INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

On the Influence of a Tensor Friction upon Diffusion in a Magnetic Field.

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I. Introduction

In a previous paper[1], Kurşunoglu discussed the effect of a constant tensor friction on the motion of a particle in a plasma. The friction tensor consisted of two parts, one due to the rotation of the particle orbits by the magnetic field, and the other due to the average deviation suffered by a particle colliding with the other particles in the plasma. In reference (1) this "dynamical friction" was assumed to be constant in time. This may be a satisfactory approximation if one wishes to treat the decelleration of a single fast particle injected into a plasma, or if the main contribution to the decelleration is due to collisions with a neutral background. But it is less satisfactory if one wishes to treat the randomization of a typical particle of the plasma due to Coulomb scattering. In the appendix, it is shown that, for a weak magnetic field, the friction tensor depends critically on the velocity distribution function. Therefore, it may be expected that the time scale in which the particle is "thermalized" should be of the same order as that in which the plasma velocity distribution approaches a Maxwellian.

II. The Equations of Motion

The equation of motion of a plasma particle in a constant magnetic field along $\hat{z}$ is:

$$\frac{d|P\rangle}{dt} = -\Lambda|P\rangle + |F(t)\rangle ,$$

$$K_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (1)$$

where

$$\Lambda(t) = f(t) - i\omega_c K_3 ,$$

f(t) is a symmetric matrix, representing the average effect of the other particles on a test particle due to Coulomb scattering. The form of this term is discussed in greater detail in the appendix. In general if the net momentum and angular momentum of the system is zero, the trace, determiant and determinants of the principal minors of f(t) must be greater than zero, in order for the system to tend toward an equilibrium. These conditions are sufficient for the eigenvalues of f(t) to be positive. The term K_3 represents the contribution of the magnetic field and

$$|F(t)\rangle = e|\mathcal{E}(t)\rangle \qquad (2)$$

represents the local fluctuating electric field and varies "fast" to relative to f(t). If the system contains several species then the friction on a particle of type "i" may be written as a sum of terms due to interaction with each of the other species, so that

$$f_i = \sum_j f_{ij} .$$

Each of the terms fij is also in general a function of the velocity of the test particle, but this dependence has been disregarded. Equation (1) may be solved by using the method of Chandrasekhar[2].

The solution to equation (1) may be written

$$|u\rangle = |P\rangle - e^{-\int_0^t \Lambda(\eta)\,d\eta}\,|P_0\rangle =$$
$$\int_0^t e^{-\int_0^t \Lambda(\eta)\,d\eta}\; e^{+\int_0^\rho \Lambda(\eta)\,d\eta}\,|F(\rho)\rangle\,d\rho =$$
$$\int_0^t \psi(\rho,t)\,|F(\rho)\rangle\,d\rho\,, \tag{2}$$

where

$$\psi(\rho,t) = e^{-\int_0^t \Lambda(\eta)\,d\eta}\; e^{+\int_0^\rho \Lambda(\eta)\,d\eta}\,. \tag{3}$$

Set

$$|u\rangle = \Sigma\,|u_n\rangle = \sum_n \psi(n\Delta t, t)\int_{n\Delta t}^{(n+1)\Delta t} |F(\rho)\rangle\,d\rho =$$
$$= \sum_n \psi(n\Delta t, t)\,|\Gamma_{n\Delta t}(\Delta t)\rangle \tag{4}$$

where $\psi(\rho,t)$ can be considered constant in the interval Δt, but Δt is large enough so that $\Gamma(\Delta t)$ is distributed statistically.

The distribution of $|u\rangle$ will tend to a Maxwellian if the probability of various occurences of $|\Gamma(t)\rangle$ is governed by the distribution function

$$\mathcal{W}_t(\Delta t) = \left[\mathrm{Det}\,(4\pi m k T \Delta t\, f(t))\right]^{-1/2} e^{-(4 m k T \Delta t)^{-1}\langle \Gamma_t(\Delta t)|\, f(t)^{-1}\,|\Gamma_t(\Delta t)\rangle}\,. \tag{5}$$

The probability of $|u_n\rangle$ may then be written

$$\eta_{n\Delta t}|u_n) = \left[\mathrm{Det}\left(4\pi m k T \Delta t\, \psi(n\Delta t)\, f(n\Delta t)\, \tilde{\psi}(n\Delta t)\right)\right]^{-1/2} \cdot$$
$$\cdot\, e^{-(4 m k T \Delta t)^{-1} \langle u_n | \left(\psi(n\Delta t)\, f(n\Delta t)\, \tilde{\psi}(n\Delta t)\right)^{-1} | u_n \rangle} . \tag{6}$$

Markoff's method gives

$$W(u, u_0, t) = (2\pi)^{-3} \int e^{-i\vec{\rho}\cdot\vec{u}}\, u(\rho)\, d\vec{\rho} \tag{7}$$

where

$$u(\rho) = \lim_{N\to\infty} \prod_{s=1}^{N} \eta(u_s)\, e^{i\vec{\rho}\cdot\vec{u}_s}\, d\vec{u}_s = e^{-\frac{1}{4}\langle \rho | C^{-1} | \rho \rangle} , \tag{8}$$

using (6) and setting

$$C^{-1} = 4 m k T \int_0^t \psi(\rho)\, f(\rho)\, \tilde{\psi}(\rho)\, d\rho . \tag{9}$$

It is therefore possible to write (7) as

$$W(u, u_0, t) = \left[\mathrm{Det}\left(4\pi m k T \int_0^t \psi(\rho)\, f(\rho)\, \tilde{\psi}(\rho)\, d\rho\right)\right]^{-1/2} \cdot$$
$$\cdot\, e^{-(4 m k T)^{-1} \langle u | \left(\int_0^t \psi(\rho)\, f(\rho)\, \tilde{\psi}(\rho)\, d\rho\right)^{-1} | u \rangle} . \tag{10}$$

III. The Average Energy

In order to calculate the average energy, the velocity distribution function (10) must first be expressed as a function of the friction coefficients through (3). Thus

$$\psi(\rho,t) = e^{-\int_0^t \Lambda(\eta)d\eta}\, e^{+\int_0^\rho \Lambda(\eta)d\eta} ,$$

and it is possible to calculate

$$\int_0^t \psi(\rho)\, f(\rho)\, \tilde{\psi}(\rho)\, d\rho \qquad (11)$$

Equation (1) gives

$$f(\rho) = \tfrac{1}{2}\left[\Lambda(\rho) + \tilde{\Lambda}(\rho)\right] . \qquad (12)$$

Now

$$\begin{aligned}\int_0^t \psi(\rho)\, f(\rho)\, \tilde{\psi}(\rho)\, d\rho &= \\ &= \tfrac{1}{2}\int_0^t \frac{d}{d\rho}\left[e^{-\int_0^t \Lambda(\eta)d\eta}\, e^{\int_0^\rho \Lambda(\eta)d\eta}\, e^{\int_0^\rho \tilde{\Lambda}(\eta)d\eta}\, e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta}\right]d\rho = \\ &= \tfrac{1}{2}\left[1 - e^{-\int_0^t \Lambda(\eta)d\eta}\, e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta}\right] ,\end{aligned} \qquad (13)$$

so that (10) may be written

$$W(\rho, \rho_0, t) = \left[\mathrm{Det}\, \pi/s\right]^{-1/2} e^{-\langle u|s|u\rangle} , \qquad (14)$$

where

$$\gamma = (2mkT)^{-1}\left[1 - e^{-\int_0^t \Lambda(\eta)d\eta}\, e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta}\right]^{-1} \tag{15}$$

The average energy may then be found from (2) and (14),

$$\begin{aligned}\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2m} \int \Big\{ & \underset{\mathrm{I}}{u^2} + \underset{\mathrm{II}}{\langle p_0 | \left[e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta}\, e^{-\int_0^t \Lambda(\eta)d\eta} \right] | p_0 \rangle} + \\ & + \underset{\mathrm{III}}{\langle u | e^{-\int_0^t \Lambda(\eta)d\eta} | p_0 \rangle} + \\ & + \underset{\mathrm{III}}{\langle p_0 | e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta} | u \rangle} \Big\}\, W(p, p_0, t)\, d^3u ,\end{aligned} \tag{16}$$

where terms III are odd in u and therefore give zero contribution. Term I may be written

$$\begin{aligned} I &= \int u^2 \left[\operatorname{Det} \pi/\gamma\right]^{-\frac{1}{2}} e^{-\langle u|\gamma|u\rangle}\, d^3u = \\ &= -\left[\operatorname{Det} \pi/\gamma\right]^{-\frac{1}{2}} \sum_i \frac{\partial}{\partial \gamma_{ii}} \int e^{-\langle u|\gamma|u\rangle}\, d^3u = \\ &= \frac{1}{2}\, \frac{\sum \text{first minors of } \gamma}{\operatorname{Det} \gamma} = \\ &= \frac{1}{4}\left[\frac{(\operatorname{trace} \gamma)^2 - \operatorname{trace} \gamma\gamma^2}{\operatorname{Det} \gamma} \right] . \end{aligned} \tag{17}$$

The average energy is therefore:

$$\begin{aligned}\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2m} \langle p_0 | e^{-\int_0^t \Lambda(\eta)d\eta}\, e^{\int_0^t \tilde{\Lambda}(\eta)d\eta} | p_0 \rangle + \\ + \frac{1}{4m} \operatorname{Det} \gamma^{-1} \cdot \sum \text{first minors of } \gamma \end{aligned} \tag{18}$$

and is in general dependent on the magnetic field.

When the integral $\int_0^t f(\eta)\,d\eta$ is diagonal,

$$\int_0^t \Lambda(\eta)\,d\eta \int_0^t \tilde{\Lambda}(\eta)\,d\eta = \int_0^t \tilde{\Lambda}(\eta)\,d\eta \int_0^t \Lambda(\eta)\,d\eta \tag{19}$$

and

$$e^{-\int_0^t \Lambda(\eta)d\eta}\, e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta} = e^{-\int_0^t (\Lambda+\tilde{\Lambda})d\eta} = e^{-2\int_0^t f(\eta)d\eta}$$

Then

$$W = \left[\mathrm{Det}\, 2\pi mkT\left(1-e^{-2\int_0^t f(\eta)d\eta}\right)\right]^{\frac{1}{2}} \cdot e^{-(2mkT)^{-1}\langle u|\left(1-e^{-2\int_0^t f(\eta)d\eta}\right)^{-1}|u\rangle}$$

and (20)

$$\left\langle \frac{p^2}{2m}\right\rangle = \frac{1}{2m}\langle P_0|e^{-2\int_0^t f(\eta)d\eta}|P_0\rangle + \frac{1}{4m}\,\mathrm{Det}\, 2mkT\left(1-e^{-2\int_0^t f(\eta)d\eta}\right) \cdot \sum \text{first minors of } \left[(2mkT)^{-1}\left(1-e^{-2\int_0^t f(\eta)d\eta}\right)^{-1}\right];$$

which is independent of the magnetic field. Note that only requirement for this to be true is that $\int_0^t f(\eta)d\eta$ is diagonal. In other words any off diagonal components of $f(\eta)$ must be oscillating functions of time which average out to zero. In a similar manner, the average energy along the field may be calculated remembering that

$$|P_z\rangle = A|P\rangle;$$

where $$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

Thus

$$\langle \frac{P_z^2}{2m} \rangle = \frac{1}{2m} \int \Big[\underset{I}{u_z^2} + \langle P_0 | e^{-\int_0^t \tilde{\Lambda}(\eta) d\eta} \underset{II}{A} e^{-\int_0^t \Lambda(\eta) d\eta} | P_0 \rangle + \tag{22}$$

$$+ \underset{III}{\langle u | A e^{-\int_0^t \Lambda(\eta) d\eta} | P_0 \rangle} +$$

$$+ \underset{III}{\langle P_0 | e^{-\int_0^t \tilde{\Lambda}(\eta) d\eta} A | u \rangle} \Big] W(\varphi, P_0, t) \, d^3u$$

Terms III do not contribute.
Term I may be written

$$\int u_z^2 \, W(\varphi, P_0, t) \, d^3u = -[\mathrm{Det}\, \pi/\gamma]^{-1/2} \frac{d}{d\gamma_{33}} \int e^{-\langle u | \gamma | u \rangle} d^3u =$$

$$= \frac{1}{2} \mathrm{Det}\, \gamma^{-1} \cdot (\text{minor of } \gamma_{33}) \tag{23}$$

and equation (23) may be written

$$\langle \frac{P_z^2}{2m} \rangle = \frac{1}{2m} \langle P_0 | e^{-\int_0^t \tilde{\Lambda}(\eta) d\eta} A e^{-\int_0^t \Lambda(\eta) d\eta} | P_0 \rangle +$$

$$+ \frac{1}{4m} \frac{\text{minor of } \gamma_{33}}{\mathrm{Det}\, \gamma} \; . \tag{24}$$

Now

$$\int_0^t \tilde{\Lambda}(\eta) d\eta \, A = \begin{pmatrix} 0 & 0 & \int_0^t f_{13} \, d\eta \\ 0 & 0 & \int_0^t f_{23} \, d\eta \\ 0 & 0 & \int_0^t f_{33} \, d\eta \end{pmatrix} = \int_0^t \Lambda(\eta) d\eta \, A \, . \tag{25}$$

$$A\int_0^t \tilde{\Lambda}(\eta)d\eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \int_0^t f_{13}d\eta & \int_0^t f_{23}d\eta & \int_0^t f_{33}d\eta \end{pmatrix} = A\int_0^t \Lambda(\eta)d\eta .$$

(26)

Therefore, when $\int_0^t f_{13}\,d\eta = \int_0^t f_{23}\,d\eta = 0$

$$A\int_0^t \Lambda(\eta)d\eta = \int_0^t \Lambda(\eta)d\eta\, A = A\int_0^t f_{33}(\eta)d\eta .$$

Under this condition $\int_0^t f_{33}(\eta)d\eta$ is an eigenvalue of $\int_0^t \Lambda(\eta)d\eta$ and $\int_0^t \tilde{\Lambda}(\eta)d\eta$,

and the average energy along the field may be written

$$\left\langle \frac{p_z^2}{2m} \right\rangle = \frac{1}{2m}\langle P_0 | e^{-2\int_0^t f_{33}(\eta)d\eta} | P_0 \rangle + \frac{1}{4m\lambda_3} \qquad (27)$$

where λ_3 is the associated eigenvalue of γ , so that

$$\frac{1}{\lambda_3} = 1 - e^{-2\int_0^t f_{33}(\eta)d\eta} ,$$

and (27) is therefore independent of the magnetic field. Consider now the integral

$$\int_{-\infty}^{+\infty} |P\rangle\langle P| \,(\mathrm{Det}\,\pi/\gamma)^{-\frac{1}{2}}\, e^{-\langle u|\gamma|u\rangle}\, d^3u =$$

$$= \int_{-\infty}^{+\infty} \left[|u\rangle + e^{-\int_0^t \Lambda(\eta)d\eta} |P_0\rangle \right] \left[\langle P_0 | e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta} + \langle u| \right] \frac{e^{-\langle u|\gamma|u\rangle}}{[\mathrm{Det}\,\pi/\gamma]^{\frac{1}{2}}}\, d^3u =$$

$$= \int_{-\infty}^{+\infty} |u\rangle\langle u| \, (\mathrm{Det}\, \pi\gamma)^{-1/2} \, e^{-\langle u|\gamma|u\rangle} \, d^3u \; +$$

$$+ \, e^{-\int_0^t \Lambda(\eta)d\eta} \, |P_0\rangle\langle P_0| \, e^{-\int_0^t \tilde{\Lambda}(\eta)\, d\eta} \tag{28}$$

Now $\gamma = S\lambda S^{-1}$, $\gamma^{-1} = S\lambda^{-1}S^{-1}$, $S^{-1} = S^*$

$|u\rangle = S|v\rangle$ for a Hermitian matrix,

and I may be written:

$$\frac{1}{(\mathrm{Det}\,\pi\gamma)^{\frac{1}{2}}} \, S \int_{-\infty}^{+\infty} |v\rangle\langle v| \, e^{-\sum_i \lambda_i v_i^2} \, d^3u \, S^{-1} =$$

$$= \frac{1}{2} S\lambda^{-1}S^{-1} = \frac{1}{2}\gamma^{-1}$$

Therefore

$$\int_{-\infty}^{+\infty} |P\rangle\langle P| \, [\mathrm{Det}\,\pi\gamma]^{-1/2} \, e^{-\langle u|\gamma|u\rangle} \, d^3u =$$

$$= \frac{1}{2}\gamma^{-1} + e^{-\int_0^t \Lambda(\eta)d\eta} \, |P_0\rangle\langle P_0| \, e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta} . \tag{29}$$

Note that

$$|P_2\rangle\langle P_2| = A P_2^2 = A|P\rangle\langle P|A \; .$$

Then

$$\langle \tfrac{p_z^2}{2m} \rangle A = \tfrac{1}{2m} \left(A \tfrac{1}{2} \gamma^{-1} A + A e^{-\int_0^t \Lambda(\eta) d\eta} |P_0\rangle \langle P_0| e^{-\int_0^t \tilde{\Lambda}(\eta) d\eta} A \right)$$

$$\left(\tfrac{p_z^2}{2m} - \tfrac{1}{4m} \gamma_{33}^{-1} \right) \langle P_0| e^{-\int_0^t \tilde{\Lambda}(\eta) d\eta} A e^{-\int_0^t \Lambda(\eta) d\eta} |P_0\rangle = \qquad (30)$$

$$= \left(\langle P_0| e^{-\int_0^t \tilde{\Lambda}(\eta) d\eta} A e^{-\int_0^t \Lambda(\eta) d\eta} |P_0\rangle \right)^2 ,$$

so that

$$\langle \tfrac{p_z^2}{2m} \rangle = \tfrac{1}{4m} \gamma_{33}^{-1} + \tfrac{1}{2m} \langle P_0| e^{-\int_0^t \tilde{\Lambda}(\eta) d\eta} A e^{-\int_0^t \Lambda(\eta) d\eta} |P_0\rangle , \qquad (31)$$

which is the same as (24) since one may show that

$$(\gamma^{-1})_{ii} = \mathrm{Det}\, \gamma^{-1} (\text{minor of } \gamma_{ii}) . \qquad (32)$$

In the limit where $t \to \infty$ (20 and (31) give us

$$\lim_{t\to\infty} \langle \tfrac{p^2}{2m} \rangle = \tfrac{1}{4m} (2mkT)^3 \cdot \tfrac{3}{(2mkT)^2} = \tfrac{3}{2} kT ,$$

$$\lim_{t\to\infty} \langle \tfrac{p_z^2}{2m} \rangle = \tfrac{1}{4m} (2mkT)^3 \tfrac{1}{(2mkT)^2} = \tfrac{1}{2} kT ,$$

which agree with the Maxwellian values.

To summarize, the average total energy of a particle in a magnetic field and subject to a tensor friction is in general a function of the field unless the time average of the friction tensor is a diagonal matrix. The component of the average energy parallel to the field is independent of the field whenever one of the principal axes of the friction tensor is parallel to the field. Both averages reduce to their Maxwellian values in the limit of infinite times for an arbitrary friction tensor.

IV. Resonance Diffusion

The probability distribution for the displacement $\vec{r}$ of a Brownian particle at time t which is at $\vec{r}_0$ at $t=0$ is found by noting that

$$|r-r_0\rangle = \frac{1}{m}\int_0^t |P(t)\rangle\, dt =$$
$$= \frac{1}{m}\int_0^t e^{-\int_0^\rho \Lambda(\eta)d\eta}\,|P_0\rangle\, d\rho \;+ \tag{40}$$
$$+\frac{1}{m}\int_0^t d\rho \int_0^\rho e^{-\int_0^\rho \Lambda(\eta)d\eta}\, e^{\int_0^\xi \Lambda(\eta)d\eta}\,|F(\xi)\rangle\, d\xi$$

using (2).

Term I may be written

$$\frac{1}{m}\int_0^t e^{-\int_0^\rho \Lambda(\eta)d\eta}\,|P_0\rangle\, d\rho = -\frac{1}{m}\int_0^t \frac{1}{\Lambda(\rho)}\frac{d}{d\rho}\, e^{-\int_0^\rho \Lambda(\eta)d\eta}\,|P_0\rangle\, d\rho =$$
$$= -\frac{1}{m\Lambda(\rho)}\, e^{-\int_0^\rho \Lambda(\eta)d\eta}\Big|_0^t\, |P_0\rangle \;-$$
$$-\frac{1}{m}\int_0^t \frac{\Lambda'(\rho)}{\Lambda(\rho)^2}\, e^{-\int_0^\rho \Lambda(\eta)d\eta}\, d\rho\, |P_0\rangle \tag{41}$$

However if $\Lambda(\rho)$ is a slowly varying function of time, the last term on the right may be neglected in comparison with the left hand term. Then I reduces to:

$$\frac{1}{m}\left(\Lambda(0)^{-1} - \Lambda(t)^{-1}\, e^{-\int_0^t \Lambda(\eta)d\eta}\right)|P_0\rangle \tag{42}$$

Similarly term II may be written

$$\frac{1}{m}\int_0^t d\rho\, e^{-\int_0^\rho \Lambda(\eta)\,d\eta}\int_0^\rho e^{\int_0^\xi \Lambda(\eta)\,d\eta}\,|F(\xi)\rangle\, d\xi = -$$

$$-\frac{1}{m}\int_0^t \frac{1}{\Lambda(\rho)}\, d\rho\, \frac{d}{d\rho}\left(e^{-\int_0^\rho \Lambda(\eta)\,d\eta}\right)\int_0^\rho e^{\int_0^\xi \Lambda(\eta)\,d\eta}\,|F(\xi)\rangle\, d\xi =$$

$$-\frac{1}{m}\int_0^t \frac{1}{\Lambda(\rho)}\, d\rho\, \frac{d}{d\rho}\left(e^{-\int_0^\rho \Lambda(\eta)\,d\eta}\int_0^\rho e^{\int_0^\xi \Lambda(\eta)\,d\eta}\,|F(\xi)\rangle\, d\xi\right) +$$

$$+\frac{1}{m}\int_0^t \frac{1}{\Lambda(\rho)}\, e^{-\int_0^\rho \Lambda(\eta)\,d\eta}\, d\rho\, \frac{d}{d\rho}\left(\int_0^\rho e^{\int_0^\xi \Lambda(\eta)\,d\eta}\,|F(\xi)\rangle\, d\xi\right). \tag{43}$$

When $\Lambda(\rho)$ varies slowly, term IIa may be written

$$-\frac{1}{m\Lambda(\rho)}\, e^{-\int_0^\rho \Lambda(\eta)\,d\eta}\int_0^\rho e^{\int_0^\xi \Lambda(\eta)\,d\eta}\,|F(\xi)\rangle\, d\xi\,\Big|_0^t = -$$

$$-\frac{1}{m\Lambda(t)}\, e^{-\int_0^t \Lambda(\eta)\,d\eta}\int_0^t e^{\int_0^\xi \Lambda(\eta)\,d\eta}\,|F(\xi)\rangle\, d\xi\,. \tag{44}$$

Term IIb reduces to

$$\frac{1}{m}\int_0^t \frac{1}{\Lambda(\rho)}\, F(\rho)\, d\rho\,. \tag{45}$$

Therefore the displacement (40) may be written

$$|R\rangle = |r - r_0\rangle - \frac{1}{m}\left(\frac{1}{\Lambda(0)} - \frac{1}{\Lambda(t)}\, e^{-\int_0^t \Lambda(\eta)\,d\eta}\right)|P_0\rangle =$$

$$= \frac{1}{m}\int_0^t \left(\frac{1}{\Lambda(\rho)} - \frac{1}{\Lambda(t)}\, e^{-\int_0^t \Lambda(\eta)\,d\eta}\, e^{\int_0^\rho \Lambda(\eta)\,d\eta}\right)|F(\rho)\rangle\, d\rho\,. \qquad (46)$$

To find the corresponding distribution, set

$$\psi = \frac{1}{\Lambda(\rho)} - \frac{1}{\Lambda(t)}\, e^{-\int_0^t \Lambda(\eta)\,d\eta}\, e^{+\int_0^\rho \Lambda(\eta)\,d\eta}\,. \qquad (47)$$

Then

$$\int_0^t \psi(\rho)\, f(\rho)\, \tilde{\psi}(\rho)\, d\rho = \underbrace{\int_0^t \frac{1}{\Lambda(\rho)}\, f(\rho)\, \frac{1}{\tilde{\Lambda}(\rho)}\, d\rho}_{I} -$$

$$- \underbrace{\int_0^t \frac{1}{\Lambda(t)}\, e^{-\int_0^t \Lambda(\eta)\,d\eta}\, e^{+\int_0^\rho \Lambda(\eta)\,d\eta}\, f(\rho)\, \frac{1}{\tilde{\Lambda}(\rho)}\, d\rho}_{II} - \qquad (48)$$

$$- \underbrace{\int_0^t \frac{1}{\Lambda(\rho)}\, f(\rho)\, e^{+\int_0^\rho \tilde{\Lambda}(\eta)\,d\eta}\, e^{-\int_0^t \tilde{\Lambda}(\eta)\,d\eta}\, \frac{1}{\tilde{\Lambda}(t)}\, d\rho}_{III} +$$

$$+ \int_0^t \frac{1}{\Lambda(t)}\, e^{-\int_0^t \Lambda(\eta)\,d\eta}\, e^{+\int_0^\rho \Lambda(\eta)\,d\eta}\, f(\rho)\, e^{\int_0^\rho \tilde{\Lambda}(\eta)\,d\eta}\, e^{-\int_0^t \tilde{\Lambda}(\eta)\,d\eta}\, \frac{1}{\tilde{\Lambda}(t)}\, d\rho\,.$$

When $\Lambda(\rho)$ varies slowly,

$$II = \frac{1}{2}\left[\frac{1}{\Lambda(t)\tilde{\Lambda}(t)} + \frac{1}{\Lambda(t)^2} - e^{-\int_0^t \Lambda(\eta)d\eta}\left(\frac{1}{\Lambda(t)\tilde{\Lambda}(0)} + \frac{1}{\Lambda(t)\Lambda(0)}\right)\right],$$

$$III = \frac{1}{2}\left[\frac{1}{\Lambda(t)\tilde{\Lambda}(t)} + \frac{1}{\tilde{\Lambda}(t)^2} - \right.$$

$$\left. - \left(\frac{1}{\tilde{\Lambda}(0)\tilde{\Lambda}(t)} + \frac{1}{\Lambda(0)\tilde{\Lambda}(t)}\right) e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta}\right];$$

$$IV = \frac{1}{2}\left[\frac{1}{\Lambda(t)\tilde{\Lambda}(t)} - \frac{1}{\Lambda(t)} e^{-\int_0^t \Lambda(\eta)d\eta} e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta} \frac{1}{\tilde{\Lambda}(t)}\right].$$

Therefore

$$\int_0^t \psi(\rho) f(\rho) \tilde{\psi}(\rho) d\rho = \int_0^t \left(\tilde{\Lambda}(\rho) f(\rho)^{-1} \Lambda(\rho)\right)^{-1} d\rho - \frac{1}{2}\left(\Lambda(t)^{-2} + \tilde{\Lambda}(t)^{-2}\right) +$$
$$+ \frac{1}{2} e^{-\int_0^t \Lambda(\eta)d\eta}\left(\frac{1}{\Lambda(t)\tilde{\Lambda}(0)} + \frac{1}{\Lambda(t)\Lambda(0)}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{\tilde{\Lambda}(0)\tilde{\Lambda}(t)} + \frac{1}{\Lambda(0)\tilde{\Lambda}(t)}\right) e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta} - \tag{49}$$
$$- \frac{1}{2}\frac{1}{\Lambda(t)}\left(1 + e^{-\int_0^t \Lambda(\eta)d\eta} e^{-\int_0^t \tilde{\Lambda}(\eta)d\eta}\right)\frac{1}{\tilde{\Lambda}(t)}.$$

In order for the distribution (14) to approach a Maxwellian as $t \to \infty$ it is necessary that

$$\lim_{t\to\infty} \int_0^t \tilde{\Lambda}(\eta)d\eta \to \infty \tag{50}$$

If $\Lambda(\rho)$ approaches zero slower than t^{-1} , for large t, (50) will hold and the first term of (49) is the leading term at large t. Then

$$\int_0^t \psi(\rho)\, f(\rho)\, \tilde{\psi}(\rho)\, d\rho \simeq \int_0^t \left(\tilde{\Lambda}(\rho)\, f(\rho)^{-1} \Lambda(\rho)\right)^{-1} d\rho \tag{51}$$

at large t and the distribution function

$$W(\vec{r},\vec{r}_0,\vec{p}_0,t) = \left[\mathrm{Det}\ \frac{4\pi kT}{m} \int_0^t \left(\tilde{\Lambda}(\rho)\, f(\rho)^{-1} \Lambda(\rho)\right)^{-1} d\rho\right]^{-1/2} \cdot$$
$$\cdot\, e^{-\frac{m}{4kT}\langle R| \left[\int_0^t \left(\tilde{\Lambda}(\rho)\, f(\rho)^{-1} \Lambda(\rho)\right)^{-1} d\rho\right]^{-1} |R\rangle}, \tag{52}$$

as may be found from equation (10).

The average mean square displacement across and along the magnetic field is

$$\langle \Delta R_\perp^2 \rangle = \frac{1}{2} \int \left[(x_1 - x_{10})^2 + (y_1 - y_{10})^2\right] W(\vec{r},\vec{r}_0,\vec{p}_0,t)\, d^3R\, , \tag{53}$$

$$\langle \Delta R_\parallel^2 \rangle = \int (z_1 - z_{10})^2\, W(\vec{r},\vec{r}_0,\vec{p}_0,t)\, d^3R\, . \tag{54}$$

Let $D_1 t$ $D_2 t$ $D_3 t$ be the eigenvalues of $\frac{kT}{m} \int_0^t \left(\Lambda(\rho)\, f(\rho)^{-1} \Lambda(\rho)\right)^{-1} d\rho$

so that

$$\left[\frac{kT}{m}\int_0^t \left(\tilde{\Lambda}(\rho)\, f(\rho)^{-1}\Lambda(\rho)\right)^{-1} d\rho\right]^{-1} = S\begin{pmatrix} \frac{1}{D_1 t} & & \\ & \frac{1}{D_2 t} & \\ & & \frac{1}{D_3 t}\end{pmatrix} S^{-1}. \tag{55}$$

Then $|R\rangle = S\,|u\rangle$ where the corresponding eigenvectors are $|u\rangle$.

$$\langle R| = \langle u|\, S^{*} = \langle u|\, S^{-1}.$$

Now investigate

$$\int |R\rangle\langle R|\; W(\vec{r},\vec{r}_0,\vec{p}_0,t)\; d^3R =$$

$$= \left[(4\pi)^3 D_1 D_2 D_3 t^3\right]^{-1/2} S \int |u\rangle\langle u|\, e^{-\frac{1}{4t}(u_i^2/D_i)}\, d^3u\, S^{-1} =$$

$$= 2S\begin{pmatrix} D_1 t & & \\ & D_2 t & \\ & & D_3 t\end{pmatrix} S^{-1} = \tag{56}$$

$$= \frac{2kT}{m}\int_0^t \left(\tilde{\Lambda}(\rho)\, f(\rho)^{-1}\Lambda(\rho)\right)^{-1} d\rho\,.$$

Thus

$$\langle \Delta R_{\parallel}^{\,2}\rangle A = \int A\, |R\rangle\langle R| A\, W(\vec{r},\vec{r}_0,\vec{p}_0,t)\, d^3R =$$

$$= \frac{2kT}{m} A \int_0^t \left(\Lambda(\rho)\, f(\rho)^{-1}\Lambda(\rho)\right)^{-1} d\rho\, A =$$

$$= \frac{2kT}{m}\left[\int_0^t \left(\Lambda(\rho)\, f(\rho)^{-1}\Lambda(\rho)\right)^{-1} d\rho\right]_{33},$$

where the matrix A is given by equation (21). The mean square displacement along the field is

$$\langle \Delta R_{\parallel}^2 \rangle = \frac{2kT}{m} \left[\int_0^t \left(\Lambda(\rho) f(\rho)^{-1} \Lambda(\rho) \right)^{-1} d\rho \right]_{33} \tag{57}$$

and the corresponding displacement across the field is

$$\langle \Delta R_{\perp}^2 \rangle = \frac{kT}{m} \sum_{i=1,2} \left[\int_0^t \left(\Lambda(\rho) f(\rho)^{-1} \Lambda(\rho) \right)^{-1} d\rho \right]_{ii} \tag{58}$$

Equation (57) leads to

$$\langle \Delta R_{\parallel}^2 \rangle = 2 D_3 t \tag{59}$$

only when

$$\left[\int_0^t \left(\tilde{\Lambda}(\rho) f(\rho)^{-1} \Lambda(\rho) \right)^{-1} d\rho \right]_{13} = 0 \, ,$$
$$\left[\int_0^t \left(\tilde{\Lambda}(\rho) f(\rho)^{-1} \Lambda(\rho) \right)^{-1} d\rho \right]_{23} = 0 \, . \tag{60}$$

Since the trace is invariant under a rotation, equation (58) may then be written as

$$(\Delta R_\perp)^2 = D_1 t + D_2 t. \tag{61}$$

It is easy to show that in general

$$\tilde{\Lambda}(\rho) f(\rho)^{-1} \Lambda(\rho) = f(\rho) + \omega_L^2 K_3 f(\rho)^{-1} K_3 , \tag{62}$$

where K_3 is defined in (1).

Assume that $f(\rho)^{-1}$ is a general symmetric tensor with elements a_{ij}, then

$$K_3 f(\rho)^{-1} K_3 = \begin{pmatrix} a_{22} & -a_{12} & 0 \\ -a_{12} & a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{63}$$

However

$$f(\rho)^{-1}_{ij} = a_{ij} = \frac{\text{minor of } f_{ij}}{\text{Det } f(\rho)} . \tag{64}$$

When the friction coefficients are constants and $f_{13} = f_{23} = 0$, then f_{33} is the third eigenvalue of both f and (62). Thus $\frac{t}{f_{33}}$ is the third eigenvalue of

$$\int_0^t \left(\tilde{\Lambda}(\rho) f(\rho)^{-1} \Lambda(\rho)\right)^{-1} d\rho ,$$ giving

$$\langle \Delta R_\parallel^2 \rangle = \frac{2KTt}{m f_{33}} \tag{65}$$

for equation (59). In this case the diffusion along the field is independent of the field. This will not be true when f_{13} and f_{23} are different from zero.

Now consider some specific examples with constant friction tensors of various forms.

A. If

$$f = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix} , \tag{66}$$

$$\tilde{\Lambda} f^{-1} \Lambda = \begin{pmatrix} \beta_1 + \frac{\omega_c^2}{\beta_2} & 0 & 0 \\ 0 & \beta_2 + \frac{\omega_c^2}{\beta_1} & 0 \\ 0 & 0 & \beta_3 \end{pmatrix} , \tag{67}$$

then

$$\langle \Delta R_{\parallel}^2 \rangle = \frac{2KTt}{m \beta_3} , \tag{68}$$

and

$$\langle \Delta R_{\perp}^2 \rangle = \frac{KTt(\beta_1 + \beta_2)}{m(\omega_c^2 + \beta_1 \beta_2)} . \tag{69}$$

B. If

$$f = \begin{pmatrix} \beta & \gamma & 0 \\ \gamma & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} , \tag{70}$$

$$\tilde{\Lambda} f^{-1} \Lambda = \begin{pmatrix} \beta + \frac{\omega_c^2 \beta}{\beta^2 - \gamma^2} & \gamma + \frac{\gamma \omega_c^2}{\beta^2 - \gamma^2} & 0 \\ \gamma + \frac{\gamma \omega_c^2}{\beta^2 - \gamma^2} & \beta + \frac{\omega_c^2 \beta}{\beta^2 - \gamma^2} & 0 \\ 0 & 0 & \beta \end{pmatrix} \tag{71}$$

$$[\tilde{\Lambda} f^{-1} \Lambda]_{33}^{-1} = \frac{1}{\beta} , \tag{72}$$

$$[\Lambda f^{-1} \Lambda]_{22}^{-1} = [\tilde{\Lambda} f^{-1} \Lambda]_{11} = \frac{\beta}{\beta^2 - \gamma^2 + \omega_c^2} ,$$

(73)

and

$$\langle \Delta R_{\parallel}^2 \rangle = \frac{2kTt}{m\beta} , \tag{74}$$

$$\langle \Delta R_{\perp}^2 \rangle = \frac{2kTt\beta}{m(\beta^2 - \gamma^2 + \omega_c^2)} , \tag{75}$$

where $\beta > \gamma$ in order that all the eigenvalues of f be positive and that the system will approach a Maxwellian equilibrium.

C. If

$$f = \begin{pmatrix} \beta & 0 & \delta \\ 0 & \beta & \delta \\ \delta & \delta & \beta \end{pmatrix} \tag{76}$$

$$\tilde{\Lambda} f^{-1} \Lambda = \begin{pmatrix} \beta + \omega_c^2 \dfrac{\beta^2-\delta^2}{\beta(\beta^2-2\delta^2)} & -\dfrac{\delta^2\omega_c^2}{\beta(\beta^2-2\delta^2)} & \delta \\ \dfrac{-\delta^2\omega_c^2}{\beta(\beta^2-2\delta^2)} & \beta + \omega_c^2 \dfrac{\beta^2-\delta^2}{\beta(\beta^2-2\delta^2)} & \delta \\ \delta & \delta & \beta \end{pmatrix}, \tag{77}$$

$$[\tilde{\Lambda} f^{-1} \Lambda]^{-1}_{33} = \frac{\beta^2+\omega_c^2}{\beta(\beta^2+\omega_c^2-2\delta^2)}, \tag{78}$$

$$[\tilde{\Lambda} f^{-1} \Lambda]^{-1}_{22} = [\Lambda f^{-1} \Lambda]^{-1}_{11} = \frac{\beta^2-\delta^2}{\beta(\beta^2+\omega_c^2)}, \tag{79}$$

$$\langle \Delta R_{\parallel}^2 \rangle = \frac{2KTt}{m} \; \frac{\beta^2+\omega_c^2}{\beta(\beta^2-2\delta^2+\omega_c^2)}, \tag{80}$$

$$\langle \Delta R_{\perp}^2 \rangle = \frac{2KTt}{m} \; \frac{\beta^2-\delta^2}{\beta(\beta^2+\omega_c^2)}, \tag{81}$$

where $\beta^2 > 2\delta^2$.

By comparing the diffusion in the three cases (66), (70), (76) it is found that for a friction of the form (70), the diffusion along the field is the same as with a scalar friction. However the diffusion across the field is increased by the ratio

$$\frac{\beta^2 + \omega_c^2}{\beta^2 + \omega_c^2 - \gamma^2}$$

This is just the ratio that would be found by substituting the eigenvalues of the friction tensor (70) into equation (69), and does therefore not represent a physical difference unless γ is a function of the field strength. For a friction of the form of (76) the diffusion across the field is smaller than for a scalar friction by $\frac{\beta^2 - \delta^2}{\beta^2}$ while

the diffusion along the field is increased by $\frac{\beta^2 + \omega_c^2}{\beta^2 + \omega_c^2 - 2\delta^2}$

These ratios can not be tranformed away by a simple diagonalization of the friction tensor (76), but represent a true mixing of the contributions of the dynamical friction and the magnetic field to the diffusion rates which may occur when none of the principal axes of the dynamical friction tensor are parallel to the field.

It is evident that the presence of a magnetic field tends to stabilize the diffusion not only by reducing the diffusion rate across the field, but also by reducing the diffusion rates of an anisotropic plasma towards those of a Maxwellian one, in the presence of a strong field.

It is thus suggested that the addition of a constant magnetic field does not lead to resonance peaks in the diffusion rates across or along the fields, but rather serves as a stabilizing influence. On the other hand, the presence of a tensor friction will alter the diffusion rates compared to those of a Maxwellian plasma. Of course, if the off-diagonal terms of the friction tensor are also dependent on the field, then resonance behavior is still possible.

It must be emphasized, however, that the assumed distribution function (5) is valid only for a stationary non-rotating system. In general an infinite plasma may also have a net drift and rotational velocity. Even a laboratory plasma that is cylindrical in shape may rotate, as seems to be the case for may plasma beams. In this case the diffusion rates across the field should be a critical function of the field strength, so that the high energy particles may be dissipated out of the field.

V. The Fokker-Planck Equation.

It is possible to derive the differential equation governing the distribution function $W(\vec{p},\vec{r},t)$ of particles whose motion is described by a generalized Langevin equation similar to(1)

$$\frac{d}{dt}|P(t)\rangle = -\Lambda(\vec{r},t)|P(t)\rangle + |G(\vec{r},t)\rangle + |F(t)\rangle ,$$

where (82)

$$\Lambda(\vec{r},t) = f(\vec{r},t) - i\omega_c(\vec{r},t)\, K_3$$

represents the dynamical friction in an anisotropic plasma and the rotation of the particle orbits due to a varying external magnetic field which is parallel to the $\hat{z}$ axis. The term $\overrightarrow{G(\vec{r},t)}$ may represent an external electric force field. It will be assumed that there exists a time Δt which is long relative to fluctuations in the local stochastic field $\overrightarrow{F}(t)$ but short compared to changes in the physical parameters. Then it is possible to set

$$|\Delta r_t\rangle = \frac{\Delta t}{m}|P(t)\rangle . \tag{83}$$

Equation (82) may be integrated in the same way as (2) to give

$$|\Delta P_t\rangle = |P_{t+\Delta t}\rangle - |P_t\rangle = -\Lambda(\vec{r},t)\,\Delta t\,|P_t\rangle + \Delta t\,|G(\vec{r},t)\rangle +$$

$$+|\Gamma_t(\Delta t)\rangle . \tag{84}$$

The distribution at time $t+\Delta t$ may be written

$$W(p,r,t+\Delta t) = \int W(\vec{p}-\Delta\vec{p},\vec{r}-\Delta\vec{r},t)\,\Psi(\vec{p}-\Delta\vec{p},\vec{r}-\Delta\vec{r},\Delta\vec{p},\Delta\vec{r})\,d\Delta\vec{p}\,d\Delta\vec{r} , \tag{85}$$

where $\Psi(\vec{p},\vec{r},\Delta\vec{p},\Delta\vec{r})$ represents the transition probability in phase space for a change $\Delta\vec{p}$, $\Delta\vec{r}$ at $\vec{p},\vec{r}$.
This may be written in terms of the momentum transition probability by using (83), such that

$$\Psi(\vec{p},\vec{r},\Delta\vec{p},\Delta\vec{r}) = \psi(\vec{p},\vec{r},\Delta\vec{p})\,\delta\!\left(\Delta\vec{r}(t) - \frac{\Delta t}{m}\vec{P}(t)\right) , \tag{86}$$

where

$$\psi(\vec{p},\vec{r},\Delta\vec{p}) = [\mathrm{Det}\ \pi/\gamma]^{-\frac{1}{2}}\ e^{-\langle \Gamma_i(\Delta t)|\gamma|\Gamma_i(\Delta t)\rangle} \tag{87}$$

and

$$\gamma = [4mkT\Delta t\ f(\vec{r},t)]^{-1},$$

using equation (5). The differential equation for the distribution function is now found by expanding (85) in a Taylor series

$$W(\vec{p},\vec{r},t) + \frac{\Delta t}{m}\ \vec{p}(t)\cdot\nabla_r\ W(\vec{p},\vec{r},t) + \Delta t\ \frac{\partial W(\vec{p},\vec{r},t)}{\partial t} + O(\Delta t^2) =$$

$$\begin{aligned} &= -\nabla_p\cdot\nabla W(\vec{p},\vec{r},t)\int\Delta\vec{p}\ \psi(\vec{p},\vec{r},\Delta\vec{p})\,d\Delta\vec{p}\ + \\ &\quad + \frac{1}{2}\nabla_p\nabla_p : W(\vec{p},\vec{r},t)\int\Delta\vec{p}\Delta\vec{p}\ \psi(\vec{p},\vec{r},\Delta\vec{p})\,d\Delta\vec{p}\ + \\ &\quad + O(\Delta\vec{p}\Delta\vec{p}\Delta\vec{p}). \end{aligned} \tag{88}$$

After performing the integrations and grouping the terms, the equation may be written:

$$\frac{\partial W(\vec{p},\vec{r},t)}{\partial t} + \frac{\vec{p}}{m}\cdot\nabla_p\ W(\vec{p},\vec{r},t) + \left(\frac{e}{mc}\ \vec{p}\times\vec{B} + G(\vec{r},t)\right)\cdot\nabla_p\ W(\vec{p},\vec{r},t) =$$

$$= \sum_{ij} f_{ij}(\vec{r},t)\,\frac{\partial}{\partial p_i}\left(\vec{p}_j\ W(\vec{p},\vec{r},t)\right) + mkT\sum_{ij} f_{ij}(\vec{r},t)\,\frac{\partial^2 W(\vec{p},\vec{r},t)}{\partial p_i\,\partial p_j} \tag{89}$$

in the limit as $\Delta t \to 0$. This is of the same form as the equation given by Kurşunoglu, and reduces to the ordinary Fokker-Planck equation for a diagonal friction tensor.

Appendix

A. On the Form of the Dynamical Friction Tensor

The Chandrasekhar method of treating the deviation of a particle interacting successively with a large number of other particles under a $(\frac{1}{r})^2$ force field may be generalized to arbitrary velocity distribution functions [3]. The particles are assumed to interact through the attractive gravitational force,

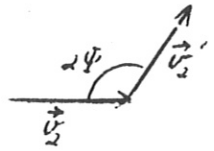
$$\vec{F} = -\frac{G m_1 m_2}{r^2}\,\vec{e}_r\,, \tag{A.1}$$

but the results are equally valid for charged particles suffering successive two body coulomb encounters, where

$$\vec{F} = \frac{e_1 e_2}{r^2}\,\vec{e}_r\,. \tag{A.2}$$

B. Derivation of the Equations of Motion

Assume that the velocity of an incident star is $\vec{v}_2$ before collision and $\vec{v}_2{}'$ after collision. The angle of deviation is $\pi - 2\Psi$



$$\cos(\pi - 2\Psi) = \frac{\vec{v}_2 \cdot \vec{v}_2{}'}{v_2\, v_2{}'} \tag{A.3}$$

In order to find the time deflection, it is necessary to express $\vec{v}_2{}'$ as a function of the initial parameters. Assume that the velocity of the star in the cm frame of reference before and after an encounter is $\vec{v}_{2g}$, $\vec{v}_{2g}{}'$. Assume that it collides with another star with respective velocities $\vec{v}_1$, $\vec{v}_1{}'$. Then the velocity of the center of gravity $\vec{V}_g$ can be related to the particles' velocities:

$$(m_1+m_2)\vec{V}_g = m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1{}' + m_2\vec{v}_2{}' \;, \tag{A.4}$$

where m_1 and m_2 are the masses of the two particles, since the total momentum of the system is invariant. The relative velocity before and after collision is

$$\vec{V} = \vec{v}_2 - \vec{v}_1 \;, \tag{A.5}$$

$$\vec{V}{}' = \vec{v}_2{}' - \vec{v}_1{}' \;. \tag{A.6}$$

Assume that $\vec{v}_2$ is parallel to the z - axis of our coordinate system and that the angle between $\vec{v}_2$ and $\vec{v}_1$ is θ. Then

$$V^2 = v_1^2 + v_2^2 - 2v_1v_2\cos\theta \tag{A.7}$$

and

$$\vec{v}_{2g} = \vec{v}_2 - \vec{V}_g = \frac{m_1}{m_1+m_2}\vec{V} \;, \tag{A.8}$$

$$\vec{v}_{2g}{}' = \vec{v}_2{}' - \vec{V}_g = \frac{m_1}{m_1+m_2}\vec{V}{}' . \tag{A.9}$$

The angle between $\vec{v}_{2g}$ and $\vec{v}_{2g}{}'$ is the same as the angle between $\vec{V}$ and $\vec{V}{}'$ since it may be shown that $|\vec{V}| = |\vec{V}{}'|$, due to the fact that energy is conserved in the collision. In the orbital plane each body describes a hyperbola about the other. At the end of the encounter, the relative velocity $\vec{V}$

is deflected by $\pi - 2\psi$ in the orbital plane, where [4]

$$\cos\psi = \frac{1}{\left(1 + \frac{D^2 V^4}{G^2(m_1+m_2)^2}\right)^{1/2}} \tag{A.9a}$$

The impact parameter is denoted by D.
Equation (A.9) gives

$$\vec{v}_2 \cdot \vec{v}_2{}' = \vec{v}_2 \cdot \left(\vec{v}_{2g}{}' + \vec{V}_g\right) \tag{A.10}$$

$$v_2 v_2{}' \cos(\pi - 2\psi) = \vec{v}_2 \cdot \vec{v}_{2g}{}' + \vec{v}_2 \cdot \vec{V}_g \tag{A.11}$$

I. Call the direction cosine of $\vec{v}_2$ with respect to $\vec{V}$ = $\cos(\Phi - \delta)$

II. Call the direction cosine of an axis in the orbital plane perpendicular to $\vec{V}$ = $-\sin(\Phi-\delta)\ \cos\Theta$,

III. Call the direction cosine of an axis perpendicular to the orbital plane and $\vec{V}$ = $\sin(\Phi-\delta)\ \sin\Theta$,

where δ is the angle between $\vec{v}_2$ and $\vec{V}g$. Since $\vec{v}_{2g}{}'$ is along $\vec{V}'$, the direction cosines of $\vec{v}_{2g}{}'$ in the same directions are

I along $\vec{V}$, = $\cos(\phi' - \phi)$

II in the orbital plane and $\perp$ to $\vec{V}$, = $\sin(\phi' - \phi)$,

III perpendicular to the orbital plane and $\vec{V}$, = 0.

Therefore

$$\vec{v}_2 \cdot \vec{v}_{2g}{}' = v_2 v_{2g}{}' \left(\cos(\phi'-\phi)\cos(\Phi-\delta) - \sin(\phi'-\phi)\sin(\Phi-\delta)\cos\Theta\right) \tag{A.12}$$

and equation (A.11) may be written

$$v_2 v_2' \cos(\pi - 2\Psi) = v_2 \left\{ v_{2g}' \left[\cos(\phi' - \phi) \cos(\Phi - \delta) - \right.\right.$$
$$\left.\left. - \sin(\phi' - \phi) \sin(\Phi - \delta) \cos\Theta \right] + V_g \cos\delta \right\} . \tag{A.13}$$

However the angle between $\vec{V}$ and $\vec{V}'$ has already been defined to be $\pi - 2\psi$

$$\pi - 2\psi = \phi' - \phi . \tag{A.14}$$

Equation(A.13) may thus be written:

$$\cos 2\Psi = \frac{1}{v_2'} \left\{ v_{2g}' \left(\cos 2\psi \cos(\Phi - \delta) + \sin 2\psi \sin(\Phi - \delta) \cos\Theta \right) - \right.$$
$$\left. - V_g \cos\delta \right\} . \tag{A.15}$$

Now the direction cosine of $\vec{v}_2$ with respect to $\vec{V}$ is just

$$\cos(\Phi - \delta) = \frac{\text{rel. vel.} \parallel \vec{v}_2}{\text{rel. vel.}} = \frac{v_2 - v_1 \cos\theta}{V} \tag{A.16}$$

and

$$\sin(\Phi - \delta) = \frac{-\text{rel. vel} \perp v_2}{\text{rel. vel.}} = \frac{v_1 \sin\theta}{V} . \tag{A.17}$$

Equation(A.4) gives

$$(m_1 + m_2) \vec{V}_g = m_1 \vec{v}_1 + m_1 \vec{v}_2 ,$$

$$(m_1 + m_2)\, v_2 V_g \cos\delta = m_1 v_1 v_2 \cos\theta + m_2 v_2^2 \quad ;$$

$$V_g \cos\delta = (m_1 + m_2)^{-1} (m_1 v_1 \cos\theta + m_2 v_2) \quad . \tag{A.18}$$

Since $|\vec{V}| = |\vec{V}'|$ equation (A.9) gives

$$\frac{v_{2g}'}{V} = \frac{m_1}{m_1 + m_2} \tag{A.19}$$

By substituting (A.16), (A.17), (A.18) and (A.19) into (A.15) this reduces to:

$$\cos 2\Psi = \frac{1}{(m_1+m_2)\, v_2'} \Big\{ 2 m_1 (v_2 - v_1 \cos\theta) \cos^2\psi +$$

$$+ 2 m_1 v_1 \sin\theta \cos\omega \sin\psi \cos\psi - (m_1 + m_2)\, v_2 \Big\} \quad . \tag{A.20}$$

The change in velocity parallel and perpendicular to the original direction of motion is

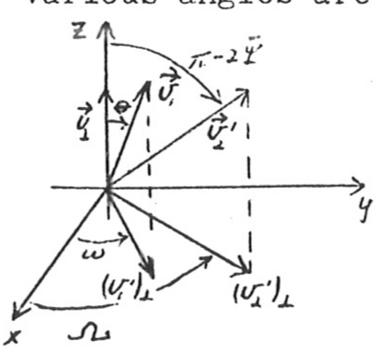
$$\Delta v_{\parallel} = v_2' \cos(\pi - 2\Psi) - v_2 \quad , \tag{A.21}$$

$$\Delta v_{\perp} = v_2' \sin(\pi - 2\Psi) \quad . \tag{A.22}$$

Combining (A.20) and (A.21),

$$\Delta v_{\parallel} = -\frac{2m_1}{m_1+m_2}\left\{(v_2 - v_1\cos\Theta)\cos\Psi + v_1\sin\Theta\cos\Theta\sin\Psi\right\}\cos\Psi \tag{A.23}$$

which is the same expression as that given by Chandrasekhar. In order to find the deviation perpendicular to the field, it is necessary to choose two specific axes to refer to since (A.22) will only give the magnitude of the deviation. The various angles are identified in the diagram.



(A.24)

Now

$$\vec{v}_2{}' = \left\{v_2'\left[\sin(\pi-2\Psi)\cos\Omega\,\vec{e}_x + \sin(\pi-2\Psi)\sin\Omega\,\vec{e}_y + \right.\right.$$
$$\left.\left. + \cos(\pi-2\Psi)\,\vec{e}_z\right]\right\}.$$

Then $\vec{v}_2 \times \vec{v}_2'$ may be calculated, using equation (A.9)

$$(\vec{v}_2 \times \vec{v}_2{}') = v_2 v_{2g}'\left\{-\sin(\Phi-\delta)\sin\Theta\sin(\phi'-\phi)\,\vec{e}_1 + \right.$$
$$+ \sin(\Phi-\delta)\sin\Theta\cos(\phi'-\phi)\,\vec{e}_2$$
$$+ \left[\cos(\Phi-\delta)\sin(\phi'-\phi) + \sin(\Phi-\delta)\cos\Theta\cos(\phi'-\phi)\right]\vec{e}_3$$
$$+ \frac{2m_1}{m_1+m_2} v_2 v_1 \sin\Theta\; \vec{e}_{v_2}\times\vec{e}_{v_1}\,, \tag{A.25}$$

where

$$\vec{v}_2\times\vec{v}_1 = v_2 v_1 \sin\Theta\; \vec{e}_{v_2}\times\vec{e}_{v_1}\,. \tag{A.26}$$

In the same way as for $\vec{v}_2$, the direction cosine of $\vec{v}_1$ respect to $\vec{V}$ is

$\cos(\Phi-\delta+\theta)$, since it lies in the plane of $\vec{v}_2$ and $\vec{V}$.

Therefore,

$$\vec{v}_1 = v_1 \{ \cos(\Phi-\delta+\theta)\, \vec{e}_1 - \sin(\Phi-\delta+\theta) \cos\Theta\, \vec{e}_2 + \\ + \sin(\Phi-\delta+\theta) \sin\Theta\, \vec{e}_3 \} \tag{A.27}$$

and

$$\vec{v}_2 \times \vec{v}_1 = v_1 v_2 \{ [\sin(\Phi-\delta) \sin\Theta \cos(\Phi-\delta+\theta) - \cos(\Phi-\delta) \sin(\Phi-\delta+\theta) \sin\Theta]\, \vec{e}_2 + \\ + [\sin(\Phi-\delta) \cos\Theta \cos(\Phi-\delta+\theta) - \cos(\Phi-\delta) \sin(\Phi-\delta+\theta) \cos\Theta]\, \vec{e}_3 \} . \tag{A.28}$$

Substituting (A.14),(A.16),(A.17) and (A.13)

$$(\vec{v}_2 \times \vec{v}_1) \cdot (\vec{v}_2 \times \vec{v}_2{}') = v_2{}^2 v_1 v_2{}' \sin\theta \sin(\pi-2\Psi) \cos(\Omega-\omega) = \\ = v_2{}^2 v_1 \frac{m_1}{m_1+m_2} \{ v_1 \sin^2\theta [1-\cos(\pi-2\psi)] + \\ + v_1 \sin\theta \cos\theta \sin(\pi-2\psi) \cos\Theta - \\ - v_2 \sin\theta \sin(\pi-2\psi) \cos\Theta , \tag{A.29}$$

$$(\vec{v}_2 \times \vec{v}_1) \times (\vec{v}_2 \times \vec{v}_2{}') = v_2{}^2 v_1 v_2{}' \sin\theta \sin(\pi-2\Psi) \sin(\Omega-\omega)\, \vec{e}_{v_2} = \\ = - v_2{}' v_1 \frac{m_1 V}{m_1+m_2} \sin\Theta \sin(\pi-2\psi) \sin\theta\, \vec{e}_{v_2} , \tag{A.30}$$

so that (A.29) and (A.30) reduce to

$$v_2' \sin\theta \sin(\pi - 2\Psi) \cos(\Omega - \omega) = \frac{m_1}{m_1 + m_2} \Big\{ v_1 \sin^2\theta \left[1 - \cos(\pi - 2\psi)\right] +$$

$$+ v_1 \sin\theta \cos\theta \sin(\pi - 2\psi) \cos\Theta -$$

$$- v_2 \sin\theta \sin(\pi - 2\psi) \cos\Theta \Big\} , \qquad (A.31)$$

$$v_2' \sin(\pi - 2\Psi) \sin(\Omega - \omega) = - \frac{m_1 V}{m_1 + m_2} \sin\Theta \sin(\pi - 2\psi) \sin\theta . \qquad (A.32)$$

However,

$$\sin\Omega = \cos\omega \sin(\Omega - \omega) + \sin\omega \cos(\Omega - \omega) \qquad (A.33)$$

$$\cos\Omega = \cos\omega \cos(\Omega - \omega) - \sin\omega \sin(\Omega - \omega) \qquad (A.34)$$

Using (A.33) and (A.34), equations (A.31) and (A.32) may be combined to give

$$v_2' \sin\theta \sin 2\Psi \sin\Omega = \frac{2m_1}{m_1 + m_2} \cos\psi \sin\theta \Big\{ -V \sin\Theta \sin\psi \cos\omega +$$

$$+ \sin\omega \left[v_1 \sin\theta \cos\psi + v_1 \cos\theta \sin\psi \cos\Theta - v_2 \sin\psi \cos\Theta \right] \Big\} , \qquad (A.35)$$

$$v_2' \sin\theta \sin 2\Psi \cos\Omega = \frac{2m_1}{m_1 + m_2} \sin\theta \cos\psi \Big\{ V \sin\Theta \sin\psi \sin\omega + \qquad (A.36)$$

$$+ \cos\omega \left[v_1 \sin\theta \cos\psi + v_1 \cos\theta \sin\psi \cos\Theta - v_2 \sin\psi \cos\Theta \right] \Big\} .$$

From (A.24) it is obvious that

$$\Delta v_{2x} = v_{2x}' = v_2' \sin 2\Psi \cos \Omega \tag{A.37}$$

and

$$\Delta v_{2y} = v_{2y}' = v_2' \sin 2\Psi \sin \Omega \tag{A.38}$$

so that

$$(\Delta v_{2\perp})^2 = (v_{2x}')^2 + (v_{2y}')^2 = \frac{4 m_1^2}{(m_1 + m_2)^2} \cos^2\psi \Big\{ v_1^2 + v_2^2 - 2 v_1 v_2 \cos\theta - \left[(v_2 - v_1 \cos\theta)\cos\psi + v_1 \sin\theta \sin\psi \cos\Theta \right]^2 \Big\} \tag{A.39}$$

which is the result derived by Chandrasekhar.

If the particle is subject to a series of collisions then (A.23), (A.37) and (A.38) may be averaged over these such that

$$\sum \Delta v_{2i} = \Delta t \int_0^\infty dv_1 \int_0^\pi d\theta \int_0^{2\pi} dw \int_0^{D_0} dD \int_0^{2\pi} \frac{d\Theta}{2\pi}\, 2\pi N(v_1, \theta, w)\, V D\, \Delta v_{2i} \tag{A.40}$$

where N is the density distribution of particles of type 1, D is the impact parameter and is related to ω through (A.9a) and it is assumed that Δt is a time such that the particle suffers many collisions but the density distribution may be considered to be unchanged. If the density distribution has a sharp gradient then it would also be a function of θ, and D. It is assumed that this dependence may be neglected. Now

$$\int_0^{2\pi} d\theta \, \Delta v_{2x} = \frac{4\pi m_1 v_1}{m_1+m_2} \cos\psi \cos\omega \sin\theta \cos\psi \, , \tag{A.41}$$

$$\int_0^{2\pi} d\theta \, \Delta v_{2y} = \frac{4\pi m_1 v_1}{(m_1+m_2)} \cos\psi \sin\omega \sin\theta \cos\psi \, , \tag{A.42}$$

$$\int_0^{2\pi} d\theta \, \Delta v_{2z} = -\frac{4\pi m_1}{m_1+m_2} \cos^2\psi \, (v_2 - v_1 \cos\theta) \, ; \tag{A.43}$$

so that (A.40) may be written

$$\Sigma \, \Delta\vec{v}_2 = \Delta t \int_0^\infty dv_1 \int_0^\pi d\theta \int_0^{2\pi} d\omega \int_0^{D_0} dD \, 2\pi N(v_1,\theta,\omega) VD \cdot \\ \cdot \frac{2m_1}{m_1+m_2} (\vec{v}_1 - \vec{v}_2) \cos^2\psi . \tag{A.44}$$

The integral

$$\int_0^{D_0} D\,dD \cos^2\psi = \frac{G^2(m_1+m_2)^2}{2V^4} \log\left(1 + \frac{D_0^2 V^4}{G^2(m_1+m_2)^2}\right) . \tag{A.45}$$

It will be noted, that it is necessary to cut this integral off at some maximum value rather than allowing the integral go to infinite impact parameters, in order to avoid divergence of the term. This is a characteristic of the long range $(\frac{1}{r})^2$ force law. In this case such a cut off is also necessary on theoretical grounds, because the problem may be treated as a series of two body collisions, only for the short range collisions. For charged particles with impact parameters greater than the Debye length,

$$\lambda_D = \left(\frac{KT}{4\pi N e^2}\right)^{1/2} , \qquad \text{(A.45a)}$$

the collective interaction will predominate.

The number of field particles per volume with velocities between $\vec{v}_1$ and $\vec{v}_1 + d\vec{v}_1$ in the element of solid angle $\sin\theta\, d\theta\ d\omega$ is $N(v_1, \theta, \omega)\ dv_1\, d\theta\, d\omega$. This may be also written as

$$N_1\, f(v_1, \theta, \omega)\ v_1\, dv_1\, v_1 \sin\theta\ d\omega\ d\theta ,$$

where $f(v_1, \theta, \omega)$ is the normalized density distribution of particles of type 1, and N_1 is the density of these particles in configuration space.

Thus

$$N(v_1, \theta, \omega) = N_1\, f(v_1, \theta, \omega)\ v_1^2 \sin\theta \qquad \text{(A.46)}$$

and using (A. 45) and (A. 46), (A.44) may be written

$$\sum \Delta\vec{v}_2 = -\Delta t\, 2\pi\, m_1 N_1 (m_1 + m_2)\, G^2 \cdot$$

$$\cdot \int_{-\infty}^{\infty} \frac{\log\left(1 + \frac{b^2 V^4}{G^2 (m_1 + m_2)^2}\right)}{V^3}\ \vec{V}\, f(\vec{v}_1)\, d\vec{v}_1 .$$

(A.47)

Similarly the change of energy of a particle due to a collision may be found from (A.23), (A.37), and (A.38)

$$\Delta KE_2 = \frac{1}{2} m_2 \left\{ \sum_{i=1}^{3} (v_{2i}')^2 - v_2^2 \right\} = \frac{1}{2} m_2 \left(\frac{2m_1}{m_1+m_2} \cos\psi \right)^2 V^2 -$$

$$- \frac{4m_1}{m_1+m_2} v_2 \cos\psi \left(\alpha_1 \cos\psi + \alpha_2 \sin\psi \cos\Theta \right) , \qquad \text{(A.48)}$$

where

$$\alpha_1 = v_2 - v_1 \cos\theta \quad ,$$

$$\alpha_2 = v_1 \sin\theta \quad .$$

The change in energy due to many such collisions is thus:

$$\sum \Delta KE_2 = -\Delta t \, 2\pi N_1 \, m_1 m_2 (m_1+m_2) G^2 \cdot$$

$$\cdot \int_{-\infty}^{\infty} \log \left(1 + \frac{D^2 V^4}{G^2 (m_1+m_2)^2} \right) \frac{\vec{V} \cdot \vec{V_2}}{V^3} f(\vec{v}_1) \, d\vec{v}_1 \qquad \text{(A.49)}$$

using (A.4).

The net change in momentum of the system due to collisions may be written as a sum over the species j.

$$\sum_j (\Sigma \Delta \vec{v}_j)\, m_j N_j f(\vec{v}_j)\, d\vec{v}_j = -$$

$$= -\Delta t \sum_{ij} m_i m_j N_i N_j (m_i + m_j) G^2 2\pi \cdot$$

$$\cdot \int_{-\infty}^{\infty} \log\left[1 + \frac{D_{ij}^2 V_{ij}^4}{G^2 (m_i + m_j)^2}\right] \frac{\vec{v}_j - \vec{v}_i}{V_{ij}^3} f(\vec{v}_i) f(\vec{v}_j)\, d\vec{v}_i\, d\vec{v}_j = 0 , \tag{A.50}$$

as may be seen by switching indices and adding. This changes the sign of the vector.

Similarly, the net change in energy of the system due to collisions is:

$$\sum_j (\Sigma (\Delta KE_j)\, N_j f(\vec{v}_j)\, d\vec{v}_j = -$$

$$= -\Delta t \sum_{ij} 2\pi N_i N_j m_i m_j (m_i + m_j) G^2 \cdot \tag{A.51}$$

$$\cdot \int_{-\infty}^{\infty} \log\left[1 + \frac{D_{ij}^2 V_{ij}^4}{G^2 (m_i + m_j)^2}\right] \frac{(\vec{v}_j - \vec{v}_i) \cdot \vec{V}_g}{V_{ij}^3} f(\vec{v}_i) f(\vec{v}_j)\, d\vec{v}_i\, d\vec{v}_j = 0 ,$$

in the same way. Thus the energy and momentum of the system is invariant under collisions. However a necessary criterion for this to hold is that we set

$$D_{ij} = D_{ji} .$$

In other words the maximum impact parameter must be chosen such that an equal number of collisions of type ij are retained, both for species i and species j. If the Debye length is used this must be some average value for collisions between ions and electrons for example. Of course one may use different maximum values for ion-ion, electron-ion and electron-electron collision.

Now consider a system of only one type of particles. If the incoming particle is travelling with the mean velocity of the system and $f(\vec{v}_1)$ is a symmmetric function about the average velocity, then (A.47) shows that it will experience no net deviation due to collisions. A particle travelling faster than with the mean velocity will be slowed down, and one travelling slower will be accelerated. Because of the difficulty associated with the integration of (A.47) it is customary to assume that v_2 is a fast particle that would lie in the "tail" of the velocity distribution of $f(\vec{v}_1)$. Then it is possible to expand the functions in powers of v_1/v_2 . Chandrasekhar justifies this procedure by the fact that major contribution to the change in velocity is due to the slower particles. This may be valid when discussing the decelleration of an incident particle, but is not as satisfactory when one wishes to follow the motion of a typical particle of the system. For example, in this paper the diffusion coefficients of a plasma were derived and found to be proportional to $\frac{1}{\beta}$. If the friction in turn is different for the fast and slow particles, it would be of interest to find out if it is the fast or the slow particles that leave the system more rapidly and if the average energy per particle, or the temperature, is decreased or increased due to diffusion.

C. The Dynamical Friction due to Coulomb Collisions.

From equations (A.1) and (A.2) it is evident that

$$G = -\frac{e_1 e_2}{m_1 + m_2} \qquad \text{(A.52)}$$

For an electrostatic interaction force(A47) and(A49) may thus be written

$$\sum \Delta \vec{v}_2 = -\Delta t \, \frac{2\pi N_1 e_1^2 e_2^2}{m_2 \mu_{12}} \int_{-\infty}^{+\infty} \log(1 + C_0 V^4) \, \frac{\vec{V}}{V^3} \, f(\vec{v}_1) \, d\vec{v}_1 \, , \qquad \text{(A.53)}$$

$$\sum \langle \Delta KE \rangle_2 = -\Delta t \, \frac{2\pi N_1 e_1^2 e_2^2}{\mu_{12}} \int_{-\infty}^{+\infty} \log(1 + C_0 V^4) \, \frac{\vec{V} \cdot \vec{V}_2}{V^3} \, f(\vec{v}_1) \, d\vec{v}_1 \, , \qquad \text{(A.54)}$$

where μ_{12} is the reduced mass of the particles and

$$C_0 = -\frac{\mu_{12}}{e_1 e_2} D_{12} \qquad \text{(A.55)}$$

It is customary to set the maximum impact parameter equal to the Debye length(A45a). The friction coefficient Λ is then defined by the equation

$$\sum \langle \Delta v_2 \rangle = -\Delta t \, \Lambda \langle v_2 \rangle \qquad \text{(A.56)}$$

From equation (A.52) it is obvious that Λ may be a tensor. Normally it is assumed that Λ is a constant, but it will be shown that this is not correct unless the velocity of the incident particle is much less than the thermal velocities of the field particles. When the distribution function is Maxwellian, then (A.56) is parallel to $\vec{v}_2$ and the friction tensor is a scalar.

In the previous work, it was assumed that the equations (A.53) and (A.54) should be integrated to some maximum impact parameter independent of the relative velocity $\vec{V}$. This is customary for Rutherford scattering of two particles when this distance may be chosen by requiring that the scattering angle be greater than some minimum value. However, in a plasma, it may be more suitable to include only collisions where the distance of closest approach of the particles is less than a specified value. For example, two-particle collisions predominate when the particles are within a Debye length of each other. Placing such a condition on the spatial integration gives "greater weight" to collisions between particles having a large relative velocity, since these can approach closer to each other than particles having less energy.

Equations (A.53) and (A.54) may then be written

$$\sum \Delta\vec{v}_2 = -\Delta t \frac{2\pi N_1 e_1^2 e_2^2}{m_2 \mu_{12}} \int \log (1+cV^2)^2 \frac{\vec{V}}{V^3} f(\vec{v}_1)\, d\vec{v}_1 , \qquad \text{(A.57)}$$

$$\sum (\Delta KE_2) = -\Delta t \frac{2\pi N_1 e_1^2 e_2^2}{\mu_{12}} \int \log (1+cV^2)^2 \frac{\vec{V}\cdot\vec{V}_g}{V^3} f(v_1)\, d\vec{v}_1 ; \qquad \text{(A.58)}$$

where

$$c = -\frac{\mu_{12}}{e_1 e_2} r_0 \qquad \text{(A.59)}$$

and r_0 is the maximum distance of closest approach giving two-body scattering. The integration over $|\vec{V}|$ must be restricted to

$$V^2 \geq \frac{2e_1 e_2}{\mu_{12} r_0} \qquad \text{(A.60)}$$

for particles having the same sign charge. Particles with a smaller relative velocity than this cannot approach to within a distance r_0 of each other and therefore will not suffer two particle collisions. For particles having opposite charge the integration is performed over all velocities as in (A.53) and (A.54).

The distance r_0 may be approximated by the effective length at which the Salpeter two-particle correlation functions[5] are reduced to $1/e$ of their unshielded values. This gives r_0, equal to

$$r_{01}^{\,2} = \left[\frac{4\pi N e^2}{kT_e} + \frac{4\pi N e^2}{kT_i}\right]^{-1} \qquad (A.61)$$

for electron-electron and ion-ion collisions and r_{02} equal to

$$\frac{T_i}{T_e}\, e^{\left(1-\frac{r_{02}}{r_{01}}\right)} + \frac{T_e - T_i}{T_e}\, e^{\left(1 - r_{02}\frac{4\pi N e^2}{kT_e}\right)} = 1$$

$$(A.62)$$

for ion-electron collisions.

D. Integration of the Scattering Functions.

In this section the integrals

$$Q_1 = \int \log(1+cV^2)^2\, \frac{\vec{V}}{V^3}\, f(\vec{v}_1)\, d\vec{v}_1 \ , \qquad (A.63)$$

$$Q_2 = \int \log(1+cV^2)^2\, \frac{\vec{V}\cdot\vec{V}_3}{V^3}\, f(\vec{v}_1)\, d\vec{v}_1 \ , \qquad (A.64)$$

which appear in the equations (A.57) and (A.58) will be calculated for a Maxwellian distribution function. The mathematical details are carried out for attracting particles, where $c > 0$, but the corresponding results for repulsing charges are also given.

Using the Fourier convolution product[6] ,(A.63) may be written

$$Q_1 = 2\int_{-\infty}^{\infty} g(\vec{v}_2 - \vec{v}_1)\, f(\vec{v}_1)\, d\vec{v}_1 = 2\int_{-\infty}^{\infty} g(\vec{k})\, f(\vec{k})\, e^{-i\vec{k}\cdot\vec{v}_2}\, d\vec{k} \ , \qquad (A.65)$$

where

$$g(\vec{v}_2-\vec{v}_1) = \text{Log}(1+cV^2)\,\frac{\vec{V}}{V^3} \qquad \text{(A.66)}$$

and $f(\vec{k})$ is the Fourier transform of the velocity distribution function.

Then

$$g(\vec{k}) = (2\pi)^{-3/2}\int_{-\infty}^{+\infty} g(\vec{V})\, e^{i\vec{k}\cdot\vec{V}}\, d\vec{V} = i\,\vec{e}_k \int_0^{\infty} \frac{\log(1+cV^2)}{\sqrt{kV}}\, J_{3/2}(kV)\, dV\ , \qquad \text{(A.67)}$$

where $J_{3/2}(kv)$ is a Bessel function of order 3/2. For a Maxwellian distribution function

$$f(\vec{k}) = (2\pi)^{-3/2}\, e^{-k^2/4\alpha}$$

$$\alpha = \frac{2KT_i}{m_i} \qquad \text{(A.68)}$$

so that

$$Q_1 = 2\,\frac{\vec{v}_2}{v_2^{3/2}} \int_0^{\infty} \frac{\log(1+cV^2)}{V^{1/2}}\, dV \int_0^{\infty} k\, J_{3/2}(kV)\, J_{3/2}(kv_2)\, e^{-k^2/4\alpha}\, dk =$$

$$= 4\alpha\,\frac{\vec{v}_2}{v_2^{3/2}}\, e^{-\alpha v_2^2} \int_0^{\infty} \frac{\log(1+cV^2)}{V^{1/2}}\, I_{3/2}(2\alpha v_2 V)\, e^{-\alpha V^2}\, dV\ , \qquad \text{(A.69)}$$

after performing the integrations over the angles and $\vec{k}$, where

$$I_\nu(z) = e^{-i\pi\nu/2}\, J_\nu(iz)\ . \qquad \text{(A.70)}$$

It is convenient to set

$$|cV^2| = t\ , \qquad \text{(A.71)}$$

$$\left|\frac{\alpha}{c}\right| = x\ . \qquad \text{(A.72)}$$

Then (A.69) may be written

$$Q_1 = 2x\alpha^{3/2}\vec{v}_2\, e^{-\alpha v_2^2}\sum_{n=0}^{\infty} x^n(\alpha v_2^2)^n \frac{1}{n!\,\Gamma(n+5/2)}\,\cdot \tag{A.73}$$
$$\cdot\int_0^{\infty} \log(1+t)\, t^n e^{-xt}\, dt\,,$$

where $I_{3/2}(2\alpha v_2 V)$ has been expanded in a power series. Note that

$$\int_0^{\infty}\log(1+t)\,t^n e^{-xt}\,dt = (-1)^n \frac{d^n}{dx^n}\int_0^{\infty}\log(1+t)\,e^{-xt}\,dt = \tag{A.74}$$
$$= (-1)^n \frac{d^n}{dx^n}\left[-\tfrac{1}{x}\, e^{x} E_i(-x)\right]\,,$$

where $E_i(-x)$ is the exponential integral,

$$E_i(-x) = -\int_x^{\infty}\frac{e^{-t}}{t}\,dt = \log \gamma x + \sum_{m=1}^{\infty}(-1)^m \frac{x^m}{m(m!)} \tag{A.75}$$

and $\quad \gamma = 1.781.$ (A.76)

Thus

$$Q_1 = 2x\alpha^{3/2}\vec{v}_2\, e^{-\alpha v_2^2}\sum_{n=0}^{\infty}(-1)^n x^n(\alpha v_2^2)^n \frac{1}{n!\,\Gamma(n+5/2)}\,\frac{d^n}{dx^n}\left[-\tfrac{1}{x}\,e^{x}E_i(-x)\right]. \tag{A.77}$$

The corresponding equation for repulsing particles is

$$Q_1' = 2x\alpha^{3/2}\vec{v}_2\, e^{-\alpha v_2^2}\sum_{n=0}^{\infty}(-1)^n x^n(\alpha v_2^2)^n \frac{1}{n!\,\Gamma(n+5/2)}\,\frac{d^n}{dx^n}\left[-\tfrac{1}{x}\,e^{-x}E_i(-x)\right]. \tag{A.78}$$

The energy scattering function (A.64), may be integrated in the same way, setting

$$Q_2 = 2\int_{-\infty}^{+\infty} \vec{g}(\vec{v}_2-\vec{v}_1)\cdot\vec{F}(\vec{v}_1)\,d\vec{v}_1 = 2\int_{-\infty}^{\infty}\vec{g}(\vec{\kappa})\cdot\vec{F}(\vec{\kappa})\,e^{-i\vec{\kappa}\cdot\vec{v}_2}\,d\vec{\kappa}\,, \tag{A.79}$$

where $g(\vec{v}_2 - \vec{v}_1)$ is given by (A.66) and

$$\vec{F}(\vec{v}_1) = f(\vec{v}_1)\, \vec{V}_g \,. \tag{A.80}$$

Then

$$\vec{F}(\vec{k}) = (2\pi)^{-3/2} \mu_{12}\, e^{-k^2/4\alpha} \left[\frac{\vec{v}_2}{m_1} + \frac{i\vec{k}}{2\alpha m_2} \right] \tag{A.81}$$

for a Maxwellian distribution and

$$Q_2 = 4\alpha\, v_2^{1/2}\, e^{-\alpha v_2^2} \int_0^\infty \frac{\log(1+cV^2)}{V^{1/2}}\, e^{-\alpha V^2} I_{3/2}(2\alpha v_2 V)\, dV -$$

$$- \frac{4\alpha \mu_{12}}{m_2 v_2^{1/2}}\, e^{-\alpha v_2^2} \int_0^\infty V^{1/2} \log(1+cV^2)\, e^{-\alpha V^2} I_{1/2}(2\alpha v_2 V)\, dV \,, \tag{A.82}$$

in the same way as with (A.69). Making the substitutions (A.71) and (A.72) and expanding the hyperbolic Bessel functions in power series gives

$$Q_2 = 2\sqrt{\alpha}\, e^{-\alpha v_2^2} \sum_{n=0}^{\infty} x^{n+1} (\alpha v_2^2)^n \frac{(-1)^n}{n!\,\Gamma(n+\frac{3}{2})} \cdot \left[\frac{\alpha v_2^2}{n+\frac{3}{2}} - \frac{\mu_{12}}{m_2} \right] \frac{d^n}{dx^n}\left(-\frac{1}{x} e^{x} E_i(-x) \right) \tag{A.83}$$

for attracting particles and

$$Q_2' = 2\sqrt{\alpha}\, e^{-\alpha v_2^2} \sum_{n=0}^{\infty} (x)^{n+1} (\alpha v_2^2)^n \frac{(-1)^n}{n!\,\Gamma(n+\frac{3}{2})} \cdot \left[\frac{\alpha v_2^2}{n+\frac{3}{2}} - \frac{\mu_{12}}{m_2} \right] \frac{d^n}{dx^n}\left(-\frac{1}{x} e^{-x} E_i(-x) \right) \tag{A.84}$$

for repulsing particles. The rates of loss of velocity and energy of a particle are thus functions of the ratio of the particle speed to thermal speed and of x,

$$x = \left|\frac{\alpha}{c}\right| = \frac{e_1 e_2/\mu_{12}}{\frac{2kT_1}{m_1} r_0} \tag{A.85}$$

where kT_1 is the average thermal energy of the field particles. This parameter x may be recognized as the ratio of the Landau length, or the distance of closest approach of the field particles, to the Debye length. This ratio is in general much smaller than one in a plasma, where small angle collisions normally predominate. When $x \ll 1$, the scattering terms for attracting and repulsing charged particles are approximately the same and

$$\frac{d^n}{dx^n}\left[-\frac{1}{x} e^{tx} E_i(-tx)\right] \rightarrow \frac{(-1)^{n+1} n!}{x^{n+1}} \log x \tag{A.86}$$

Therefore equations (A.77), (A.78), and (A.84) reduce to

$$Q_1 = 2\alpha^{3/2} \vec{v}_2\, e^{-\alpha v_2^2} \sum_{n=0}^{\infty} (\alpha v_2^2)^n \frac{\log(1/x)}{\Gamma(n+5/2)} = \frac{2\vec{v}_2}{v_2^3} \log\frac{1}{x}\left[1 - \frac{1}{\Gamma(3/2)}\Gamma(3/2, \alpha v_2^2)\right] \tag{A.87}$$

$$Q_2 = 2\sqrt{\alpha}\, e^{-\alpha v_2^2} \sum_{n=0}^{\infty} (\alpha v_2^2)^n \frac{\log\frac{1}{x}}{\Gamma(n+3/2)}\left[\frac{\alpha v_2^2}{n+3/2} - \frac{\mu_{12}}{m_2}\right] =$$

$$= \frac{2}{v_2} \log\frac{1}{x}\left\{\left[1 - \frac{\Gamma(3/2, \alpha v_2^2)}{\Gamma(3/2)}\right] - \frac{\mu_{12}}{m_2}\left[1 - \frac{\Gamma(1/2, \alpha v_2^2)}{\Gamma'(1/2)}\right]\right\} =$$

$$= \frac{2\mu_{12}}{v_2} \log\frac{1}{x}\left\{\frac{-2}{m_2\sqrt{\pi}}\sqrt{\alpha v_2^2}\, e^{-\alpha v_2^2} + \frac{1}{m_1}\left[1 - \frac{\Gamma(3/2, \alpha v_2^2)}{\Gamma(3/2)}\right]\right\}$$

where

$$\Gamma(a, x) = \int_x^{\infty} e^{-t} t^{a-1} dt = \Gamma(a) - \int_0^x e^{-t} t^{a-1} dt\,. \tag{A.89}$$

Using (A.63) and (A.64), equations (A.57) and (A.58) may be written

$$\sum \Delta\vec{v}_2 = -\Delta t \frac{4\pi N_1 e_1^2 e_2^2}{m_2 \mu_{12}} \frac{\vec{v}_2}{v_2^3} \log\frac{1}{x}\left[1 - \frac{\Gamma(3/2, \alpha v_2^2)}{\Gamma(3/2)}\right] =$$

$$= -\Delta t \frac{4\pi N_1 e_1^2 e_2^2}{m_2 \mu_{12}} \frac{\vec{v}_2}{v_2^{-3}} \log\frac{1}{x}\left[\frac{2}{\sqrt{\pi}}\int_0^{\alpha v_2^2} \sqrt{t}\, e^{-t} dt\right], \tag{A.90}$$

$$\sum (\Delta KE)_2 = -\Delta t \frac{4\pi N_1 e_1^2 e_2^2}{v_2} \log\frac{1}{x}\left[\frac{1}{m_1}\left\{1 - \frac{\Gamma(3/2, \alpha v_2^2)}{\Gamma 3/2}\right\} - \right.$$

$$\left. - \frac{2}{m_2\sqrt{\pi}} \sqrt{\alpha v_2^2}\, e^{-\alpha v_2^2}\right] = -\Delta t \frac{4\pi N_1 e_1^2 e_2^2}{v_2} \log\frac{1}{x} \cdot$$

$$\cdot \frac{2}{\sqrt{\pi}}\left[\frac{1}{m_1}\int_0^{\alpha v_2^2} \sqrt{t}\, e^{-t} dt - \frac{1}{m_2}\sqrt{\alpha v_2^2}\, e^{-\alpha v_2^2}\right]. \tag{A.91}$$

These equations hold for all values of αv_2^2 when $x \ll 1$.

If the velocity distribution function $f(\vec{v}_2)$ is also Maxwellian, then equation (A.91) may be multiplied by $N_2 f(\vec{v}_2)$ and integrated over $\vec{v}_2$ to give the average change in energy of particles "2" due to collisions with particles "1". The resulting expression,

$$\left\langle \sum (\Delta KE)_2 \right\rangle_{av.} \propto T_1 - T_2 , \tag{A.92}$$

demonstrates that the particles "2" will gain energy on the average due to collisions with the field particles if they have a lower temperature than the field, and loose energy if they are "hotter" than the field particles.

The asymptotic expressions of the velocity diffusion coefficient (A.90) and (A.91) for fast and slow particles may also be given.

When $\alpha v_2^2 \ll 1$ (A.93)

$$\Gamma(3/2, \alpha v_2^2) \longrightarrow \Gamma(3/2) - \frac{2}{3}(\alpha v_2^2)^{3/2} ,$$

so that (A.90) and (A.91) become

$$\sum \Delta \vec{v}_2 = -\Delta t \cdot \frac{16\sqrt{\pi}\, N_1 e_1^2 e_2^2}{3\, m_2 \mu_{12}} \left(\frac{m_1}{2kT_1} \right)^{3/2} \vec{v}_2 \log \frac{1}{\chi} \tag{A.94}$$

and

$$\sum (\Delta KE)_2 = -\Delta t\, \frac{8\sqrt{\pi}}{3} N_1 e_1^2 e_2^2 \left(\frac{m_1}{2kT_1} \right)^{1/2} \frac{2}{2kT_1 m_2} \cdot \left[\frac{1}{2} m_2 v_2^2 - \frac{3}{2} kT_1 \right]. \tag{A.95}$$

The friction coefficient (A.56) is thus a constant when the incident particle is slower than the thermal velocity of the field particles.

When $\alpha v_2^2 >> 1$,

$$I(\tfrac{3}{2}, \alpha v_2^2) \rightarrow \sqrt{\alpha v_2^2}\, e^{-\alpha v_2^2} \left[1 + \frac{1}{2\alpha v_2^2} \right] \tag{A.96}$$

and

$$\sum \Delta \vec{v}_2 = -\Delta t \frac{4\pi N_1 e_1^2 e_2^2}{m_2 \mu_{12}} \frac{\vec{v}_2}{v_2^3} \log \frac{1}{\chi}, \tag{A.97}$$

$$\sum (\Delta KE)_2 = -\Delta t \frac{4\pi N_1 e_1^2 e_2^2}{v_2} \log \frac{1}{\chi} \left[\frac{1}{m_1} - \frac{2\sqrt{\alpha v_2^2}}{\mu_{12}\sqrt{\pi}} e^{-\alpha v_2^2} \right]. \tag{A.98}$$

For a fast incident particle the friction coefficient is proportional to v_2^{-3} and the energy loss due to binary collisions goes as v_2^{-1}.

Chandrasekhar[3)] has performed a similar integration of (A.47) and (A.49). For a Maxwellian plasma under the assumption that $c_0 V^4 >> 1$ and that the relative velocity may be replaced by some average value in the logarithmic term before performing the integration over v_1 . Furthermore, the integration over v_1 , is limited to $v_1 \leq v_2$. The resulting expressions correspond to (A.97) and (A.98) for a fast incident particle. In deriving his formula for the resistivity of a plasma, Spitzer[7] used expressions of this type for the velocity diffusion coefficients $\langle \Delta v \rangle_{av}$ and $\langle \Delta v^2 \rangle_{av}$. [8],[9],[10].

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