INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

Kinetic Equations for Plasmas.
Part II.

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IPP/6/15 February 1964

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Institut für Plasmaphysik GmbH und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

Summary.

The kinetic equation for a one species homogeneous plasma in the presence of an external homogeneous magnetic field is derived with the aid of the extension method of Sandri[1]. Although collective effects are not taken into account the collision integral converges for large interaction distances due to the fact that the Larmor radius plays the role of a natural cut-off. This is explained by considering the collision process itself in some detail.

A new contribution to the kinetic equation is found exhibiting an anisotropic character and due to the long duration of the interaction of two gyrating particles with a small relative velocity of the guiding centres.

Interesting results are obtained for the relaxation into the kinetic regime.

1. Introduction

We consider a homogeneous electron gas embedded in a continous neutralizing background and in the presence of a homogeneous magnetic field.

We want to derive a kinetic equation for this system by means of the extension method of Sandri [1,2,3,4]. This can be done with the weak coupling expansion as discussed in Part I, section 5. This means that collective effects are not taken into account. It turns out, however, that the Larmor radius plays the role of a natural cut-off, as has been observed also by Silin [5], so that the method is satisfactory if the Larmor radius is smaller than the Debye length. Nevertheless the neglect of collective effects is only justified if the zero order distribution function is stable in the sense of Part I, section 5.

Another restriction is that the zero order distribution should not give rise to currents because otherwise additional magnetic fields would be present which are not taken into account actually.

In section 2 we introduce our basic equations and the notation, in sections 3,4,5 we treat the zero-, first-, and second order theory respectively, in section 6 we discuss the first order correlation, in section 7 the collision integral and in section 8 the collision process of two electrons in a magnetic field. This gives a good understanding of the convergence of the collision integral for large interaction distances without collective effects.
In section 9 we treat the relaxation into the kinetic regime and in section 10 finally we collect our main results.

2. Basic equations and notation.

We have to complete the equations (2.14) and (2.15) of Part I with terms representing the Lorentz force. For the distribution function we have

$$\frac{\partial F(1)}{\partial t} - \frac{e}{mc}\vec{v}_1 \times \vec{B}\cdot\frac{\partial F(1)}{\partial \vec{v}_1} = \frac{n}{m}\frac{\partial}{\partial \vec{v}_1}\cdot\int \frac{\partial \Phi_{12}}{\partial \vec{x}_1} g(1,2)\,d\xi_2 \qquad (2.1)$$

and for the pair correlation function

$$\left\{\frac{\partial}{\partial t} + (\vec{v}_1 - \vec{v}_2)\cdot\frac{\partial}{\partial \vec{x}_1} - \frac{e}{mc}\left(\vec{v}_1\times\vec{B}\cdot\frac{\partial}{\partial \vec{v}_1} + \vec{v}_2\times\vec{B}\cdot\frac{\partial}{\partial \vec{v}_2}\right)\right\} g(1,2)$$

$$- \frac{\partial \Phi_{12}}{m\partial \vec{x}_1}\cdot\left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2}\right)\left\{g(1,2) + F(1)F(2)\right\}$$

$$(2.2)$$

$$- \frac{n}{m}\frac{\partial F(1)}{\partial \vec{v}_1}\cdot\int\frac{\partial \Phi_{13}}{\partial \vec{x}_1} g(2,3)\,d\xi_3 - \frac{n}{m}\frac{\partial F(2)}{\partial \vec{v}_2}\cdot\int\frac{\partial \Phi_{23}}{\partial \vec{x}_2} g(1,3)\,d\xi_3 =$$

$$= \frac{n}{m}\int\left(\frac{\partial \Phi_{13}}{\partial \vec{x}_1}\cdot\frac{\partial}{\partial \vec{v}_1} + \frac{\partial \Phi_{23}}{\partial \vec{x}_2}\cdot\frac{\partial}{\partial \vec{v}_2}\right) g_3(1,2,3)\,d\xi_3$$

where Φ_{ij} is the Coulomb potential

$$\Phi_{ij} = e^2\left(|\vec{x}_i - \vec{x}_j|\right)^{-1} \qquad (2.3)$$

and the correlations are defined as in Part I, eq.(2.13). The notation is the same as that of Part I.
We choose the z-axis parallel to the magnetic field and write

$$\vec{B} = B_0\,\vec{e}_z \qquad \omega = \frac{eB_0}{mc} \qquad (2.4)$$

Furthermore we introduce relative distances and velocities by

$$\vec{x} = \vec{x}_1 - \vec{x}_2 \qquad \vec{u} = \vec{v}_1 - \vec{v}_2 \qquad (2.5)$$

Frequently Fourier transforms will be used, i.e.

$$\Phi_{12}(x) = \int \hat{\Phi}_{12}(k) \exp(i\vec{k}\cdot\vec{x})\, d^3k \tag{2.6}$$

and similarly for the pair correlation.

It is convenient to use cylindrical coordinates in velocity space and k-space with the axis parallel to the magnetic field. We use the notation

$$\vec{v}_1 = (v_\perp, \varphi_1, v_z), \quad \vec{u} = (u_\perp, \varphi, u_z), \quad \vec{k} = (k_\perp, \psi, k_z) \tag{2.7}$$

As already has been remarked we consider the plasma as a weakly coupled system and apply the extension method of Sandri[1]. In Part I, section 7, we have seen that under certain conditions for the initial correlation functions the system relaxes into the kinetic regime according to the results of the so called simple initial value problem defined by

$$g_s(t=0) = 0$$

$$F^{(0)}(t=0) = F(t=0) \qquad F^{(\nu)}(t=0) = 0 \quad (\nu \geq 1) \tag{2.8}$$

i.e. all intitial correlations are zero and the initial distribution function is incorporated in its zeroth order. In this paper we assume that all conditions of this kind are satisfied so that we can restrict ourselves to the initial conditions (2.8).

As in Part I we do not distinguish typographically the extended functions from the original ones and we usually do not write down the time coordinates τ_0, τ_1, τ_2 etc. as arguments of the extended functions.

3. Zero order theory

In zeroth order of the weak coupling expansion we have (compare Part I, section 5)

$$\frac{\partial F^{(0)}}{\partial \tau_0} + \omega \frac{\partial F^{(0)}}{\partial \varphi_1} = 0 \tag{3.1}$$

and, because of eq.(2.8),

$$g_s^{(0)} = 0 \tag{3.2}$$

for all s.

As is well known the Lorentz force gives rise to a term respresenting a derivative with respect to the azimuth in velocity space the axis being parallel to the magnetic field.

The solutinn of eq.(3.1) is simply

$$F^{(0)}(\vec{v_1}, \tau_0) = F^{(0)}(v_\perp, \varphi_1 - \omega\tau_0, v_z, \tau_0 = 0) \tag{3.3}$$

It would be natural to assume that the initial $F^{(0)}$ does not depend on φ_1 . In that case $F^{(0)}(\vec{v_1}, \tau_0)$ is constant in the time coordinate τ_0.

We do not want, however, this loss of generality. It will be seen that the general solution (3.3) hardly complicates the calculations in the following sections. It is easily seen that the restriction to vanishing zero order electric currents, mentioned in section 1, means

$$\begin{aligned} &\int F^{(0)}(v_\perp, \varphi_1, v_z)\, v_\perp \exp(i\varphi_1)\, d^3v = 0 \\ &\int F^{(0)}(v_\perp, \varphi_1, v_z)\, v_z\, d^3v = 0 \end{aligned} \tag{3.4}$$

If the properties (3.4) are satisfied initially they are true for all τ_0 as is clear from eq.(3.3).

The compensation of the charge density of the electrons by the continuous background may be expressed by the normalisation

$$\int F^{(0)}(v_\perp, \varphi_1, v_z)\, d^3v = 1 \tag{3.5}$$

4. First order theory.

The equation for the first order distribution function is

$$\frac{\partial F^{(1)}}{\partial \tau_0} + \frac{\partial F^{(0)}}{\partial \tau_1} + \omega \frac{\partial F^{(1)}}{\partial \varphi_1} = 0 \tag{4.1}$$

and has the solution

$$F^{(1)} = -\tau_0 \frac{\partial}{\partial \tau_1} F^{(0)}\left(v_\perp, \varphi_1 - \omega\tau_0, v_z, \tau_0 = 0\right) \tag{4.2}$$

if eqs.(2.8) and (3.3) are taken into account. Following Sandri[1] we require, however, that $F^{(1)}$ remains small for all τ_0. Therefore

$$\frac{\partial F^{(0)}}{\partial \tau_1} = F^{(1)} = 0 \tag{4.3}$$

For the Fourier transform of the first order pair correlation function we find the equation

$$\left\{\frac{\partial}{\partial \tau_0} + i\vec{k}\cdot\vec{u} + \omega\left(\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \varphi_1}\right)\right\}\hat{g}^{(1)}(\vec{k}, \vec{u}, \vec{v}_1, \tau_0) = \tag{4.4}$$
$$= \frac{1}{m}\hat{\Phi}_{12}(\vec{k})\, i\vec{k}\cdot\vec{G}$$

where

$$G(\vec{u}, \vec{v}_1) \equiv F^{(0)}(\vec{v}_1 - \vec{u})\frac{\partial F^{(0)}(\vec{v}_1)}{\partial \vec{v}_1} - F^{(0)}(\vec{v}_1)\frac{\partial F^{(0)}(\vec{v}_1 - \vec{u})}{\partial \vec{v}_1} \tag{4.5}$$

$\vec{G}$ and $F^{(0)}$ being functions of τ_0 too due to eq. (3.3). It may be noted that $\vec{G}$ satisfies the differential equation

$$\frac{\partial \vec{G}}{\partial \tau_0} + \omega \left(\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \varphi_1} \right) \vec{G} = 0$$

and therefore

$$\vec{G} = \vec{G}\left(u_\perp, \varphi - \varphi_1, u_z, v_\perp, \varphi_1 - \omega \tau_0, v_z \right)$$

This can be seen also directly from the definition (4.5)

Writing out the scalar product in the right hand side of eq.(4.4) we are now able to solve this equation. The result is under the initial condition of eq.(2.8)

$$\hat{g}^{(1)}(\vec{k}, \vec{u}, \vec{v}_1, \tau_0) = i\, m^{-1} \hat{\Phi}(k) \int_0^{\tau_0} ds \, \{ k_z G_z +$$

$$+ k_\perp G_\perp \cos(\psi - \varphi_1 + \omega s) + k_\perp G_\varphi \sin(\psi - \varphi_1 + \omega s) \} \cdot$$

$$\cdot \exp\left[-i \left\{ k_z u_z s - \frac{k_\perp u_\perp}{\omega} \left[\sin(\psi - \varphi) - \sin(\psi - \varphi + \omega s) \right] \right\} \right] \tag{4.6}$$

For large τ_0 this function approaches an asymptotic limit $\hat{g}_A^{(1)}$ in the sense of generalized functions[6] as is proved in section 6. This $\hat{g}_A^{(1)}$ may still be a periodic function of τ_0 through $\varphi_1 - \omega \tau_0$.

5. Second order Theory.

We only treat the second order equation for the distribution function which already leads to the kinetic equation.

$$\frac{\partial F^{(2)}}{\partial \tau_0} + \frac{\partial F^{(1)}}{\partial \tau_1} + \frac{\partial F^{(0)}}{\partial \tau_2} + \omega \frac{\partial F^{(2)}}{\partial \varphi_1} = -8\pi^3 i \frac{n}{m}$$

$$\frac{\partial}{\partial \vec{v}_1} \cdot \int d^3k\, d^3u\, \vec{K}\, \hat{\Phi}(k)\, \hat{g}^{(1)}(k_\perp, k_z, u_\perp, u_z, , v_\perp, v_z, \psi - \varphi, \varphi - \varphi_1, \varphi_1 - \omega\tau_0, \tau_0) \tag{5.1}$$

The second term of the left hand side vanishes because of eq. (4.3).

The solution for $F^{(2)}$ as a function of τ_0 is then easily seen to be with eq. (2.8)

$$F^{(2)} = -\tau_0 \frac{\partial}{\partial \tau_2} F^{(0)}(v_\perp, \varphi_1 - \omega\tau_0, v_z, \tau_0 = 0) - 8\pi^3 i \frac{n}{m}$$

$$\frac{\partial}{\partial \vec{v}_1} \cdot \int_0^{\tau_0} d\tau_0' \int d^3k\, d^3u\, \vec{K}\, \hat{\Phi}(k)\, \hat{g}^{(1)}(k_\perp, --, \varphi_1 - \omega\tau_0, \tau_0')$$

For large τ_0 we obtain

$$F^{(2)} \rightarrow -\tau_0 \left[\frac{\partial F^{(0)}}{\partial \tau_2} + 8\pi^3 i \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int d^3k\, d^3u\, \vec{K}\, \hat{\Phi}(k)\, \hat{g}_A^{(1)} \right] \tag{5.2}$$

The condition that $F^{(2)}$ should be small for all τ_0 now gives immediately the kinetic equation

$$\frac{\partial F^{(0)}}{\partial \tau_2} = -8\pi^3 i \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int d^3k\, d^3u\, \vec{K}\, \hat{\Phi}(k)\, \hat{g}_A^{(1)}(k_\perp, --, \varphi_1 - \omega\tau_0) \tag{5.3}$$

The properties of the collision integral of eq. (5.3) are discussed further in section 7.

6. The first order Correlation Function.

By means of elementary trigonometric idendities and integration by parts eq. (4.6) may be rewritten as

$$\hat{g}^{(1)}(\vec{k},\vec{u},\vec{v}_1,\tau_0) = m^{-1}\hat{\Phi}(k)\Big[u_\perp^{-1}G_1\{1-h(\tau_0)\} + ik\int_0^{\tau_0} ds\, h(s)\{(G_z - u_z u_\perp^{-1}G_1)\cos\theta + G_2\sin(\alpha+\omega s)\sin\theta\}\Big] \quad (6.1)$$

where

$$k_z = k\cos\theta,\; k_\perp = k\sin\theta,\; \alpha = \psi - \varphi,\; \beta = \varphi - \varphi_1$$

$$G_1 = G_\perp\cos\beta + G_\varphi\sin\beta \qquad G_2 = -G_\perp\sin\beta + G_\varphi\cos\beta \quad (6.2)$$

$$h(s) = \exp\Big[-ik\{u_z s\cos\theta - u_\perp\omega^{-1}\sin\theta[\sin\alpha - \sin(\alpha+\omega s)]\}\Big]$$

The introduction of G_1 and G_2 means simply a rotation of the axis in velocity space over the angle β in the plane perpendicular to the z-axis. The vector G is given in eq.(4.5). The Fourier transform of the Coulomb potential is given by

$$\hat{\Phi}(k) = \frac{e^2}{2\pi^2k^2} \quad (6.3)$$

By virtue of a well known expansion we may express h(s) as

$$h(s) = \exp\Big[-ik\{u_z s\cos\theta - u_\perp\omega^{-1}\sin\theta\sin\alpha\}\Big]\cdot$$
$$\cdot\sum_{-\infty}^{+\infty} J_n(u_\perp\omega^{-1}k\sin\theta)\exp\{-in(\alpha+\omega s)\} \quad (6.4)$$

Taking the limit of eq.(6.1) for $\tau_0\to\infty$ we obtain therefore

$$\hat{g}_A^{(1)}(\vec{k},\vec{u},\vec{v}_1,\tau_0) = m^{-1}\hat{\Phi}(k)\Big[u_\perp^{-1}G_1 + ik\exp(ia\sin\alpha)\cdot$$
$$\cdot\sum_{-\infty}^{+\infty}\exp(-in\alpha)\,\delta^-(ku_z\cos\theta + n\omega)\{(G_z - u_z u_\perp^{-1}G_1)\cdot \quad (6.5)$$
$$\cdot J_n(a)\cos\theta + iG_2J_n'(a)\sin\theta\}\Big]$$

where

$$a = k\, u_\perp\, \omega^{-1} \sin\theta$$

J_n is the Besselfunction of order n and J_n' its derivative. The negative frequency part of the delta function $\delta^-(x)$ has been defined in Part I, eq.(5.11). Note that $\hat{g}_A^{(1)}$ may still be a function of τ_0 through $\vec{E}$.

The limit (6.5) certainly exists as a generalized function. This is easily seen after multiplication of eq.(6.5) with some function of, for instance, u_z and integration over all u_z. The resulting series converges because the Bessel functions $J_n(z)$ decrease very fast for large n, namely as $z^n/n!$. We do not use the expression (6.5) in this paper. It turns out that it is more convenient to use directly the asymptotic correlation given by eq.(6.1) with $h(\tau_0) \to 0$ in the first term of the right hand side and the integral extended to infinity.

Let us consider now the limit $\omega \to 0$.
The function h(s) reduces to

$$h(s) = \exp\left(-i\,\vec{k}\cdot\vec{u}\,s\right)$$

and a simple calculation using eq. (6.2) leads to

$$\hat{g}^{(1)}(\vec{k}, \vec{u}, \vec{v}_1, \tau_0) = \hat{\Phi}(k)\, \frac{1 - \exp\left(-i\,\vec{k}\cdot\vec{u}\,\tau_0\right)}{m\; \vec{k}\cdot\vec{u}}\, \vec{k}\cdot\vec{E} \tag{6.6}$$

in agreement with Part I, eq. (5.9).

Another interesting check on the calculations is a Maxwellian zero order distribution, i.e.

$$F^{(0)}(\vec{v}_1) = C \exp\left(-\lambda v_1^2\right) \tag{6.7}$$

It is easily seen that in this case

$$G_1 = -2\lambda u_\perp F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_1 - \vec{u})$$
$$G_2 = 0$$
$$G_z = -2\lambda u_z F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_1 - \vec{u}) \tag{6.8}$$

and therefore, from eq. (6.1),

$$\hat{g}^{(1)}(\vec{k}, \vec{u}, \vec{v}_1, \tau_0) = -2\lambda m^{-1} \hat{\phi}(k) \cdot$$
$$\cdot F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_1 - \vec{u}) \{1 - h(\tau_0)\} \tag{6.9}$$

This represents the relaxation of the correlation function from the initial value zero to its asymptotic value in equilibrium. It is seen immediately that this asymptotic correlation is independent of ω . Indeed it is well known that a magnetic field does not change the properties of thermodynamic equilibrium.

Of course, we do not find a shielded correlation function because collective effects have not been taken into account.

7. The Collision Integral

In order to make the kinetic equation more explicit we substitute the asymptotic value of eq.(6.1) into eq. (5.3). It is easily seen that the first term of the right hand side of eq. (6.1) does not contribute.

Substituting the Coulomb potential (6.3) we obtain

$$\frac{\partial F^{(0)}}{\partial \tau_2} = \frac{2e^4 n}{\pi m^2} \frac{\partial}{\partial \vec{v}_1} \cdot \int \vec{j}\, d^3 u \tag{7.1}$$

where

$$\vec{j} = \begin{pmatrix} j_\perp \\ j_\varphi \\ j_z \end{pmatrix} = \int_0^\infty dk \int_0^\infty ds \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\alpha \begin{pmatrix} \sin\theta\cos(\alpha+\beta) \\ \sin\theta\sin(\alpha+\beta) \\ \cos\theta \end{pmatrix} \cdot$$

$$\cdot h(s)\left\{\left(G_z - u_z u_\perp^{-1} G_1\right)\cos\theta + G_2 \sin(\alpha+\omega s)\sin\theta\right\} \tag{7.2}$$

The notation is clear from eqs.(6.2) and (4.5).

The α-, θ-, and k- integrations are carried out in Appendix A. The result is

$$\vec{j} = \pi^2 \int_0^\infty \frac{d\tau}{W\tau}\left[\left(u_\perp G_z - u_z G_1\right)\frac{\sin\tau}{W\tau}\vec{j}_1 + G_2 \vec{j}_2\right] \tag{7.3}$$

with

$$\tau = \frac{\omega s}{2} \qquad W = \left\{u_z^2 + u_\perp^2\left(\frac{\sin\tau}{\tau}\right)^2\right\}^{1/2} \tag{7.4}$$

and

$$\begin{aligned} j_{1\perp} &= -u_z W^{-1}\cos(\beta-\tau) \\ j_{1\varphi} &= -u_z W^{-1}\sin(\beta-\tau) \\ j_{1z} &= u_\perp (W\tau)^{-1}\sin\tau \end{aligned} \tag{7.5}$$

$$
\begin{aligned}
j_{2\perp} &= \sin(\beta - 2\tau) + u_\perp^2 (W\tau)^{-2} \sin^3\tau \cos(\beta-\tau) \\
j_{2\varphi} &= -\cos(\beta-2\tau) + u_\perp^2 (W\tau)^{-2} \sin^3\tau \sin(\beta-\tau) \\
j_{2z} &= u_z u_\perp (W^2\tau)^{-1} \sin^2\tau
\end{aligned}
\tag{7.6}
$$

The integral in eq. (7.3) converges for large τ, but not for small τ. The convergence for large τ obtained without taking into account the collective behaviour is due to the confinement perpendicular to the magnetic field. The fact that the direction parallel to the magnetic field does not disturb the convergence is related to the circumstance that in an one-dimensional gas no energy randomisation is possible and therefore the collision term vanishes. This becomes clearer in the next section.

For small τ we need a cut-off τ_ℓ, say, corresponding to the distance of closest approach of two colliding particles. It is shown in Appendix B that

$$
\begin{aligned}
\tau_\ell &= \frac{2e^2\omega}{m|u_z|^3} \quad \text{if} \quad \frac{4e^2}{m u_z^2} \gg \frac{u_\perp}{\omega} \\
\tau_\ell &= \frac{2e^2\omega}{m u^3} \quad \text{if} \quad \frac{4e^2}{m u_z^2} \ll \frac{u_\perp}{\omega}
\end{aligned}
\tag{7.7}
$$

It should be observed that

a) our procedure differs from the usual one in this respect that we cut off the τ-integral instead of the k-integral. This is also justified in Appendix B.

b) we are interested in the case $\omega \gg \omega_p$ (ω_p :plasma frequency). In this case the Debye length does not play a role in the integrals in eq. (7.3). This is shown in Appendix C.

If $\omega < \omega_p$ the magnetic field does not influence the collision term appreciably.

We assume also that $\tau_L \ll 1$ where τ_L corresponds to the average distance of closest approach

$$
\tau_L = \frac{2e^2\omega}{m v_T^3} \tag{7.8}
$$

(v_T : thermal velocity)

c) The results of the integrations are evaluated in logarithmic accuracy, i.e. terms of order unity compared with the Coulomb logarithm are neglected. A higher accuracy has no sense because of the somewhat arbitrary cutting off procedure.

Because $\tau_L \ll 1$ we may neglect that part of the $\vec{u}$ -integral for which $2e^2\omega(m u^3)^{-1}$ is not much smaller than unity. This is possible because the integrand remains finite for small u. The remainder of $\vec{u}$ -space consists of three interesting regions.

I. $|u_z| \geqq u_\perp$. The dominant contributions to the τ -integrals come from the divergence, the region of small τ. They lead to the modified Coulomb logarithm $\ln \tau_L^{-1} = \ln r_c r_L^{-1}$ if r_c and r_L are the average Larmor radius and distance of closest approach respectively.

$$r_c = \frac{V_T}{\omega} \qquad\qquad r_L = \frac{V_T}{\omega \tau_L} \tag{7.9}$$

We have replaced τ_l by τ_L in the slowly varying logarithm. The result is

$$\vec{j}_I = \pi^2 \ln \frac{1}{\tau_L} \left[\frac{u_\perp G_z - u_z G_\perp}{u^3} \begin{pmatrix} -u_z \cos\beta \\ -u_z \sin\beta \\ u_\perp \end{pmatrix} + \frac{G_\perp}{u} \begin{pmatrix} \sin\beta \\ -\cos\beta \\ 0 \end{pmatrix} \right] \tag{7.10}$$

or, introducing Cartesian coordinates and using index notation,

$$j_{I\alpha} = \pi^2 \ln \frac{1}{\tau_L} \, \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3} \, G_\beta \tag{7.11}$$

This substituted in eq. (7.1) gives exactly the Landau equation (I. 5.23) with a modified Coulomb logarithm. It represents the collisions which cannot be influenced considerably by the magnetic field.

II. $\left(\frac{4e^2\omega}{m\,u_\perp}\right)^{1/2} \ll |u_z| \ll u_\perp$ In this region we still have the contribution of eq. (7.10) or (7.11). But now the region of large τ gives also some interesting results. For small u_z some integrals give a contribution proportional to $|u_z|^{-1}$. This additional contribution has the form

$$\vec{j}_{II} = \frac{\pi^2}{|u_z|}\left[G_z(u_z{=}0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0{,}50\, G_2(u_z{=}0)\begin{pmatrix} \sin\beta \\ -\cos\beta \\ 0 \end{pmatrix}\right] \tag{7.12}$$

or, in cartesian coordinates

$$\begin{aligned} j_{II\alpha} &= 0{,}50\,\frac{\pi^2}{|u_z|}\,\frac{u_\perp^2\delta_{\alpha\beta} - u_\alpha u_\beta}{u_\perp^2}\,G_\beta(u_z{=}0) \\ &\qquad\qquad \alpha,\beta = 1,2 \\ j_{IIz} &= \frac{\pi^2}{|u_z|}\,G_z(u_z{=}0) \end{aligned} \tag{7.13}$$

This contribution has clearly an anisotropic character. Physically its origin lies in the long duration of the interaction between two gyrating particles with a small component of the relative velocity parallel to the magnetic field.

III. $|u_z| \ll \left(\frac{4e^2\omega}{m\,u_\perp}\right)^{1/2}$ Here $|u_z|$ is so small that τ_ℓ as given in the first line of eq. (7.7) is much larger than unity.

Therefore the lower limit of the τ-integrals is larger and these integrals become negligible.

Therefore the quantity $\left(\frac{4e^2\omega}{m\,u_\perp}\right)^{1/2}$ acts as a lower cut off in the u_z-integration over region II. We obtain

$$\int_{II} \frac{d\,u_z}{|u_z|} = \ln\frac{m\,u_\perp^3}{4e^2\omega} \simeq \ln\frac{1}{\tau_L} \tag{7.14}$$

where we have replaced relative velocity components by the thermal velocity in the slowly variable logarithm, exactly as has been done in eq. (7.10).
Therefore the terms $\vec{j}_I$ and $\vec{j}_{II}$ in

$$\vec{j} = \vec{j}_I + \vec{j}_{II} \tag{7.15}$$

are of the same order of magnitude.

All the details leading to eqs. (7.10) -(7.13) are described in Appendix C.

8. Collision of two electrons in a homogeneous magnetic field

In order to understand better the convergence of the collision integral found in the preceding section without taking into account collective effects we consider now more directly the collision process itself.
We treat the weak interaction of two identical particles in the presence of a homogeneous magnetic field.

We introduce coordinates and velocities of the center of gravity and the relative motion by

$$\vec{R} = \frac{1}{2}\left(\vec{x}_1 + \vec{x}_2\right) \qquad \vec{V} = \frac{1}{2}\left(\vec{v}_1 + \vec{v}_2\right)$$
$$\vec{r} = \vec{x}_1 - \vec{x}_2 \qquad \vec{u} = \vec{v}_1 - \vec{v}_2 \tag{8.1}$$

The equations of motion together with eq.(2.4) are easily seen to yield

$$\frac{\partial \vec{V}}{\partial t} = -\omega \vec{V} \times \vec{e}_z \tag{8.2}$$

describing a simple gyration of the center of gravity around a field line, and

$$\frac{\partial \vec{u}}{\partial t} = -\frac{2e}{m}\vec{E} - \omega \vec{u} \times \vec{e}_z \tag{8.3}$$

where the electric field is given by

$$\vec{E} = e\frac{\partial}{\partial \vec{r}}\left(\frac{1}{r}\right) = -\frac{e\vec{r}}{r^3} \tag{8.4}$$

The relative motion may be visualized as the motion of a "reduced" particle with charge -e and mass $\frac{1}{2}$ m under the influence of the magnetic field $\frac{1}{2} B_o \vec{e}_z$ and a fixed charge -e at the origin r = 0.

In the weak interaction approximation the guiding center of the "reduced" particle runs approximately along a field line with constant velocity. Let this straight line be in the x, z-plane at a distance D from the z-axis and let the guiding center pass the point x = D, z = 0 at t = 0, say. See fig.1.

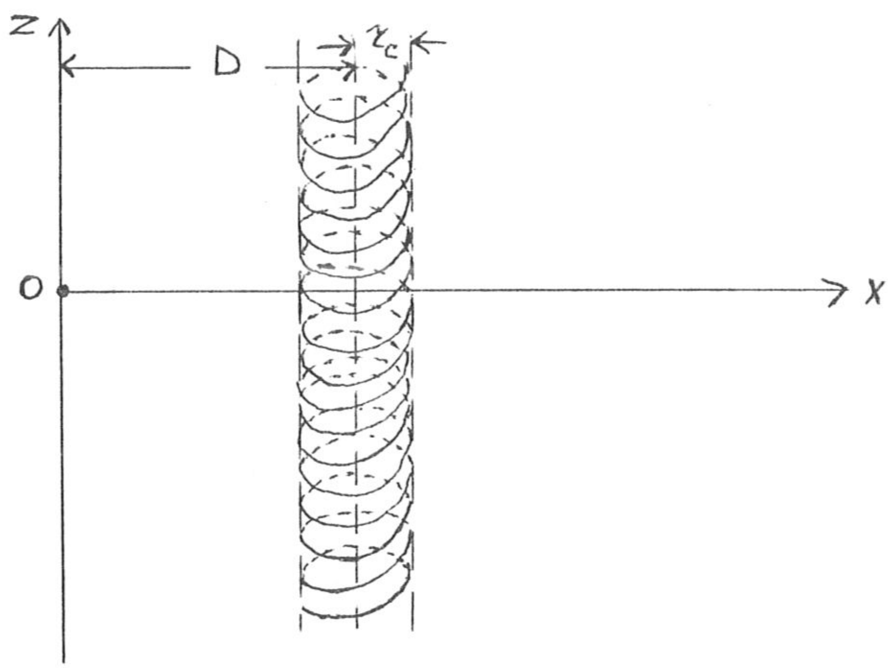

Fig.1 Motion of "reduced" particle

The magnitude of the relative velocity $\mathcal{U}$ is an exact collision invariant, i.e. $\mathcal{U}(t=+\infty) = \mathcal{U}(t=-\infty)$
If, moreover, the electric field felt by the "reduced" particle is nearly constant during one cyclotron period, i.e.

$$r_c = \mathcal{U}_\perp / \omega \ll D \qquad \text{and} \qquad |\mathcal{U}_z| / \omega \ll D \tag{8.5}$$

then the magnetic moment [7,8,9] $\mu = \mathcal{U}_\perp^2 / B$, and therefore $\mathcal{U}_\perp$ itself, is an approximate collision invariant.

We calculate the actual change of $\mathcal{U}_\perp$ because this is connected with the relaxation towards thermal equilibrium.

If we measure distances, velocities and times in units of D, $|U_{oz}|$ and $D/|U_{oz}|$ respectively, U_{oz} being the unperturbed z-component of the relative velocity, we may write eqs.(8.3), (8.4) in the dimensionless form

$$\frac{\partial \vec{U}}{\partial t} = \varepsilon \frac{\vec{r}}{r^3} - \Omega \vec{U} \times \vec{e}_z \tag{8.5}$$

where ε is the parameter of weak interaction and Ω a dimensionless cyclotron frequency

$$\varepsilon = \frac{2e^2}{mD U_{oz}^2} \qquad \Omega = \frac{\omega D}{|U_{oz}|} \tag{8.6}$$

We expand $\vec{U}$ in powers of ε and apply the extension method, just as an illustration of this method

$$\vec{U} = \vec{U}_0 + \varepsilon \vec{U}_1 + \cdots \qquad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_1} + \cdots$$

In zero order we get

$$\frac{\partial \vec{U}_0}{\partial \tau_0} = -\Omega \vec{U}_0 \times \vec{e}_z \tag{8.7}$$

with the solution

$$\vec{U}_0 = \xi \left\{ \vec{e}_x \cos(\Omega \tau_0 + \chi) + \vec{e}_y \sin(\Omega \tau_0 + \chi) \right\} + \vec{e}_z \tag{8.8}$$

and therefore

$$\vec{r}_0 = \vec{e}_x \left\{ 1 + \xi/\Omega \sin(\Omega \tau_0 + \chi) \right\} - \vec{e}_y \, \xi/\Omega \cos(\Omega \tau_0 + \chi) + \vec{e}_z \tau_0 \tag{8.9}$$

where

$$\xi = U_{o\perp}/|U_{oz}| \qquad \xi/\Omega = r_c/D \tag{8.10}$$

The first order equation is

$$\frac{\partial \vec{U}_1}{\partial \tau_0} + \frac{\partial \vec{U}_0}{\partial \tau_1} = \frac{\vec{r}_0}{r_0^3} - \Omega \vec{U}_1 \times \vec{e}_z \tag{8.11}$$

Multiplying the perpendicular components of eqs.(8.7) and (8.11) with $\vec{u}_{1\perp}$ and $\vec{u}_{0\perp}$ respectively and adding we derive

$$\frac{\partial(\vec{u}_{0\perp}\cdot\vec{u}_{1\perp})}{\partial\tau_0}+\frac{1}{2}\frac{\partial u_{0\perp}^2}{\partial\tau_1}=\frac{\vec{r}_{0\perp}\cdot\vec{u}_{0\perp}}{r_0^3}$$

or, because $u_{0\perp}^2$ is independent of τ_0 according to eq.(8.7)

$$\vec{u}_{0\perp}\cdot\vec{u}_{1\perp}=-\frac{1}{2}\tau_0\frac{\partial u_{0\perp}^2}{\partial\tau_1}+\int_{-\infty}^{\tau_0}\frac{\vec{r}_{0\perp}(\tau_0')\cdot\vec{u}_{0\perp}(\tau_0')}{r_0^3(\tau_0')}d\tau_0'$$

Because $\vec{u}_{0\perp}\cdot\vec{u}_{1\perp}$ must be finite for all τ_0 we have to require

$$\frac{\partial u_{0\perp}^2}{\partial\tau_1}=0$$

and find for the total change of $u_\perp^2$ during the collision in first order of the weak interaction parameter

$$\Delta u_\perp^2=2\varepsilon\int_{-\infty}^{+\infty}\frac{\vec{r}_{0\perp}(\tau_0)\cdot\vec{u}_{0\perp}(\tau_0)}{r_0^3(\tau_0)}d\tau_0 \qquad (8.12)$$

Along quite similar lines we find for the change of u_z^2

$$\Delta u_z^2=2\varepsilon\int_{-\infty}^{+\infty}\frac{\tau_0}{r_0^3(\tau_0)}d\tau_0 \qquad (8.13)$$

From eqs.(8,8) and (8.9) we may substitute

$$\vec{r}_{0\perp}\cdot\vec{u}_{0\perp}=\xi\cos(\Omega\tau_0+\chi)$$

$$r_0^2=1+\xi^2/\Omega^2+2\,\xi/\Omega\,\sin(\Omega\tau_0+\chi)+\tau_0^2$$

By adding eqs.(8.12) and (8.13) it is easily seen then that

$$\Delta u_\perp^2+\Delta u_z^2=0$$

as has been anticipated.

Using the first inequality of (8.5) we obtain

$$\Delta u_\perp^2 = 2\varepsilon\xi \int_{-\infty}^{+\infty} \frac{\cos(\Omega\tau_0 + \chi)}{(1+\tau_0^2)^{3/2}}\, d\tau_0$$

$$= 4\varepsilon\xi\Omega K_1(\Omega)\cos\chi$$

where $K_1(\Omega)$ is the Neumann function of first order of purely imaginary argument.

If the second inequality of (8.5) is also valid we may use the asymptotic expression for $K_1(\Omega)$ and write

$$\Delta u_\perp^2 = 2\varepsilon\xi(2\pi\Omega)^{1/2}\exp(-\Omega)\cos\chi \tag{8.14}$$

i.e. the changes of the gyration- and therefore also of the parallel - relative velocity due to the weak interaction collision process are exponentially small. This could have been expected on basis of the invariance of the magnetic moment to all orders[7,8]. This fact explains the convergence of the collision integral found in the preceding section for large interaction distances, i.e. large D.

9. Relaxation into the kinetic regime

As in Part I, section 6, we treat here the influence of transient correlation function on the relaxation of the distribution function to a solution of the kinetic equation.

It follows from eqs. (5.1), (4.3) and (5.3) that this process is described by

$$\frac{\partial F^{(2)}}{\partial\tau_0} + \omega\frac{\partial F^{(2)}}{\partial\varphi_1} = -8\pi^3 i\frac{n}{m}\frac{\partial}{\partial\vec{v}_1}\cdot\int d^3k\, d^3 u\, \vec{k}\,\hat{\Phi}(k)\times \times\left(\hat{g}^{(1)} - \hat{g}_A^{(1)}\right) \tag{9.1}$$

For the sake of simplicity we restrict ourselves to a Maxwellian $F^{(0)}$. The result, however, is generally valid. We found already the relaxation of the correlation function to its asymptotic value for this case in section 6. Eq.(6.9) leads to

$$\hat{g}^{(1)} - \hat{g}_A^{(1)} = \frac{2\lambda}{m} \hat{\Phi}(k) F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_1 - \vec{u}) h(\tau_0) \tag{9.2}$$

and the Maxwellian distribution (6.7) implies

$$F^{(0)}(\vec{v}_1 - \vec{u}) = \left(\frac{\lambda}{\pi}\right)^{3/2} \exp\left[-\lambda\left\{(v_z - u_z)^2 + v_\perp^2 + u_\perp^2 - 2 v_\perp u_\perp \cos\beta\right\}\right] \tag{9.3}$$

It is now advantageous to expand $h(\tau_0)$ as

$$h(\tau_0) = \exp(-i k_z u_z \tau_0) \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} (-1)^m J_m\left(\frac{k_\perp u_\perp}{\omega}\right) \cdot \tag{9.4}$$
$$\cdot J_n\left(\frac{k_\perp u_\perp}{\omega}\right) \exp\left[i\left\{(m+n)\alpha + m\omega\tau_0\right\}\right]$$

After substitution of eqs.(9.2), (9.3) and (9.4) into eq.(9.1) the α-, ß-, and u_z -integrations are easily carried out. The α-integration has been used already in eq.(9.4) and introduces Bessel functions J_m , the ß-integration gives rise in a similar way to the Bessel functions of purely imaginary argument $I_{0,1}$, and the u_z -integration is elementary. The result is

$$\frac{\partial F^{(2)}}{\partial \tau_0} + \omega \frac{\partial F^{(2)}}{\partial \varphi_1} = \mathrm{Const.} \left\{\frac{\partial l_z}{\partial v_z} + \frac{\partial (v_\perp l_\perp)}{v_\perp \partial v_\perp}\right\} \tag{9.5}$$

with

$$l_z = \int_{-\infty}^{+\infty} dk_z \int_0^\infty dk_\perp k_\perp \int_0^\infty du_\perp u_\perp \hat{\Phi}^2(k) \exp\left[-\lambda(v_z^2 + 2v_\perp^2 + \right. \tag{9.6}$$
$$\left. + u_\perp^2) - k_z^2 \tau_0^2 / 4\lambda\right] \sin(k_z v_z \tau_0) k_z .$$

$$\cdot\, I_0\left(2\lambda v_\perp u_\perp\right)\sum_{-\infty}^{+\infty} J_m^2\left(\frac{k_\perp u_\perp}{\omega}\right)\cos m\omega\tau_0 \tag{9.6}$$

and

$$\begin{aligned}\ell_\perp = \int_{-\infty}^{+\infty} dk_z \int_0^\infty dk_\perp k_\perp \int_0^\infty du_\perp u_\perp\, \hat{\Phi}^2(k) \exp\Big[-\lambda\left(v_z^2+2v_\perp^2+\right.&\\ \left. + u_\perp^2\right) - k_z^2\tau_0^2/4\lambda\Big]\cos\left(k_z v_z \tau_0\right) k_\perp \,\cdot&\\ \cdot\, I_1\left(2\lambda v_\perp u_\perp\right)\sum_{-\infty}^{+\infty} J_m\left(\frac{k_\perp u_\perp}{\omega}\right) J_{m+1}\left(\frac{k_\perp u_\perp}{\omega}\right)\sin m\omega\tau_0&\end{aligned} \tag{9.7}$$

We now get the asymptotic expansion of ℓ_z and $\ell_\perp$ by introducing the new integration variable $q = k_z\tau_0$ and expanding $\hat{\Phi}^2\left(k_\perp^2 + q^2/\tau_0^2\right)$ for small q/τ_0.

It is seen immediately that the leading asymptotic terms have the form

$$\begin{aligned}\ell_z &\longrightarrow \frac{1}{\tau_0^2}\sum_0^\infty A_m \cos m\omega\tau_0\\ \ell_\perp &\longrightarrow \frac{1}{\tau_0}\sum_1^\infty B_m \sin m\omega\tau_0\end{aligned} \tag{9.8}$$

The relaxation of $\partial F^{(2)}/\partial\tau_0$ is slower than the τ_0^{-4} -law found in Part I, section 6. On the other hand it has been found without taking into account collective effects, while the application of the theory leading to the τ_0^{-4}-law requires a cut-off at the Debye length in the plasma case.

The expressions (9.8) assure anyway that $F^{(2)}$ remains a finite correction for all τ_0 .

10. Conclusions

We derived the kinetic equation for a one species homogeneous plasma in a constant external magnetic field by means of the weak interaction treatment and the extension method of Sandri [1].

The neglect of self consistent magnetic fields implies that no zero order electric currents are present.
The weak interaction treatment is only correct if the zero order distribution function is stable and if the Debye screening does not play an essential role. The latter condition appears to be satisfied if the Larmor radius is smaller than the Debye length because the Larmor radius acts as a natural cut-off in the collision integral which converges for large interaction distances. This fact is connected with the adiabatic invariance of the magnetic moment in collisions.

There is an interesting new contribution to the kinetic equation due to the possibility of long interaction times. Expressing the results in one time coordinate only we write

$\partial/\partial t$ instead of $\partial/\partial\tau_0 + \varepsilon\,\partial/\partial\tau_1 + ---$ and

find

$$\frac{\partial F^{(0)}}{\partial t} - \frac{e}{mc}\vec{V}\times\vec{B}\cdot\frac{\partial F^{(0)}}{\partial\vec{V}} = \frac{2e^4 n}{\pi m^2}\frac{\partial}{\partial\vec{V}}\cdot\int\vec{j}\left\{F^{(0)};\vec{u}\right\}d^3u \qquad (10.1)$$

where the general expression for $\vec{j}$ is given in eqs.(7.3)-(7.6)

The relaxation into the kinetic regime is oscillatory and slow but guarantees that the corrections to the kinetic equation remain always small.

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Appendix A. Evaluation of collision integral

The α-integrals in eq.(7.2) are the first few coefficients of the Fourier expansion of $h(s)$ given in eq.(6.2)

$$h(s) = \exp(-ik\,U_z \cos\theta) \sum_{-\infty}^{+\infty} (-i)^m J_m(Z) \exp\{im(\alpha + \omega s/2)\}$$

where

$$Z = \frac{2kU_\perp}{\omega} \sin\theta \sin \omega s/2$$

The result is

$$\begin{pmatrix} j_z \\ j_\perp \\ j_\varphi \end{pmatrix} = 2\pi \int_0^\infty dk \int_0^\infty ds \int_0^\pi d\theta \sin\theta \exp(-ik\,U_z\cos\theta) \left[\left(G_z - U_z/U_\perp\, G_1\right) \cdot \right.$$

$$\cdot \begin{pmatrix} \cos\theta\, J_0(Z) \\ -i \sin\theta\, J_1(Z) \cos(\beta - \omega s/2) \\ -i \sin\theta\, J_1(Z) \sin(\beta - \omega s/2) \end{pmatrix} + G_2 \sin\theta \,\cdot$$

$$\left. \cdot \begin{pmatrix} i\cos\theta\, J_1(Z) \sin \omega s/2 \\ 1/2 \sin\theta \{ J_0(Z) \sin(\beta - \omega s) + J_2(Z) \sin\beta \} \\ -1/2 \sin\theta \{ J_0(Z) \cos(\beta - \omega s) + J_2(Z) \cos\beta \} \end{pmatrix} \right] \qquad (A.1)$$

The Θ - integration can be carried out by means of Gegenbauer's finite integral [10]:

$$\int_0^\pi \exp(ix\cos\theta\cos\psi) J_{\nu-1/2}(x\sin\theta\sin\psi)\, C_r^\nu(\cos\theta) \sin^{\nu+1/2}\theta\, d\theta =$$

$$= \left(\frac{2\pi}{x}\right)^{1/2} i^r \sin^{\nu-1/2}\psi\; C_r^\nu(\cos\psi)\, J_{\nu+r}(x) \qquad (A.2)$$

where $C_\tau^\nu(t)$, the Gegenbauer function, is the coefficient of a^τ in the expansion of $(1-2at+a^2)^{-\nu}$ in ascending powers of a (ν arbitrary).

The Θ - integrals in eq.(A.1) may be written as combinations of integrals of the type (A.2). The calculation results in

$$\begin{pmatrix} \dot{j}_z \\ \dot{j}_\perp \\ \dot{j}_\varphi \end{pmatrix} = 2\pi \int_0^\infty dk \int_0^\infty ds \left(\frac{2\pi}{x}\right)^{1/2} \Bigg[\left(G_z - \frac{u_z}{u_\perp} G_1\right) \cdot$$

$$\cdot \begin{pmatrix} x^{-1} J_{3/2}(x) - u_z^2 W^{-2} J_{5/2}(x) \\ -u_z u_\perp W^{-2} \tau^{-1} \sin\tau \, J_{5/2}(x) \cos(\beta-\tau) \\ -u_z u_\perp W^{-2} \tau^{-1} \sin\tau \, J_{5/2}(x) \sin(\beta-\tau) \end{pmatrix} + \qquad \text{(A.3)}$$

$$+ G_2 \begin{pmatrix} u_z u_\perp W^{-2} \tau^{-1} \sin^2\tau \, J_{5/2}(x) \\ x^{-1} J_{3/2}(x) \sin(\beta-2\tau) + u_\perp^2 W^{-2} \tau^{-2} \sin^3\tau \, J_{5/2}(x) \cos(\beta-\tau) \\ -x^{-1} J_{3/2}(x) \cos(\beta-2\tau) + u_\perp^2 W^{-2} \tau^{-2} \sin^3\tau \, J_{5/2}(x) \sin(\beta-\tau) \end{pmatrix} \Bigg]$$

where

$$x = ksW \ , \quad \tau = \frac{\omega s}{2}$$

and W is defined in eq.(7.4)

The k-integration is now simple and leads immediately to eqs.(7.3) - (7.6).

Appendix B. Landau cut-off

It is clear that the closest approach of two particles is realized if they gyrate around the same field line. As long as the particles are far from each other $u_\perp$ is an adiabatic invariant (see section 8). Therefore the conservation of energy may approximately be written as

$$\frac{m u_z^2}{4} + \frac{e^2}{r} = \frac{m u_{z\infty}^2}{4} \tag{B.1}$$

where $u_{z\infty}$ is the longitudinal relative velocity before the interaction process. It is obvious that eq.(B.1) leads to a distance of closest approach

$$r_\ell = \frac{4e^2}{m u_{z\infty}^2} \tag{B.2}$$

This result is only correct, however, if this distance is still much larger than the Larmor radius $u_\perp/\omega$. If this is not the case we can only state that the distance of closest approach is not smaller than

$$r_\ell = \frac{4e^2}{m u^2} \tag{B.3}$$

following from the exact conservation of the total energy in the relative motion.

Now we consider the transition of the Landau cut-off from the k-integration to the r -integration. We modify the Coulomb potential in order to avoid the difficulties due to the singularity for small distances. We may write for instance

$$\Phi(x) = \frac{e^2}{x}\,\frac{2}{\pi}\int_0^{k_\ell x} K_0(x')\,dx'; \quad k_\ell = r_\ell^{-1} \tag{B.4}$$

For $k_\ell x \gg 1$ we have $\Phi(x) \to \dfrac{e^2}{x}$ and for $k_\ell x \ll 1$

$\Phi(x) \longrightarrow -\frac{2e^2 \kappa_\ell}{\pi} \ln \frac{\kappa_\ell x}{2}$. This singular behaviour is mild enough for our purposes.

We find for the Fourier transform

$$\hat{\Phi}(k) = \frac{e^2 \kappa_\ell}{2\pi^2 k^2 \sqrt{k^2 + \kappa_\ell^2}} \tag{B.5}$$

This means that we should correct the integrand in eq.(A.3) by the factor

$$\frac{\kappa_\ell^2}{k^2 + \kappa_\ell^2}$$

The simplicity of this factor is the reason for the rather strange choice (B.4).
The k-integration in eq.(A.3) is still simple if one uses the residue theory. The result is

$$(2\pi)^{3/2} \int_0^\infty dk \, \frac{J_{3/2}(ksW)}{(ksW)^{3/2}} \, \frac{\kappa_\ell^2}{k^2 + \kappa_\ell^2} = \frac{\pi^2}{sW x_\ell^2} \Big\{ x_\ell^2 - 2 + 2\exp(-x_\ell)(x_\ell + 1) \Big\} \tag{B.6}$$

$$(2\pi)^{3/2} \int_0^\infty dk \, \frac{J_{5/2}(ksW)}{(ksW)^{1/2}} \, \frac{\kappa_\ell^2}{k^2 + \kappa_\ell^2} = \frac{\pi^2}{sW x_\ell^2} \Big\{ x_\ell^2 - 6 + 2\exp(-x_\ell)(x_\ell^2 + 3x_\ell + 3) \Big\}$$

where

$$x_\ell = \kappa_\ell \, s \, W \tag{B.7}$$

The s-integrals now converge for small s as can be verified easily.

The use of the expressions in the curly brackets in eq.(B.6) is roughly equivalent to a cutt-off $\tau_\ell = \omega s_\ell/2$ given by

$$K_\ell \tau_\ell \left\{ u_z^2 + u_\perp^2 \left(\frac{\sin \tau_\ell}{\tau_\ell} \right)^2 \right\}^{1/2} = \frac{\omega}{2} \qquad \text{(B.8)}$$

It can be shown easily that eq.(B.8) together with the alternatives (B.2) and (B.3) leads in good approximation to eq.(7.7).

Appendix C. Evaluation of the integrals in eq.(7.3)

The integrals in eq.(7.3) are provided with the cut-off τ_ℓ and split up in the intervals (τ_ℓ, π) and (π, ∞) We have seen in section 7 that τ_ℓ is alsways small in regions of U-space which are of interest.

Those integrands which diverge as τ^{-1} for small τ give rise to the result $\ln \frac{1}{\tau_\ell}$ in the interval (τ_ℓ, π). The error is of order unity.
We consider as an example the integral containing $\dot{j}_{1z}$ of eq.(7.5). It is

$$I_1 = \frac{1}{u_\perp^3} \int_{\tau_\ell}^{\infty} \frac{\sin^2 \tau}{(\delta^2 \tau^2 + \sin^2 \tau)^{3/2}} d\tau, \quad \delta = \frac{|u_z|}{u_\perp} \qquad \text{(C.1)}$$

The interval (τ_ℓ, π) gives the logarithmic contribution and the interval (π, ∞) yields, as may be seen easily, a term smaller than $(2\, u_\perp^3 \delta^3)^{-1}$. This is unimportant in the region I of section 7. In region II, however, we must consider carefully*)

$$I_1^* = \int_{\pi}^{\infty} \frac{\sin^2 \tau}{(\delta^2 \tau^2 + \sin^2 \tau)^{3/2}} \qquad \text{for} \quad \delta \to 0 \qquad \text{(C.2)}$$

*) The exact value of the lower limit of the integral is immaterial.

We write

$$I_1^* = 2^{1/2} \int_{\pi}^{\infty} \frac{1-\cos 2\tau}{(1+2\delta^2\tau^2)^{3/2}} \left\{ 1 - \frac{\cos 2\tau}{1+2\delta^2\tau^2} \right\}^{-3/2} d\tau$$

and expand the expression in curly brackets in a Taylor series. We obtain

$$I_1^* = 2^{1/2} \sum_{n=0}^{\infty} \int_{\pi}^{\infty} d\tau \frac{(2n)!\cos^n 2\tau}{2^{2n-1}(n!)^2(1+2\delta^2\tau^2)^{n+1/2}} \left\{ \frac{2n+1}{2(1+2\delta^2\tau^2)} - n \right\} \quad \text{(C.3)}$$

In lowest order of δ we can take the average of $\cos^n 2\tau$ over a period, i.e.

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^{2n-1} x\, dx = 0 \qquad \frac{1}{2\pi} \int_0^{2\pi} \cos^{2n} x\, dx = \frac{(2n)!}{2^{2n}(n!)^2} \quad \text{(C.4)}$$

After that we observe that

$$\int_{\pi}^{\infty} \frac{d\tau}{(1+2\delta^2\tau^2)^{n+1/2}} = \frac{2^{2n-3/2}(n!)^2}{\delta\,(2n)!\,n} \quad \text{(C.5)}$$

Substituting eqs.(C.4) and (C.5) inot eq.(C.3) we see that every term except the n = 0 one vanishes and therefore

$$I_1^* \longrightarrow \frac{1}{\delta} \quad \text{for} \quad \delta \longrightarrow 0 \quad \text{(C.6)}$$

Therefore I_1 of eq.(C.1) leads to a $|u_z|^{-1}$ divergence.

The only other integral leading to such a divergence (it should be noticed that only an even divergence as $|u_z|^{-1}$ and not e.g. u_z^{-1} gives an important contribution to the $\vec{u}$ -integral in eq.(7.1)) is

$$I_2^* = \int_{\pi}^{\infty} \frac{\cos 2\tau}{(\delta^2\tau^2 + \sin^2\tau)^{1/2}} d\tau \quad \text{(C.7)}$$

appearing in eq.(7.3) due to the first term of the right hand sides of eq.(7.6) for $\dot{j}_{2\perp}$ and $\dot{j}_{2\varphi}$. Applying the same method as the one leading to eq.(C.6) we obtain

$$I_2^* \longrightarrow \frac{1}{4\delta}\sum_{n=0}^{\infty} a_n \,, \quad a_n = \frac{(2n)!}{2^{2n}(n!)^2(n+1)} \tag{C.8}$$

Using Stirling's formula it is easily seen that

$$a_n \longrightarrow \pi^{-1/2} n^{-3/2} \quad \text{for} \quad n \longrightarrow \infty \tag{C.9}$$

A numerical calculation using eq.(C.9) and

$$2N^{-1/2} < \sum_{N}^{\infty} n^{-3/2} < 2(N-1)^{-1/2}$$

for $N \simeq 20$ leads to

$$I_2^* \longrightarrow \frac{0{,}50}{\delta} \quad \text{for} \quad \delta \longrightarrow 0 \tag{C.10}$$

It is hard to obtain a more accurate numerical result because of the slow convergence of the series in eq.(C.8).

Finally we show that the Debye length does not influence the results. We could assign an upper limit to the integrals I_1^* and I_2^* by means of eq.(B.8) with the subscripts d instead of ℓ and

$$k_d^2 = \frac{4\pi n e^2}{m V_T^2} \tag{C.11}$$

We then obtain

$$\delta \tau_d \left(1 + \frac{\sin^2 \tau_d}{\delta^2 \tau_d^2}\right) = \frac{\omega}{2 k_d u_\perp} = \frac{r_d}{r_c} \gg 1$$

It follows immediately that

$$\tau_d \gg \frac{1}{2\delta} \tag{C.12}$$

Under these circumstances the upper limit plays no role at all. This is understandable because for decreasing $|u_z|$ the time needed to pass the Debye sphere in the z-direction increases.