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Excitation Level of Plasma Oscillations.

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Abstract:

In order to get more insight in the phenomena of plasma oscillations we compare the amplitudes of the different contributions in the free oscillation problem (poles due to the plasma and the initial conditions). We find that for small k only the Landau poles are excited and they will play the dominant role already at time t = 0 (but, may be, not indefinitely). In the problem of forced oscillations by a macroscopic electric field we also find that the Landau solutions are dominant for frequency close to ω_p. Finally the behaviour of the transient cloud around an initially correlationless test particle is shown to be dominated by the waves with a group velocity equal to the test particle velocity.

I. Introduction

The problem of plasma oscillations is an old one. After their discovery by Langmuir around 1928 a first theory was worked out by Langmuir himself. Finally the correct equation was given by Vlasov [1] and the correct treatment of this equation by Landau [2] who introduced the so called Landau damping. From this time (1946) many article have been written on this question, most of them supporting the existence of the damping and trying physical pictures of the phenomena and some pointing out difficulties which appear in the Landau treatment. Among the most interesting contributions an article by Van Kampen [3] focused the attention to the difficult problem of initial conditions - a point superficially treated in the Landau's paper (although Landau made very clear that this problem has to be studied). In [4] a class of initial conditions was precised, which, although peculiar, is probably one of the few, describing an experimental situation: in this case there is no more problem on initial conditions which is an interesting result. The purpose of this paper is to try again to broaden the subject and to show that the study of the poles of the dispersion relation (i.e. the frequency and the damping of the waves) has to be completed by the computations of the amplitudes of the different poles. This problem must be solved if we want to define experiments which exhibit the interesting phenomena.

This paper is divided in three parts. The first one is devoted to the problem of free oscillations, i.e. to the problem of the behaviour of a plasma after a perturbation has been introduced at initial time. The second part deals with the problem of forced longitudinal oscillations, a problem very similar to the problem treated in the second part of the Landau's paper, but in some respect corresponding to a more "experimental" situation. The third part considers again a problem with initial condition corresponding to the formation of the asymptotic cloud around an initially correlationless test particle.

II. Free oscillations

A) Poles of the dispersion relation

We are dealing with a uniform, infinite, electrons plasma with a smeared motionless neutralising background; ω_p is the plasma frequency; F(v) is the one component velocity distribution integrated on the two directions perpendicular to the given k vector. The double Fourier Laplace transform of the electrons density is given by the following expression where g(k, v) is the initial perturbation

$$
\begin{aligned}
& n(\vec{k}, s) = I(\vec{k}, s) / \mathcal{E}(\vec{k}, s) \\
(1) \qquad & I(\vec{k}, s) = \int g(\vec{k}, v)(s - ikv)^{-1} dv \\
& \mathcal{E}(k, s) = 1 + [i\omega_p^2/k] \int \left(\frac{dF}{dv}\right)(s - ikv)^{-1} dv
\end{aligned}
$$

Originally $\mathcal{E}$ and I are defined only for real $s > 0$ and have a cut off along the imaginary axis (or from $-ik\,a$ to $i\,k\,a$ if F and g have cut off for velocites $\pm$ a i.e. $F \equiv g \equiv 0$ if $|v| > a$). The Landau method consists in an analytical continuation of $\mathcal{E}$ and I for real $s < 0$ (we suppose that there is no pole of $\mathcal{E}$ for real $s > 0$ i.e. we suppose the distribution stable). Next we examine the poles of $\mathcal{E}$ and singularities of I and we compute n(k, t) by the residue method. Different results can be found

1. F(v) and g(k,v) are "nice" functions (for example $(v^2+a^2)^{-n}$ with n integer 2). The integrals are easily computed- the analytical combination is simple - the number of poles of $\mathcal{E}$ and sigularities of I is finite.

2. F(v) and g(k,v) are Maxwellian distributions. This fundamental case is, unhappily, more complicated. Research of the zeros of the dispersion relation involves the study of the error function for complex arguments.

3. F(v) and g(k,v) have a cut off. As pointed out in [4] the poles - for all wavenumbers - are very sensitive to

the value of the derivative d F/d v at the cut off. If $V_\phi = \frac{s}{ik} = \frac{\omega}{k}$ is the phase velocity of the wave and supposing that $V_\phi > a$ we get

$$(2) \qquad k^2 = -\omega_p^2 \int_{-a}^{a} \frac{dF/dv}{V_\phi - v}\, dv$$

If (dF/dv) $\neq 0$ ("abrupt" cut off) the integral in (2) can take value as large as we like and consequently for all k an undamped mode will appear. If k is very large V_ϕ has to be very close to **a** in order to satisfy the dispersion relation and, consequently, for very large k $\omega_k \approx k\, a$.

Of course such an undamped oscillation is unphysical if the cut off velocity is much larger than the mean square root velocity. We have to eliminate these poles.

The above considerations and similar ones expressed in [5] show that the positions of the poles introduced by the analytical continuation technique are too sensitive to the exact form of F(v). Also we would like to be able to say something for a general g(k,v) using only the properties of g(k,v) for real, physically meaning, velocities.
The key of the problem is that the different poles and singularities have very different importances. Some correspond to the mathematical treatment of the problem but have no physical meaning. Others describe a physical phenomena and play a fundamental role. Then we have not only to consider the poles but to compute the amplitude of the different excited waves.

B) General solutions for n(k,t)

We are going to treat in an unsymmetric way the singularities due to initial conditions and the poles of ε . The reason is that we refrain to take any analytical expression for

$g(k, v)$ and consequently we can really say nothing about the singularities of $I(k, s)$. But we know much better $F(v)$ - very often it will be a Maxwellian, and then we have to take advantage of the powerful technique of analytical continuation and of the results already obtained by mathematicians on error functions.

Consequently we consider $n(k, s)$ as the product of two functions. The first one is $I(k, s)$ (the inverse Laplace transform of which is easy to perform) and the second is $1/\mathcal{E}(k, s)$. The poles of the analytical continuation of $\mathcal{E}(k,s)$ are s_k^j Now $n(k,t)$ is the convolution product of the two inverse Laplace transforms. Another way to get this result which is more physical is to write.

$$(3) \quad g(k,v) = \int_{-\infty}^{\infty} \delta(v-\xi)\, g(k,\xi)\, d\xi$$

i.e. to consider $g(k,v)$ as the sum of beams. Now for the beam $\delta(v-\xi)$

$$(4) \quad n(k,s) = \frac{1}{\mathcal{E}(k,s)} \, \frac{1}{s - ik\xi}$$

The inverse Laplace transform of (4) is

$$n_\xi(k,t) = \frac{1}{\mathcal{E}(k, ik\xi)} \exp ik\xi t + \sum_j A_k^j \, s_k^j \, \frac{1}{s_k^j - ik\xi} \exp s_k^j t$$

$s_k^j A_k^j$ is the residue relative to pole s_k^j .

In the general case taking into account (3)

$$(5) \quad n(k,t) = \int_{-\infty}^{\infty} \frac{g(k,\xi) \exp ik\xi t \, d\xi}{\mathcal{E}(k, ik\xi)} + \sum_j A_k^j \exp s_k^j t \int_{-\infty}^{\infty} \frac{s_k^j \, g(k,\xi)\, d\xi}{s_k^j - ik\xi}$$

$$A_k^j = \left[s^{-1} \left(\frac{\partial \mathcal{E}}{\partial s} \right)^{-1} \right]_{s = s_k^j}$$

the calculation of A_k^j is given in Appendix I.

$$(6) \qquad A_k^j = \frac{1}{2}\left[1 - \frac{ds_k^j/dk}{s_k^j/k}\right] = \frac{1}{2}\left[1 - \frac{V_G}{V_\phi}\right]$$

$V_\phi = s_k/ik$ and $V_G = ds_k/idk$ are the phase and group velocities of the wave.

(5) shows that $n(k,t)$ is the sum of two contributions. The first is the integral on ξ and can be considered as a Fourier transform. A priori, nothing very general can be said about such contribution and especially we do not know the asymptotic behaviour. It is not surprising and is simply due to our refusal to consider the analytical continuation of $g(k,v)$ and $I(k,s)$. In the second term we get the different damped waves. To go further we have now to put a central restriction: namely we will be interested only in long wavelengths components.

C) Case of long wavelengths

We suppose k small (more precisely for a Maxwellian $kD \ll 1$). Then $\varepsilon(k, ik\xi)^{-1}$ goes to zero like $k^2/k^2 + \chi_\xi^2$ with

$$\chi_\xi^2 = \omega_p^2 \fint \frac{dF/dv}{\xi - v}\, dv + i\pi\omega_p^2 \frac{dF}{d\xi} = M_\xi + i N_\xi$$

$\fint$ means that we have to take the principal part of the integral.

Consequently we can state that the level of the first contribution in (5) goes to zero like k^2, still we have first to take care of the resonance for $\xi = \omega_k/k$ where $k^2 + M_\xi = 0$ we have to study

$$k^2 \int_{-\infty}^{\infty} \frac{\exp ik\xi t \; g(k,\xi)\, d\xi}{A(\xi - \omega_k/k) + iB}$$

with

$$A = \left.\frac{\partial M_\xi}{\partial \xi}\right)_{\xi = \omega_k/k} = 2k^3/\omega_k \qquad B = N_\xi\Big)_{\xi = \omega_k/k}$$

The integral is easily computed by the residue method: We find

$$2\pi i\, g(\omega_k/k)\, \exp i\omega_k t \quad \exp\left(k\frac{B}{A}t\right) A^{-1}$$

$k\frac{B}{A}$ is the Landau damping $-\gamma_k$. The contribution of the resonant particles is consequently $\pi i\, g(\omega_k/k)\, \frac{\omega_k}{k} \exp(-\gamma_k + i\omega_k)t$

We will assume that for $v \to \infty$

$v\, g(k,v)$ goes to zero faster than v^{-2}

Provided this reasonable constraint we can assume that the level of the first integral goes to zero like k^2. Of course we know nothing about its time behaviour except that it will eventually be destroyed by phase mixing. In the worst case (a δ shaped initial perturbation) this contribution will be undamped but at a very low level.

We look now to the second term in (5). If k goes to zero two of the poles go to $\pm i\omega_p$ (the Landau poles) and the other go to zero (see appendix II) like k.
For the Landau poles $V_G \to 0$ and $V_\phi \to \infty$ and accordingly to (6) $A_k^J \to 1/2$. On the other hand the integral $\int s_k\, g(k,\xi)(s_k - ik\xi)^{-1} d\xi$ indicates how the initial conditions match with the plasma resonance. If we take into account that $s_k = -\gamma_k + i\omega_k$ with $\gamma_k \to 0$ and $\omega_k \to \omega_p$ we can write

$$(7) \int \frac{s_k\, g(k,\xi)}{s_k - ik\xi}\, d\xi = \int \frac{g(k,\xi)\, d\xi}{1 - \xi/V_\phi} - i\pi\, \frac{\omega_k}{k}\, g(k, \omega_k/k)$$

$V_\phi \to \infty$ and we can expand $\left[1 - \xi/V_\phi\right]^{-1}$ becomes

$$(8) \int_{-\infty}^{\infty} g(k,\xi)\, d\xi + \frac{k}{\omega_k}\int_{-\infty}^{\infty} g(k,\xi)\, \xi\, d\xi$$

We have already supposed that the "resonance term" $[\omega_k/k]\, g(k,\omega_k/k)$ $\to 0$ much faster than k^2. Consequently the two terms of (8) are dominant.

What about the other poles? For small k the dispersion relation (see Appendix II) is:

$$(9) \qquad S_k^j = \alpha^j k + \beta^j k^3$$

α^j is equal to the thermal velocity multiplied by a factor of order unity; $\beta^j = V_T D^2 B$ (V_T is the thermal velocity, D the Debye length and B a constant). Carrying (9) into (6) we find that the level of excitation is very small $A_k^j = -\beta^j k^2 / \alpha^j + \beta^j k^2$ and is of order k^2.

Let us resume the results we get for small k.

The solution for the charge density is the sum of four contributions:

1) The contribution of the Landau poles $-\gamma_k \pm i\omega_k$ with $\omega_k \to \omega_p$ for $k \to 0$. They are excited at the level one and their contribution is dominant already at time $t = 0$. This contribution is:

$$n(k,t) = \left\{ n(k,0) \cos \omega_k t + \frac{1}{\omega_k} \left(\frac{dn_k}{dt} \right)_{t=0} \sin \omega_k t \right\} \exp - \gamma_k t$$

We notice that we need only $n(k,0)$ and $[dn(k,t)/dt]_{t=0} = ik\, J(k,0)$ (J is the current). Strictly speaking dn_k/dt is of first order in k and could be neglected. We keep it because we will consider later a special but interesting initial condition where $n(k,0) = 0$ (Excitation by an impulse of electric field between two plane grids). We consequently see that the long wavelengths are self consistent.

2) The contribution of the other poles. It is a complicated expression but the level of excitation is very small (it goes to zero like k^2 for a Maxwellian). Usually also the damping

of these poles is bigger than the Landau damping. If we have introduced (by a cut-off in the tail of the distribution) other (undamped) poles their level of excitation will be very small because of the form of the dispersion relation $\omega_k \approx ka$ and although they will, strictly speaking, give the asymptotic behaviour they take over at a so small level that they are not interesting.

In the two preceeding contributions the initial perturbation does not play an important role: the level of excitation is a quantity of order $n(k,0)$ multiplied by a factor independent of the initial condition which describes how much the plasma "likes" the waves.

3) The third contribution comes from the initial perturbations it can be considered as the sum of undamped waves and will be more or less slowly destroyed by phase mixing - depending critically on the shape of the perturbation. It may ultimately take over the Landau contribution but again - for long enough wavelengths - at a very low level of order k^2.

4) A contribution coming from the particles of the initial perturbation in resonance with the waves. We have to suppose that as k goes to zero $[\omega_k/k]\, g(k, \omega_k/k) \ll n(k,0)$ and more precisely is at least of order k^2. Then this contribution is quite negligible and, moreover, is Landau damped. This last requirement is the only one we have had to make on $g(k,v)$ and is quite reasonable in the case of small k.

This treatment completely justify the Landau claim that for small k only the first (Landau) pole has to be taken into consideration and moreover this property holds already at time $t=0$
On the other hand strictly speaking other contributions may take over but at negligible level and more precisely, the longer the wavelength the smaller is the level. For large k (as soon as γ_k is no more small before ω_p) the Landau poles loses its interest.

D) The double stream instability

It has been found (10) that for a velocity distribution as indicated fig.I. we get an instability for $k < k_M$.

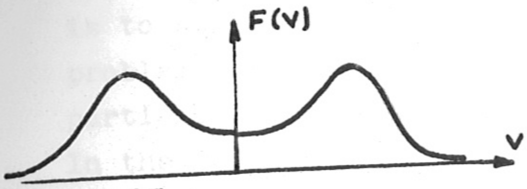

FIG I

This instability is nonconvective and the root of the dispersion relation is $s_k = \gamma_k + i\omega_k$ with $\omega_k = 0$
For $k \to 0$ $\gamma_k \to 0$ with $\gamma_k = \alpha k$
α is given by the equation

$$\int \frac{v\, dF/dv}{\alpha^2 + v^2}\, dv = 0$$

Due to the dispersionless character of the phase velocity of these waves they will be poorly excited. More precisely we have to find the next term of the approximation. It is easy to see that for small k

$$s_k = \alpha k + \beta k^3$$

β is given by $-2\beta\alpha\,\omega_p^2 \int v \left[\frac{dF}{dv}\right]\left[\alpha^2 + v^2\right]^{-2} dv = 1$

The solution is

$$n(k,t) = \frac{k^2}{\omega_p^2 \alpha^2 \int_{-\infty}^{\infty} \frac{v\,(dF/dv)}{[\alpha^2+v^2]^2}\, dv} \left[\alpha^2 \left\{ \int \frac{g(k,v)\, dv}{\alpha^2+v^2} \right\} + i\,\xi \left\{ \int \frac{v\, g(k,v)\, dv}{\alpha^2+v^2} \right\} \operatorname{th}\gamma_k t \right] \operatorname{ch}\gamma_k t$$

Consequently the long wavelengths will be excited ~~only~~ mainly through non linear interactions. Consequently it seems unlikely that the excitation found by Kofoid [6] with long wavelengths are due to double stream instability. Of course the other poles at the usual plasma frequency are fully excited.

III. Forced oscillations

In a previous paper [4] we noticed that there is practically only two ways to excite plasma oscillations. The first is to send a modulated beam in a plasma. At the limit the beam problem becomes the problem of an initially correlationless test particle suddenly created in a plasma (for example by β decay). In the next paragraph we will study how fast correlation are established. The second method is to apply between two grids distant of Δ an electric field E during a time interval τ. In [4] Δ was supposed to be much smaller than the Debye distance and the applied electric field was a pulse $E\tau\Delta = K$. We found a decrease of the electric field accordingly an $(\omega_p t)^{-1/2}$ law and a diffusion of a zone where the electric field was in phase. The dimension of this zone was growing like $t^{1/2}$.

From an experimental point of view such a signal is difficult both to produce and to measure, and the answer to a sinusoidal signal (after the transients have died) is a much easier quantity to measure. We will consequently consider the excitation signal $E \exp i\omega_0 t$ applied between two grids distant of Δ (with $\Delta \ll D$). We have already shown that the answer to the impulse during a very short time τ of the applied field E was $(2\pi)^{-1} \varepsilon(k,s)^{-1} E\tau\Delta$

Consequently the Laplace Fourier transform of the answer to a sinusoidal excitation is

$$E\Delta\,(2\pi)^{-1}(s-i\omega_0)[\varepsilon(k,s)]^{-1}$$

For the stationary problem only the poles $s = i\omega_0$ matters. After integration on k the other (plasma) poles give a contribution which decreases as $(\omega_p t)^{-1/2}$. We have to compute:

$$(10) \quad E(x) = \lim_{\gamma \to 0_+} \frac{E\Delta}{2\pi} \int_{-\infty}^{\infty} \frac{\exp -ikx \, dk}{\varepsilon(k, s = i\omega_0 + \gamma)}$$

To compute the integral let us consider the branch lines in the k plane (Fig.2). γ is a small positive quantity and will go ultimately to zero. Also we introduce on $F(v)$ a cut off at velocity $\pm a$ $\quad F(v) \equiv 0$ if $|v| > a$ - we will let a go to ∞ The path of integration is the real k axis. The branch cut off is indicated (Fig.2)

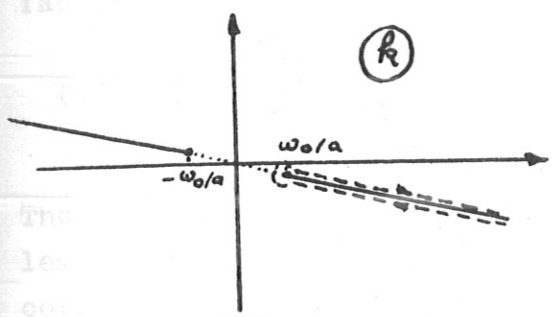

FIG II

In this problem we have no difficulty with the numerator of the integrand and we could expect to take fully advantage of an analytical continuation technique. Unhappily this is impossible because of the position of the branch cut with respect to the path of integration. The left part of the branch cut is above the integration path while the right part is under. If we remove the cut branch we consequently modify ε (say for $\mathrm{Im}.\,k < 0$ and we do not compute (10) see [7].

We consequently decide not to proceed to the analytical continuations but to change the path of integration as described Fig 2. So doing we have no more Landau-type poles but we can get a pole for $k = -i\chi$ if $\omega_0 < \omega_p$. Now on the two sides of the branch cut the real part of ε are equal and the imaginary part are opposite. Consequently (10) can be written

$$(11)\ E(x) = \text{eventual contribution of the pole} + \frac{E\Delta}{\pi}\int_0^\infty \frac{(-i)\,\varepsilon''}{|\varepsilon(k, i\omega_0)|^2}\, \exp - ikx \, dk$$

The general case is difficult to treat but we can get results in some extreme cases.

A) $\omega_0 \gg \omega_p$. There is no pole and, due to the fact that $\omega_0 \gg \omega_p$, for all values of k $|\varepsilon|$ is close to 1. We are from now going to specialize to the case of a Maxwellian distribution $F(v) = (2\pi)^{-1/2} v_T^{-1} \exp - v^2/2v_T^2$ We introduce $v = \omega_0/k$ as a new variable and we obtain:

$$(12) \quad E(x) = \frac{E\Delta\,\omega_p^2}{i\omega_0}\int_0^\infty \frac{dF}{dv}\, \exp - \frac{i\,\omega_0 x}{v}\, dv$$

For large x (12) is computed by the method of steepest descent (see Appendix III). We obtain within an unimportant phase factor

$$(13) \quad E(x) = \frac{E\Delta}{\sqrt{3}}\,\frac{\omega_p^2}{v_T\,\omega_0}\left(\frac{x\,\omega_0}{v_T}\right)^{1/3} \exp - \frac{3}{4}\left(\frac{x\,\omega_0}{v_T}\right)^{2/3} \exp - i\,\frac{3\sqrt{3}}{4}\left(\frac{x\,\omega_0}{v_T}\right)^{2/3}$$

The exponential factor is the same as in the second Landau problem [2] (but not the factor in front of it). The two problems correspond to two different excitations. Our type of excitation is somewhat simpler in an experimental apparatus. We make two remarks about the solution.

- It is a very heavily damped one (although the damping is not exponential).

- The fact that we identified $|\varepsilon|^2$ with 1 but kept ε'' in (12), corresponds to an expansion of the dielectric constant in s^{-2}.

We have

$$\varepsilon(k,s) = 1 + \frac{i\omega_p^2}{k}\int \frac{\partial F/\partial v}{s - ikv}\, dv = 1 - \omega_p^2 \int v\, \frac{dF/dv}{s^2 + k^2 v^2}\, dv$$

$$\varepsilon^{-1}(k,s) \simeq 1 + \omega_p^2 \int v\, \frac{dF/dv}{s^2 + k^2 v^2}\, dv$$

where we have taken into account that $F(v)$ is an even function

Then (10) is written

$$(14) \quad E(x) = \frac{E\Delta\,\omega_p^2}{2\pi}\int_{-\infty}^{\infty} v\,\frac{dF}{dv}\, dv \int_{-\infty}^{\infty} \frac{\exp - ikx\; dk}{s^2 + k^2 v^2}$$

The last integral in (14) is easily performed. We obtain

$$(15) \quad E(x) = \frac{E\Delta\,\omega_p^2}{s}\int_0^\infty \frac{dF}{dv}\, \exp - \frac{sx}{v}\, dv$$

(15) is identical to (12) if we replace s by $i\omega_0$

B) $\omega_0 > \omega_p$ and $\omega_0 - \omega_p \ll \omega_p$

We have no pole but a resonance in the k integration of (11) when the real part of ε is zero, i.e. for $k = k_0$ with

$$\omega_0^2/\omega_p^2 = 1 + \frac{3 V_T^2}{V_\phi^2} + \frac{15 V_T^4}{V_\phi^4} \qquad V_\phi = \frac{\omega_0}{k_0}$$

We will call $d\omega_0/dk_0 = V_G$ and $V_\phi = \omega_0/k_0$. We take the Taylor extension around $k = k_0$. One can show

$$\left.\frac{\partial \varepsilon'}{\partial k}\right|_{k=k_0} = -\frac{2}{\omega_0} \frac{V_G}{1 - V_G/V_\phi}$$

After some algebra we obtain the following result

$$E(x) = E\Delta \frac{i\omega}{2V_G} \exp -(\chi_0 + ik_0)x$$

χ_0 is the spatial Landau damping. It is related to the temporal Landau damping γ_0 (for a given k_0) by the very general relation : *See also Appendix 4*

(15) $V_G \chi_0 = \gamma_0$

The contribution of the resonance is very large and goes to $+\infty$ when $\omega \to \omega_p$ (as V_G^{-1} i.e. $(\omega_0^2 - \omega_p^2)^{-1/2}$). Of course for very large x a contribution varying as $\exp - \alpha x^{2/3}$ will ultimately takes over but at a very low level.

C) The next case to treat is $\omega_0 < \omega_p$ with $\omega_p - \omega_0 \ll \omega_p$ we get a pole for $k = -i\chi_0$ with

$$\omega_p^2 - \omega_0^2 = 3 V_T^2 \chi_0^2$$

The contribution of the pole is

$$E(x) = -\frac{\omega_0^2 E\Delta}{2\chi_0 3V_T^2} \exp - \chi_0 x$$

The damping is now much bigger but the excitation of the wave is still very high and goes to ∞ as $(\omega_p^2 - \omega_0^2)^{-1/2}$. The integral along the positive part of the k axis is difficult to perform, but is at much smaller level. As in the preceeding case for very large x it will give a contribution varying as

$\exp - \alpha x^{2/3}$ which will take over but at a very low and uninteresting level.

D) The last case is $\omega_0 \ll \omega_p$. This is probably the case where the nonlocal character of the dielectric constant (the fact that ε is not only a function of ω but also of k) brings the most interesting result.

To compute the contribution of the pole we can neglect the dynamic aspect of the dielectric constant and write $\varepsilon(\omega_0, k) \simeq \varepsilon(0, k) = 1 + \frac{1}{k^2 D^2}$. We get the pole at $k = -i D^{-1}$ and the contribution E_1 is

$$E_1 = - \frac{\Delta E}{2D} \exp - x/D$$

This solution is quickly damped. As we could guess the screening distance is the Debye length.

But in the integral on k we cannot completely neglect the dynamical aspect. More precisely, although ω_0 is very small compared to ω_p, for very long wavelengths the dielectric constant $\varepsilon(k, \omega_0)$ is different from the static approximation and small, but interesting effects will occur which disappear only in the strict limit $\omega_0 = 0$.

We get for the contributions of the integral along real $k > 0$

$$E \Delta \frac{i \omega_p^2 \omega_0}{\sqrt{2\pi}\; V_T^3} \int_0^\infty \frac{1}{k^3} \exp - \frac{1}{2} \frac{\omega_0^2}{k^2 V_T^2} \frac{1}{|\varepsilon|^2} \exp - ikx \, dk$$

We introduce $V = \omega_0 / k$ as new variable and integrate for large x by the method of steepest descent. Apart from a phase factor we get

$$(16) \quad E(x) = \frac{E\Delta}{\sqrt{3}} \frac{\omega_0^3}{\omega_p^2 V_T} \left[\frac{\omega_0 x}{V_T}\right]^{1/3} \exp - \frac{3}{4} \left[\frac{\omega_0 x}{V_T}\right]^{2/3} \exp - i \frac{3\sqrt{3}}{4} \left[\frac{\omega_0 x}{V_T}\right]^{2/3}$$

(16) has the same behaviour as (13) (obtained for $\omega_0 \gg \omega_k$); but now the damping coefficient is very small if ω_0 is small enough. Of course the level of excitation of such waves is small but still they will quickly dominate the Debye screened solution. Analog results have been reported for transmission of waves in a solid state plasma (8). Although the physics is quite different, the mathematics are similar and the anomalous transmission is also due to a nonlocal effect.

IV. Transient cloud around a test particle

If the preceding excitation was interesting for possible experiments, the problem of an initially correlationless test particle is very important from a theoretical point of view because this problem is closely connected to the question of relaxation of the two bodies correlation f_{12} to a functional involving only for f_1 according Bogolinbov hypothesis. The test particle problem is a simplification of this more general problem but in compensation is more tractable. Some results have been indicated in (8) and by Tchen (private communication).

We consider here a one dimensional plasma (i.e. plane sheets) with only one species of particle and a motionless neutralising smeared background. We will sketch briefly at the end how the results are modified when going from the one dimensional plasma to the real (3 dimensions) one.
The density $n(k,s)$ is given by:

$$n(k,s) = \frac{1}{s - ik\xi} \; \frac{1}{\varepsilon(k,s)}$$

ξ is the velocity of the test particle and the inverse transforms by:

$$n(k,t) = \frac{\exp ik\xi t}{\varepsilon(k, ik\xi)} + \sum_j A_k^j \frac{s_k^j}{s_k^j - ik\xi} \exp s_k^j t$$

$$(17) \quad n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp -ik(x-\xi t)}{\varepsilon(k, ik\xi)}\, dk + \frac{1}{2\pi}\sum_j \int_{-\infty}^{\infty} A_k^j \frac{s_k^j}{s_k^j - ik\xi} \exp[s_k^j t - ikx]\, dk$$

$$A_k^j = \frac{1}{s_k^j}\left(\frac{\partial \varepsilon}{\partial s}\right)^{-1}_{s = s_k^j}$$

The first integral in (17) is a function of $x - \xi t$ and describes the test particle plus its asymptotic cloud. The transient solution is given by the sum of all the plasma poles and the integration on k; we take into account the value of A_k^j previously obtained and we take a system moving with the velocity of the test particle $x = \xi t + y$. We get for $n^{tr}(y,t)$

$$(18) \quad 2\pi\, n^{tr}(y,t) = \sum_j \frac{1}{2}\int_{-\infty}^{\infty} \frac{s_k^j - k\frac{ds_k^j}{dk}}{s_k^j - ik\xi} \exp[(s_k^j - ik\xi)t] \exp -iky\, dk$$

We see that the transient time will be given by a combination of two effects: First the Landau damping which will decrease very quickly the small wavelength contributions. Secondly the phase mixing of different wavelengths will secure a damping also in the case where there is no Landau damping. What effect comes first, depends on the velocity of the test particle and we have to study more carefully the problem.

A) Case of a square distribution.

To get some insight we first treat a simple example which we have already used and which turned out to give rather general results. We take a square distribution $F(v) = (2a)^{-1}$ for $|v| < a$ and 0 elsewhere. Then for all wavelengths

$$\omega_k^2 = \omega_p^2 + k^2 a^2$$

There are only the two Landau poles witn no damping. We have to compute (R for real part)

$$(19)\ R\left[\frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega_k}\ \frac{\exp i(\omega_k - k\xi)t}{\omega_k - k\xi}\ \exp -iky\ dk\right]$$

Exact integration can be obtained for $\xi = 0$. Asymptotic results in the general case.

For $\xi = 0$

$$(20)\quad n^{tr}(y,t) = \frac{\omega_p}{2a}\int_{\omega_p t}^{\infty} J_0\left(u^2 - y^2\omega_p^2/a^2\right)^{1/2} du \text{ for time } t > y/a$$

(20) will be useful as a check of the asymptotic theory. To compute (19) for $\xi \neq 0$ let us first make y = 0. We suppose that $\xi < a$.

We plot $\Omega_k = \omega_k - k\xi$ as a function of k (see Fig 3). For a value k_ξ Ω_k has a minimum Ω_ξ. In an integral such as

$$\int A_k \exp i\Omega_k t\ dk$$

the asymptotic solution is obtained by the stationary phase integration method. In such a method we only take into account the wavenumbers in the neighbourhood of the minimum of Ω_k which is given by

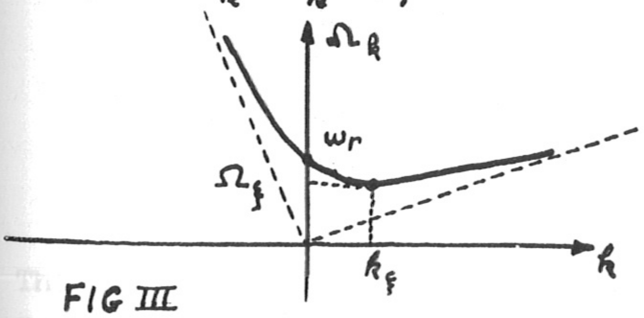

FIG III

$$\frac{d\Omega_k}{dk} = \frac{d\omega_k}{dk} - \xi = 0$$

These waves with a group velocity equal to the velocity of the test particle play the dominant role in the damping of the transient cloud. The fact that the group velocity and not the phase velocity of the plasma waves matters, could have been suspected in a transient phenomena. We find as solution

$$n^{tr}(y=0,t) = (2\pi\chi_\xi t)^{-1/2}\ \frac{\omega_p^2}{\omega_{k_\xi}\Omega_\xi}\ \cos\left(\Omega_\xi t + \frac{\pi}{4}\right)$$

with $k_\xi a = \omega_p[\xi/a][1 - \xi^2/a^2]^{-1/2}$ $\quad \omega_{k_\xi} = \omega_p\left[1 - \xi^2/a^2\right]^{-1/2}$

$$\Omega_\xi = \omega_p \left[1 - \xi^2/a^2\right]^{1/2} \qquad \chi_\xi = \left.\frac{d^2\omega_k}{dk^2}\right)_{k_\xi} = a^2/\omega_p \left[1 - \xi^2/a^2\right]^{3/2}$$

To get the general solution $y \neq 0$ we write

$\xi t + y = (\xi + \Delta\xi)t$ with $\Delta\xi = \frac{y}{t}$ and y/t should go to zero

$$(21) \quad \underset{t\to\infty}{n^{tr}(y,t)} = \frac{1}{(2\pi\chi_\xi t)^{1/2}} \cos\left[\Omega_\xi t + \frac{d\Omega_\xi}{d\xi} y + \frac{1}{2}\frac{d^2\Omega_\xi}{d\xi^2}\frac{y^2}{t} + \frac{\pi}{4}\right]$$

We introduce the dimensionless variable

$$\omega_p t = T \qquad y\omega_p/a = Y \qquad \alpha = \left[1 - \xi^2/a^2\right]^{1/2}$$

$$(22) \quad \underset{T\to\infty}{n^{tr}(Y,T)} = \frac{\omega_p}{a}\frac{1}{\left[2\pi\alpha^3 T\right]^{1/2}} \cos\left[\alpha T - \frac{\sqrt{1-\alpha^2}}{\alpha} Y - \frac{1}{2\alpha^3}\frac{Y^2}{T} + \frac{\pi}{4.}\right]$$

This solution should be compared with the asymptotic solution
In this case

$$n^{as}(Y) = \frac{\omega_p}{a}\left[\delta(Y) - \frac{1}{2\alpha}\exp - \frac{Y}{\alpha}\right]$$

We consequently find that in this special case (A square distribution)

- Only the phase mixing of wavelengths with a group velocity equal to the test particle velocity secure the damping of the transient,

- The transient could, for this one dimensional plasma, goes to zero as $t^{-1/2}$(The corresponding law is $t^{-3/2}$ for the 3 dimensional plasma)

- Near the origine the transient cloud is negligible after a time $\tau \simeq \omega_p^{-1}\left[1 - \xi^2/a^2\right]^{-1/2}$

Of course this result is interesting only if we can generalise it.

B) General distribution: Case of small velocity

What was missing in the preceding example was the Landau damping and the existence of poles others than the Landau ones. But if the group velocity for $k = k_\xi$ is much smaller than the thermal velocity and, consequently, if $\xi \ll v_T$ we can expect that the Landau damping will play no role. More precisely let us plot (Fig 4) Real and Imaginary part of $\phi = s_k - ik\xi$ For $\xi \ll v_T$ the minimum of $\mathrm{Im}\,\phi$ occurs for a value of k where the Landau damping is so small that the wavenumber phase mixing is the dominant phenomena and brings a $t^{-1/2}$ law for the damping of the transient. Then the contribution of the two Landau poles is exactly given by (21) and (22) where we have just to replace a^2 by $3v_T^2$ ($3\theta/m$ in the Maxwellian case). Of course we have to compute the contribution of the poles others than the Landau ones. Then we can forecast that the asymptotic behaviour will be given by the integration on small wavenumbers because of the quick increase of the damping with k (proportional to k). We seek only the solution for We write:

$$s_k^j = \alpha_k^j k$$

and the contribution of j^{th} pole is

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{-k\,[d\alpha_k^j / dk]}{\alpha_k^j - i\xi} \exp kt(\alpha_k^j - i\xi)\, dk$$

For small k $\alpha_k^j = \alpha^j + \beta^j k^2 + \dots$ if $k > 0$ and $-\alpha^j - \beta^j k^2 \dots$ if $k < 0$ α^j and β^j are complex numbers. We find that the contribution of this pole is

$$2\beta^j t^{-3}\left[(\alpha^j - i\xi)^{-4} + (\alpha^j + i\xi)^{-4}\right]$$

This expression is valid for all velocities ξ.

We consequently see that the contributions of the other poles, after the k integration, vary as t^{-3}. They are very small because -

- The large damping (proportional to k) of these poles
- The poor excitation of the long wavelengths.

C) Case of large velocities

We have already estimated the contribution of the poles others than the Landau ones. For those poles the method of the saddle point integration does not work any more. We can compute k such that $ds_k/dk = i\xi$ and get an asymptotic behaviour $t^{-1/2} \exp(s_{k_\xi} - ik_\xi \xi)t$ but this term is quickly damped because for large ξ the solution of $ds_k/idk = \xi$ brings a large k and a large damping; consequently the main contribution is no more given by these wavenumbers whose group velocity equal the velocity of the particle. In fact there is no more wavenumber playing a dominant role. One could expect that these wavenumbers which cancel the real part of $s_k - ik\xi$ (with a very small imaginary part), i.e. the wavenumbers around $k = \omega_p/\xi$ with a phase velocity equal to the test particle velocity, will play the dominant role. But this is not true. If we neglect in (18) the term of order k^2 we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\omega_p}{\omega_p - k\xi + i\gamma} \exp i(\omega_p - k\xi)t \; dk = 0$$

because the pole $k = \omega_p/\xi + i\gamma/\xi$ does not contribute. There is no other possibility but to perform the integral for all wavelengths or at least for these wavelengths which are not too heavily damped.

As we can see in Appendix III the possibility of finding easily the asymptotic form of the transient cloud for small ξ comes from the fact that we have been able to find a contour in the complex plane of k which replaces the real axis, taking only into account the developpement around k = 0 of $\omega_k = \omega_p + [\alpha/2] k^2$ Then it turned out that the leading contribution comes from a zone where our approximation is still valid (and consequently the method is consistent). We cannot do that here. The saddle point ξ/α being for large ξ much larger than the inverse of the Debye length.

The saddle point method would work if we could find a new possible contour going for example through k = 0. Such a case is impossible except if the damping is proprtional to k.

$$s_k = i\omega_k + \gamma_k \qquad \text{with } \gamma_k = -|k|\, a$$

A distribution giving such a dispersion relation is $F(v) = [a/\pi][a^2 + v^2]^{-1}$ but such a distribution is very objectionable because of an infinite amount of kinetic energy.

Nevertheless in this case the terms $-ik\xi$ and $-ka$ combine to give as possible path going through k = 0: the two straight lines (see Fig 4) $k = k' + ik''$

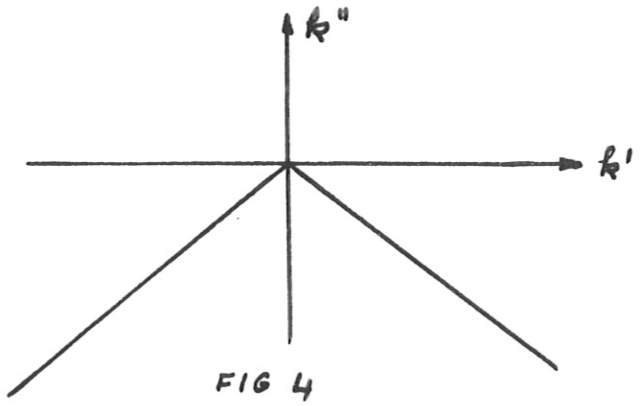

FIG 4

$$k'' = -k' \frac{\xi}{a} \quad \text{if } k' > 0$$

$$k'' = k' \frac{\xi}{a} \quad \text{if } k' < 0$$

On the path $s_k - ik\xi = i\omega_p - k' \frac{\xi^2 + a^2}{a}$ and for large t the k' integration give a t^{-1} dependence for the transient cloud. The absence of any dispersion relation for the Imaginary part of s_k leads to a wrong result for small ξ. The possibility of generalising the obtained above dependence for large ξ is an entirely open question.

The problem of three dimensional plasma can be treated in the same way. Simple dimensional analysis of the integrals shows a decay law in $(\omega_p t)^{-3/2}$ for small velocity test particle.

Appendix I

We compute $s^{-1}\left[\partial\varepsilon/\partial s\right]^{-1}$ for $s = s_k$

We have

$$(23)\quad \left.\frac{\partial\varepsilon}{\partial s}\right)_{s=s_k} = -\frac{i\omega_p^2}{k}\int\frac{\partial F/\partial v}{(s_k - ikv)^2}\,dv$$

We compute the integral I in (23) by deriving with respect to k the dispersion relation (s_k is there considered as a function of k)

$$-k/i\omega_p^2 = \int(\partial F/\partial v)(s_k - ikv)^{-1}\,dv$$

$$\frac{1}{i\omega_p^2} = \frac{ds_k}{dk}\int\frac{\partial F/\partial v}{(s_k - ikv)^2}\,dv - i\int\frac{v[\partial F/\partial v]\,dv}{(s_k - ikv)^2} = \frac{ds_k}{dk}\,I + \frac{1}{k}\int\frac{\partial F/\partial v}{s_k - ikv}\,dv - \frac{s_k}{k}\,I$$

The integral on the right handside is $-k/i\omega_p^2$. We obtain

$$s_k^{-1}\left(\frac{\partial\varepsilon}{\partial s}\right)^{-1}_{s=s_k} = \frac{1}{2}\left(1 - \frac{ds_k/dk}{s_k/k}\right)$$

Appendix II

Taking into account $(s - ikv)^{-1} = \int_0^\infty \exp -st \exp ikvt\,dt$
the dispersion relation can be written after some transformations

$$k^2 + \omega_p^2\int_0^\infty \xi \exp -\left(\frac{s}{k}\xi\right) d\xi \int_{-\infty}^{\infty} F(v) \exp iv\xi\,dv = 0$$

In the case of a Maxwellian the last integral is $\exp - \xi^2 v_T^2/2$
where v_T^2 is the mean square velocity θ/m . We obtain

$$k^2 + \omega_p^2\int_0^\infty \xi \exp - \frac{\xi^2 v_T^2}{2} \exp - \frac{s_k}{k}\xi\;d\xi = 0$$

Or introducing $K = kD$ and $s_k = s_k/\omega_p$.

$$K^2 + \int_0^\infty \xi \exp - \frac{\xi^2}{2} \exp - \frac{s_k}{K} \xi \, d\xi = 0$$

The last integral has a meaning also for real $s < 0$. It gives the analytical continuation of the dispersion relation. If $K \to 0$ the integral should also go to zero. If $s_k/K \to \infty$ the integral goes to zero like K^2 and we recover the plasma poles $s = \pm i$. It is impossible that $s_k/K \to 0$ (the value of the integral in this case is $\sqrt{\pi/2}\ \ V_T^{-1}$). We suppose now that s_k/K goes to a limit α and we try the expansion

$$s_k/K = \alpha + \beta K^2$$

Developping $\exp -(s_k/K)\xi = \exp - \alpha\xi (1 - \beta K^2 \xi)$ we obtain

$$\int_0^\infty \xi \exp - \alpha\xi \exp - \frac{\xi^2}{2} d\xi + k^2 \left(1 - \beta \int_0^\infty \xi^2 \exp - \alpha\xi \exp - \frac{\xi^2}{2} d\xi\right) = 0$$

α is given by the solution of

$$(24) \quad \int_0^\infty \xi \exp - \alpha\xi \exp - \frac{\xi^2}{2} d\xi = 0$$

β is given by $\beta = \left(\int_0^\infty \xi^2 \exp - \alpha\xi \exp - \frac{\xi^2}{2} d\xi\right)^{-1}$

To study eq.(24) we use the tables of the error function computed by Fried and Conte [9]. After some manipulations equation (24) is shown to be equivalent to

$$(25) \quad 1 + \theta Z(\theta) = -\frac{1}{2} \frac{dZ}{d\theta} = 0$$

with $\alpha = -i\sqrt{2}\,\theta$ and $Z(\theta) = 2i \exp - \theta^2 \int_{-\infty}^{i\theta} \exp - \mu^2 d\mu$

$Z(\theta)$ and $\frac{dZ}{d\theta}$ are tabulated in [9]. Fig 5 shows in the complex plane $\theta = x + iy$ the curves Real $dZ/d\theta = 0$ and $\mathrm{Im}\ dZ/d\theta = 0$
There is an infinity of solutions to equation (25). The less damped corresponds to $\alpha = -1.74 \pm i\,3.61$ and $\alpha = -2.88 \pm i\,4.45$

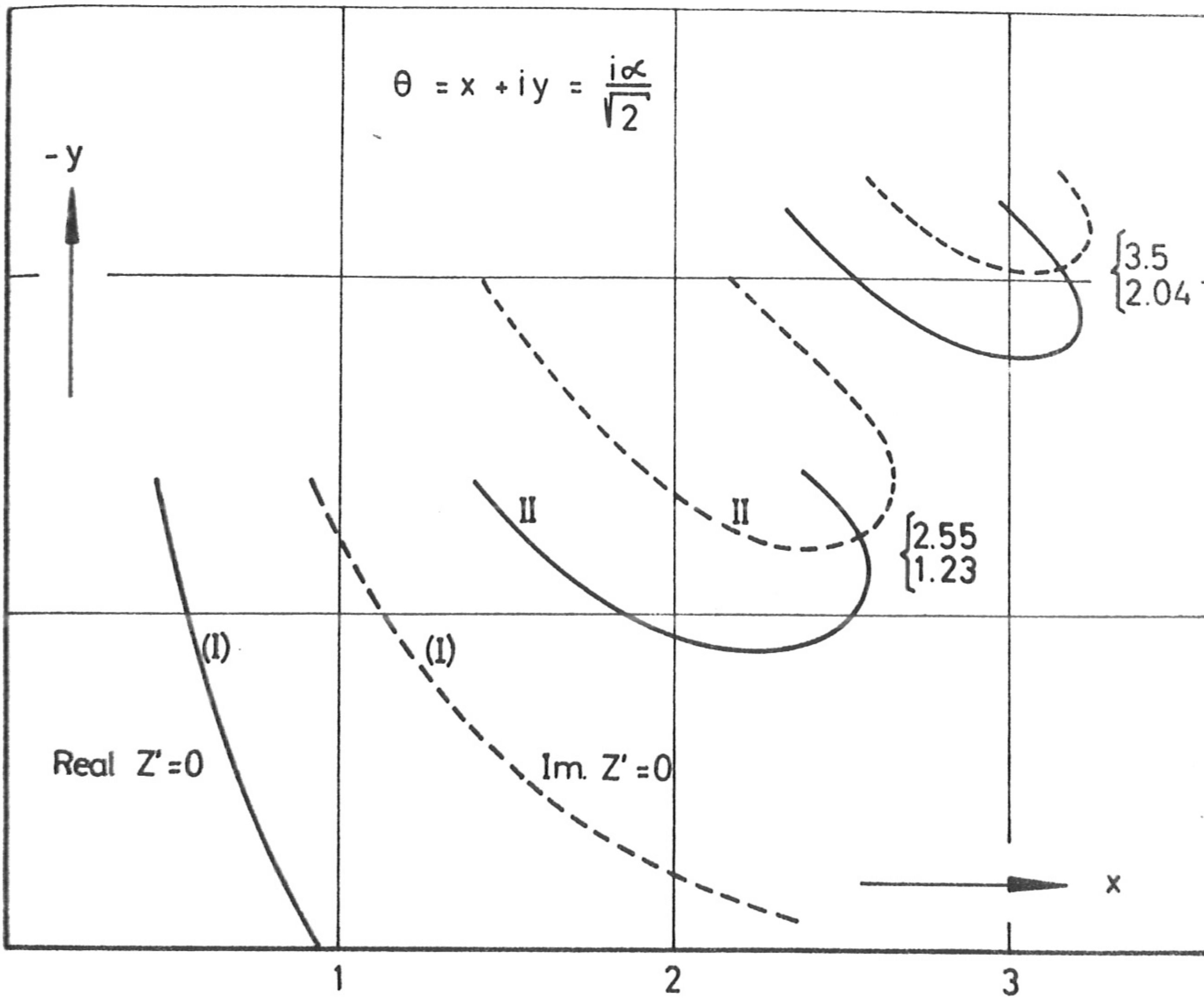

Fig. 5

Appendix III

Computation of the transient cloud for small ξ

We apply the general saddle point method to the computation of the integral in (18). For small ξ we think that only the small wavelengths are important. If this is true we can write

$$s_k = -\gamma + i\omega_p + \frac{\alpha}{2} k^2$$

with $\alpha = 3v_T^2/\omega_p$ γ is an arbitrary small and positive number then for the Landau pole $i\omega_p$ the integral in (18) is written

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\omega_p - [\alpha/2]k^2}{\omega_p + [\alpha/2]k^2 - k\xi} \exp i\left(\omega_p + \frac{\alpha}{2}k^2 - k\xi\right)t \; dk$$

We write $k = k' + ik''$. The pathes in the complex k plane are such that

$$\text{Real}\left[\omega_p + \frac{\alpha}{2} k^2 - k\xi\right] = \omega_p - \frac{\xi^2}{2\alpha}$$

are the two straight lines $\pm k'' = k' - \frac{\xi}{\alpha}$ (see Fig.6)

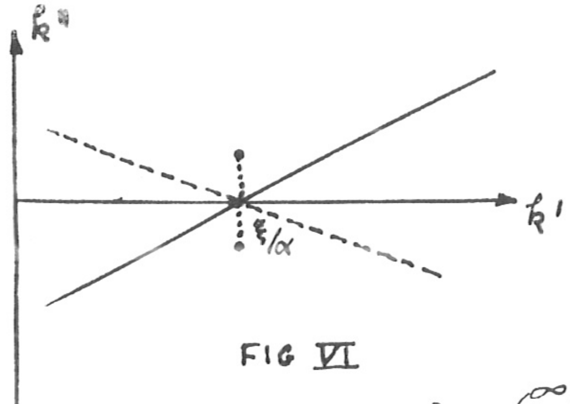

FIG VI

The poles are $k = \xi/\alpha \pm i\gamma$
Conditions at $|k| \to \infty$ select the line $k'' = k' - \xi/\alpha$
The poles give no contribution to the integral which is

$$(26) \quad \frac{1+i}{2} \exp i\left(\omega_p - \frac{\xi^2}{2\alpha}\right)t \int_{-\infty}^{\infty} \frac{(\omega_p - \xi^2/2\alpha) - k'\xi - i\alpha k'^2 - ik'\xi}{\omega_p - \xi^2/2\alpha + i\alpha k'^2} \exp -\alpha k'^2 t \, dk'$$

For large t only the region around $k' = \frac{\xi}{\alpha}$ matters and the integral in (26) is just $\sqrt{\pi/\alpha t}$. Taking into account the contribution of the other poles we get

$$n^{tr}_{t \to \infty}(y=0,t) = (2\pi\alpha t)^{-1/2} \cos\left[\left(\omega_p - \frac{\xi^2}{2\alpha}\right)t + \frac{\pi}{4}\right]$$

Appendix IV

We should be careful to compute the damping

$$\gamma_0 = \omega_p \sqrt{\frac{\pi}{8}}\,(k_\omega D)^{-3}\, \exp - \frac{1}{2V_T^2}(\omega/k_\omega)^2$$

due to the fact that $k_\omega \to 0$ it is not sufficient to use the approximate formula $k_\omega^2 = (\omega^2 - \omega_p^2)[3V_T^2]^{-1}$ to get the exact value of the term $\exp -[1/2V_T^2][\omega/k_\omega]^2$. We need the next term in the dispersion relation.

$$1 - \frac{\omega_p^2}{\omega^2} - 3\left[\frac{kV_T}{\omega}\right]^2 \frac{\omega_p^2}{\omega^2} - 15\left[\frac{kV_T}{\omega}\right]^4 \frac{\omega_p^2}{\omega^2} = 0$$

and the term

$$\frac{1}{2V_T^2}\,\frac{\omega^2}{k_\omega^2} = \frac{3}{2}\,\frac{\omega^2}{\omega^2-\omega_p^2} + 1 = \frac{3}{2}\,\frac{\omega_p^2}{\omega^2-\omega_p^2} + \frac{5}{2}$$

In the same way in the computation of γ_k for a given k, the term $(2V_T^2)^{-1}\,\omega_k^2/k^2 = [1/2]\,(k^2D^2)^{-1} + 3/2$.

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