside equilibrium. In so doing an interesting and physically meaningful relation is obtained between the two particle correlation function and the space time fluctuations. For simplicity we treat a one species electron plasma with a neutralizing smeared motionless background.



Correlations and Fluctuations Frequency Spectrum

Figure 1 summarizes the different theoretical descriptions obtained up to now. Essentially we have to consider two different aspects of the problem.

1. A "static" one. We write

(2) $$F_{12}(1,2) = F_1(1)\,F_1(2) + G(1,2).$$

We suppose F_1 normalized $\int F_1\, d\underline{v}_1 = 1$. $G(1,2)$ describes the correlation for a uniform plasma where G is a functional of $F_1 = F$ and is a function of

$\underline{x}_1 - \underline{x}_2 = \underline{x}$, $\underline{v}_1$ and $\underline{v}_2$. Its Fourier transform is $G(\underline{k}, \underline{v}_1, \underline{v}_2)$. An interesting quantity is the function $H(\underline{k}, u)$ obtained by integration of $G(\underline{k}, \underline{v}_1, \underline{v}_2)$ on $d\underline{v}_2$ and on the two components of $\underline{v}_1$ perpendicular to $\underline{k}$. The component parallel to $\underline{k}$ is called u.

$$H(\underline{k}, u) = \int G(\underline{k}, \underline{v}_1, \underline{v}_2)\, d\underline{v}_2\, d\underline{v}_{1\perp}.$$

Lenard [6] gave the integral equation for $H(\underline{k}, u)$. In [7-9] its solution is developed by different methods.

$G(\underline{k}) = \int H(\underline{k}, u)\, du$ gives the spatial part of the correlation function.

If we consider a fluctuating quantity such as

$$\rho(\underline{k}, t) = V^{-1} \sum_{i=1}^{N} \exp(i\, \underline{k} \cdot \underline{x}_i(t)),$$

the average value of $\langle |\rho(\underline{k}, t)|^2 \rangle_t = \langle |\rho(\underline{k})|^2 \rangle$ is connected to $G(\underline{k})$:

(3) $$\langle |\rho(\underline{k})|^2 \rangle = n + n^2 G(\underline{k}).$$

2. A dynamical one. We study not only $\langle |\rho(\underline{k}, t)|^2 \rangle_t$ but also $\langle \rho^*(\underline{k}, t)\, \rho(\underline{k}, t+\tau) \rangle_t$. The frequency spectrum

is $\langle |\rho(\underline{k},\omega)|^2 \rangle$. The integral on ω gives the value for $\tau = 0$, i.e. $\langle |\rho(\underline{k})|^2 \rangle$ where

(4) $$\langle |\rho(\underline{k})|^2 \rangle = \int \langle |\rho(\underline{k},\omega)|^2 d\omega .$$

(3) and (4) already provide a connection between the two descriptions through integrations on $\underline{v}_1$, $\underline{v}_2$ and ω but, we are looking for a connection between less reduced quantities.

Now, in the full phase space of the system, we can pose an initial value problem for the Liouville equation

(5) $$\frac{\partial D}{\partial t} = - \{ \mathcal{H}, D \}$$

where $\{ \mathcal{H}, D \} = \sum_{j=1}^{N} \left(\frac{\partial \mathcal{H}}{m \partial \underline{v}_j} \cdot \frac{\partial D}{\partial \underline{x}_j} - \frac{\partial \mathcal{H}}{\partial \underline{x}_j} \cdot \frac{\partial D}{m \partial \underline{v}_j} \right)$

and $\mathcal{H}$ is the Hamiltonian, the solution of which gives us directly $\langle \rho^*(\underline{k}) \rho(\underline{k},t) \rangle$:

(6) Let $D = D_0 (1 + \rho^*(\underline{k})/n)$ for $t = 0$.

The formal solution of (5) is

(7) $$D = \exp\{-iLt\} D_0 (1 + \rho^*(\underline{k})/n),$$

where L is the Liouville operator

$$iLA(\underline{v}_i, \underline{x}_i) = \{ \mathcal{H}, A \} .$$

Let D_0 be the phase space density of a spatially homogeneous system (metaequilibrium). The expectation value of the density is now the wanted correlation function. Because of the spatial homogeneity, the contribution from D_0 vanishes and the part due to the initial perturbation is

$$\langle \rho(\underline{k}) \rangle_1 = \frac{1}{n} \int d\Gamma \exp\{-iLt\} \left(D_0 \rho^*(\underline{k}) \right) \rho(\underline{k})$$

(8)

$$= \frac{1}{n} \int d\Gamma \, D_0 \rho^*(\underline{k}) \exp\{iLt\} \rho(\underline{k})$$

$$= \frac{1}{n} \langle \rho^*(\underline{k}) \rho(\underline{k}, t) \rangle,$$

where $d\Gamma$ is the volume element of phase space. The second line follows if we remember that the operator $\exp\{iLt\}$ describes the natural motion of the system which leaves $d\Gamma$ invariant.

Obviously, the time correlation of any fluctuating quantity can be gained from the solution of a corresponding initial value problem for the Liouville equation. Indeed, Rostoker's [4] joint probability of finding the system in the state X at $t=0$ and in X' at time t leads just to the most general one of these.

Reduction to the Vlasov equation.

For the problem at hand, our restricted formulation is of course much easier to deal with for $\langle \rho(\underline{k}) \rangle_1$ is just an integral over the Fourier transform of $F^{(1)}$, the perturbed 1-particle function:

(9) $$\langle \rho(\underline{k}) \rangle_1 = n \int F^{(1)}(\underline{k}, \underline{v}, t) \, d\underline{v} \, .$$

From (6), we have initially

(10) $$F^{(1)}(\underline{k}, \underline{v}, 0) = \frac{1}{n} \left[F(\underline{v}) + n \int d\underline{v}' \, G(\underline{k}, \underline{v}', \underline{v}) \right],$$

where $F(\underline{v})$ and G are the 1-particle and correlation function of our metaequilibrium D_0.

To lowest order, this is an initial value problem for the linearized Vlasov equation and yields after Fourier-Laplace transformation

(11)
$$\langle \rho^*(\underline{k})\rho(\underline{k},t)\rangle = n\langle\rho(\underline{k})\rangle_1 = \frac{n}{2\pi}\int_{-\infty-i\delta}^{\infty-i\delta}\frac{d\omega}{\epsilon(k,\omega)}\exp(i\omega t)\int_{-\infty}^{+\infty}du\,\frac{f^{(0)}(k,u)}{i(\omega-ku)}$$

with

(12)
$$f^{(0)}(k,u) = \int\left[F(\underline{v}) + n\int d\underline{v}'G(\underline{k},\underline{v}',\underline{v})\right]d\underline{v}_\perp = F(u) + n\,H(k,u).$$

For unstable equilibria, the correlation function contains exponentially growing terms from the zeros of $\epsilon(k,\omega)$ in the lower ω -half plane, which places a limit on the time over which linearization is possible. Our result agrees with one recently given by Rostoker [10].

For stable equilibria, we may integrate along the real ω -axis and obtain the spectrum

(13)
$$\langle\rho^*(k)\rho(k,t)\rangle = \int d\omega\langle|\rho(k,\omega)|^2\rangle\exp i\omega t$$
$$\langle|\rho(k,\omega)|^2\rangle = \frac{n}{\pi}\,Re\left\{\frac{1}{\epsilon(k,\omega)}\int du\,\frac{f^{(0)}(k,u)}{i(\omega-ku)}\right\}.$$

Now we have

(14)
$$\int du\,\frac{f^{(0)}(k,u)}{i(\omega-ku)} = P\int du\,\frac{f^{0}(k,u)}{i(\omega-ku)} + \frac{\pi}{k}f^{(0)}(k,\omega/k),$$
$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k^2}\left[P\int\frac{k\,dF/du}{\omega-ku} + i\pi\,(dF/du)_{u=\omega/k}\right],$$

i.e. with (12), we find after writing $\xi = \omega/k$

$$\langle|\rho(k,\omega)|^2\rangle = \frac{n}{k\,|\epsilon|^2}\Big\{F(\xi) + n\,H(\xi) \tag{15}$$

$$+ \frac{\omega_p^2}{k^2}\int\frac{du}{\xi-u}\Big[\frac{dF}{du}\big(F(\xi)+nH(\xi)\big) - \frac{dF}{d\xi}\big(F(u)+nH(u)\big)\Big]\Big\}.$$

But $H(\xi)$ is given in [6] by the equation

$$H(\xi) = \frac{\omega_p^2}{k^2}\int\frac{du}{\xi-u}\Big[\frac{dF}{d\xi}\big(F(u)/n + H(u)\big) - \frac{dF}{du}\big(F(\xi)/n + H(\xi)\big)\Big], \tag{16}$$

so all the terms between $\{\ \}$ in (15) cancel except the first one, $F(\xi)$:

$$\langle|\rho(k,\omega)|^2\rangle = \frac{n}{k\,|\epsilon(k,\omega)|^2}\,F(\omega/k), \tag{17}$$

which is the usual result.

Now in the case of a Maxwellian distribution, $H(k,u) = F(u)\,G(k)$, the case treated by Hagfors [2], but we have shown that the derivation of $\langle|\rho(k,\omega)|^2\rangle$ really involves $H(k,u)$ and not $G(k)$, although the final result is formally the same.

Our method can be extended to higher order: As (8) is an exact relation, we only have to determine the perturbed 1-particle function to higher order, which requires in turn a better approximation to the 2-particle function. The initial value of the latter can again be obtained from (6), but with the possible exception of thermal equilibrium, the problem seems quite formidable.

CONCLUSIONS

Our main result is the establishment of a relation between $H(k,u)$ and $\langle|\rho(k,\omega)|^2\rangle$. It is a direct proof that the Vlasov equation provided with adequate initial conditions describing the microscopic fluctuations at a given time will give their frequency spectrum. In a sense, it is a generalization of the Nyquist theorem: with a static description of the microscopic phenomena and a "macroscopic" coefficient - or equation - describing the irreversible process - here the Laudan damping - a dynamic picture of the microscopic fluctuations is obtained. Recently Rostoker [10] has shown how the test particle picture was able to give also the function $G(\underline{k}, \underline{v}_1, \underline{v}_2)$. Here we deal with the reverse problem and show how the dynamical picture - usually provided through the motions of quasiparticles is recovered with the Vlasov equation.

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