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# Radial Distribution Functions in a Two Component Plasma

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# E R R A T A

page 10 After eq. (3.2) read: Then we have[1,6,10)]

In the middle read: Rosenbluth and Rostoker[13)]

In the line above eq.(3.4) read: (cf.Ref.13 and 10)

page 11 In the third line from the bottom read:
eqs. (2.15) , (2.8)

page 12 In the 8th line from the bottom read: Renau[3,4)]

page 13 The exponential function in the denominatorof the first line of eq. (4.3) is to be replaced by
$\exp\{(1-\varepsilon^2)w^2\}$

page 17 In the 6th line from bottom read: limit $\varepsilon \to 0$[8)]
instead of the ref. in the 3rd line from the bottom.

page 21 In the last term of the formula in the middle
$F_i$ is to be replaced by $F^i$

General expressions for the radial distribution functions in an electron-ion-plasma outside thermal equilibrium are obtained from the hierarchy equations in first order of the plasma parameter.

It is shown that these expressions can also be derived in the framework of the test particle problem.

In the special case of Maxwellian distributions with different temperatures for electrons and ions and under the mild restriction that the thermal velocity of the ions is much smaller than that of the electrons the results reduce to expressions first given by Salpeter [1].

The results of Kadomtsev [2] and Renau [3,4] for the electron-electron radial distribution function $W_{ee}(r)$ disagree with ours and those of Salpeter. The reason for this is discussed.

A curious phenomenon, which may be called "anti-shielding", is shown to follow from $W_{ee}$ in the case that the electron temperature is smaller than the ion temperature and is studied in some detail.

Finally the influence of a magnetic field is shown to vanish in the special case of two Maxwellians.

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1) E.E. Salpeter, J. Geophys. Res. 68, 1321 (1963)
2) B.B. Kadomtsev, Sovj. Phys. J.E.T.P. 6, 117 (1958)
3) J. Renau, J. Geophys. Res. 67, 3624 (1962)
4) J. Renau, Z. f. Physik 177, 458 (1964)

## 1. Introduction

Studies of the ionosphere on one hand and of the interaction of laser beams and laboratory plasmas on the other hand, have drawn a strong interest in the scattering of electromagnetic waves by a plasma. The total power that is scattered by a plasma volume $V$ into a given solid angle is proportional to the mean square of the Fourier transform of the electron density fluctuations $\langle |\varrho_e(\vec{k})|^2 \rangle$ [5]. This quantity is related to the Fourier transform of the electron-electron radial distribution function $\hat{w}_{ee}(\vec{k})$ by eqs. (3.1), (3.3) with both subscripts $a, b$ representing electrons.

Several authors have presented calculations of $w_{ee}(r)$ but complete agreement exists only in the case of thermodynamic equilibrium [6]. Another case of great interest, however, consists of Maxwellian distributions with different temperatures $\theta_i$ and $\theta_e$ as pointed out by Salpeter [1], Renau [3,4], and Kegel [7].

Salpeter [1] and Renau [3,4] treat this problem with the aid of the test particle picture. Kadomtsev [2] and Ecker and Kröll [8] on the other hand start from a more fundamental kinetic description, i.e. from the hierarchy equations.
Renau [3,4] and Kadomtsev [2] find

$$w_{ee}(r) = -\, e^2/\theta_e r \, \exp(-r/D) \tag{1.1}$$

whereas Salpeter [1,*)] and Ecker and Kröll [8] give

$$w_{ee}(r) = -\frac{e^2}{\theta_e r}\left\{\frac{\theta_i}{\theta_e}\exp\left(-\frac{r}{D}\right) + \frac{\theta_e-\theta_i}{\theta_e}\exp\left(-\frac{r}{D_e}\right)\right\} \tag{1.2}$$

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*) Salpeters definition differs from ours by a factor $V$


5) M.N. Rosenbluth and N. Rostoker, Phys.Fluids 5, 776 (1962)
6) cf. e.g. E.E. Salpeter, Phys. Rev. 120, 1528 (1960)
7) W.H. Kegel, I.P.P. Report 6/21 (1964)
8) G. Ecker and W. Kröll, to be published in Phys. Fluids

where $e$ is the elementary charge, $\theta_e$ and $\theta_i$ electron and ion temperature respectively (in ergs), D the total Debye length, i.e.

$$\frac{1}{D^2} = \frac{1}{D_e^2} + \frac{1}{D_i^2} \qquad D_e^2 = \frac{\theta_e}{4\pi n e^2} \qquad D_i^2 = \frac{\theta_i}{4\pi n z e^2} \tag{1.3}$$

n the electron density and $ze$ the ion charge ($n_e = z n_i \equiv n$)
Our result agrees with eq. (1.2)

The result of Kadomtsev is due to an error in the mathematical analysis as already has been observed by Ecker and Kröll [8]. The same can be said with respect to the treatment of Renau [3,4]. This is shown in section 3.

It should be noted that the kinetic treatments of Kadomtsev [2] and Ecker and Kröll [8] specialize right from the beginning to the special case of two Maxwellians with the additional restriction

$$\theta_i/M \ll \theta_e/m \tag{1.4}$$

(M: ion mass, m: electron mass)
This enables these authors to separate the velocity and space dependence of the correlation functions and to end up with ordinary differential equations for the radial distribution functions.

The present paper, however, avoids such assumptions and treats in section 2 the general "meta-equilibrium state" of a plasma in first order of the plasma parameter $(n D^3)^{-1}$. In this "meta-equilibrium" [9,10] based on the ideas of Bogoliubov [11] all correlation functions are functionals of the velocity distribution which itself is approximately constant in time. General expressions for the radial distribution functions,

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9) M. Feix and P.P.J.M. Schram, Proc. VIth Conference on ionisation phenomena in gases, Part I, p. 189
10) P.P.J.M. Schram, Thesis (Part III), University of Utrecht, 1964, Euratom Report 1805.e
11) N.N. Bogoliubov, in "Studies in Statistical Mechanics", edited by J. de Boer and G.E. Uhlenbeck (North Holland Publishing Corporation, Amsterdam, 1962), Vol I, Part A.

i.e. correlation functions integrated over all velocity variables, are obtained as solutions of a system of integral equations of the type of Lenard. [12]

In section 3 the picture of dressed particles as used by Rosenbluth and Rostoker [13,14] is applied to the problem and the results are seen to agree with those of the preceding section.
In section 4 the special case of two Maxwellians with the restriction (1.4) is discussed. Special attention is paid to the phenomenon of "anti-shielding". In section 5 finally the effect of a magnetic field is studied.

## 2. Kinetic theory of the radial distribution functions.

We consider a homogeneous neutral plasma consisting of electrons and ions with densities $n$ and $n/z$ respectively.

We start from the hierarchy equations and the usual definition of correlation functions. [13] In first order of the plasma parameter $(n D^3)^{-1}$ the system of equations may be closed by neglecting triple correlations. In this approximation the electron-electron correlation function $g_{ee}(\vec{v}_1, \vec{v}_2, \vec{x}_1 - \vec{x}_2, t) \equiv g_{ee}(1,2)$ satisfies the following equation:

$$\left\{ \frac{\partial}{\partial t} + (\vec{v}_1 - \vec{v}_2) \cdot \frac{\partial}{\partial \vec{x}_1} \right\} g_{ee}(1,2) - \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_1} \cdot \left( \frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) F_e(1) F_e(2)$$

$$- \frac{n}{m} \frac{\partial F_e(1)}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{13}}{\partial \vec{x}_1} \left\{ g_{ee}(2,3) - g_{ei}(2,3) \right\} d^3x_3 d^3v_3 \qquad (2.1)$$

$$- \frac{n}{m} \frac{\partial F_e(2)}{\partial \vec{v}_2} \cdot \int \frac{\partial \phi_{23}}{\partial \vec{x}_2} \left\{ g_{ee}(1,3) - g_{ei}(1,3) \right\} d^3x_3 d^3v_3 = 0$$

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12) A. Lenard, Ann. Phys. 3, 390 (1960)
13) N. Rostoker and M.N. Rosenbluth, Phys. Fluids 3, 1 (1960)
14) N. Rostoker, Nuclear Fusion 7, 101 (1961)

where $F_e$ is the electron distribution function and

$$\phi_{ij} = \frac{e^2}{|\vec{x}_i - \vec{x}_j|} \tag{2.2}$$

The correlation functions $g_{ei}$, $g_{ie}$ and $g_{ii}$ satisfy equations of the same type.

Asymptotic Fourier transformed correlation functions may be defined by

$$\hat{g}_{ee}(\vec{k}, \vec{v}_1, \vec{v}_2) = \lim_{s \to 0} s \int_0^\infty dt \int d^3z \exp(-st - i\vec{k}\cdot\vec{z}) g_{ee}(\vec{v}_1, \vec{v}_2, \vec{z}, t) \tag{2.3}$$

and similarly for $\hat{g}_{ei}$ etc.

These functions obey equations of the type of eq. (3) in Lenards paper [12]. Furthermore we write

$$H_{ee}(\vec{k}, u) = \int d^3v_1 d^3v_2 \, \hat{g}_{ee}(\vec{k}, \vec{v}_1, \vec{v}_2) \delta\left(u - \frac{\vec{k}\cdot\vec{v}_1}{k}\right) \tag{2.4}$$

and so on for $H_{ei}, H_{ie}, H_{ii}$. Note that the velocity component $u$ belongs to a particle of the kind indicated by the first subscript of the functions $H_{ee}$ etc.

We now introduce the formalism of positive and negative frequency parts of functions. For real $u$ these are defined by the relations [15,16]

$$\begin{aligned} \psi^+(u) + \psi^-(u) &= \psi(u) \\ \psi^+(u) - \psi^-(u) &= \frac{1}{\pi i} P \int_{-\infty}^{+\infty} \frac{\psi(u')}{u' - u} du' \end{aligned} \tag{2.5}$$

($P$ means "principle value of")

It is easily seen that $\psi^+(u)$ and $\psi^-(u)$ can be continued analytically in the complex $u$-plane as

$$\begin{aligned} \psi^+(u) &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\psi(u')}{u' - u} du' \\ \psi^-(u) &= -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\psi(u')}{u' - u} du' \end{aligned} \tag{2.6}$$

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15) N.I. Muskhelishvili, Singular Integral Equations, Noordhoff, Groningen, 1953

16) N.G. van Kampen, Physica 21, 949 (1955)

The symbols $\oint$ and $\fint$ denote integration below and above the pole $u'=u$ respectively.
An important feature of $\psi^+(u)$ and $\psi^-(u)$ is that they are holomorphic in upper- and lower half plane respectively.
We define one-dimensional velocity distributions by

$$F^{e,i}(\vec{x},u)=\int F_{e,i}(\vec{v})\,\delta(u-\vec{x}\cdot\vec{v})\,d^3v, \qquad \vec{x}=\frac{\vec{k}}{k} \tag{2.7}$$

and use the abbreviations

$$\begin{aligned} H_1 &= H_{ee}-H_{ei} & F_1 &= F^e+zF^i \\ H_2 &= H_{ii}-H_{ie} & F_2 &= F^e+\frac{mz}{M}F^i \\ H &= H_1+H_2 & u_p^2 &= \frac{4\pi n e^2}{m k^2} \end{aligned} \tag{2.8}$$

We omit the arguments of all functions for the sake of brevity and arrive finally at the system of equations for $H_{ee}$ etc.

$$H_{ee}=2\pi i u_p^2\left[\frac{1}{n}\left\{F^e\fint\frac{\partial F^e}{\partial u}-F^e\oint\frac{\partial F^e}{\partial u}\right\}+\frac{\partial F^e}{\partial u}(H_1^*)^- -\frac{\partial F^e}{\partial u}H_1\right] \tag{2.9}$$

$$H_{ei}=2\pi i u_p^2\left[-\frac{z}{n}\left\{F^i\fint\frac{\partial F^e}{\partial u}-F^e\frac{m}{M}\oint\frac{\partial F^i}{\partial u}\right\}-\frac{\partial F^e}{\partial u}(H_2^*)^- +\frac{mz}{M}\frac{\partial F^i}{\partial u}H_1\right] \tag{2.10}$$

$$H_{ii}=2\pi i u_p^2\frac{mz}{M}\left[\frac{z}{n}\left\{F^i\fint\frac{\partial F^i}{\partial u}-F^i\oint\frac{\partial F^i}{\partial u}\right\}+\frac{\partial F^i}{\partial u}(H_2^*)^- -\frac{\partial F^i}{\partial u}H_2\right] \tag{2.11}$$

$$H_{ie}=2\pi i u_p^2\left[-\frac{z}{n}\left\{F^e\fint\frac{m}{M}\frac{\partial F^i}{\partial u}-F^i\oint\frac{\partial F^e}{\partial u}\right\}-\frac{mz}{M}\frac{\partial F^i}{\partial u}(H_1^*)^- +\frac{\partial F^e}{\partial u}H_2\right] \tag{2.12}$$

An asterisk denotes the complex conjugate. It should be noted that according to eqs. (2.6) differentiation and taking positive or negative frequency parts are commuting operations and that $\psi^+(u^*)=\{\psi^-(u)\}^*$.

Adding eq. (2.10) to eq. (2.9) and eq. (2.12) to eq. (2.11) we obtain

$$H_1 Z^- = 2\pi i u_p^2 \left[ \frac{1}{n} \left\{ F_1^- \frac{\partial F^e}{\partial u} - F^e \frac{\partial F_2^-}{\partial u} \right\} + \frac{\partial F^e}{\partial u} H^- \right] \quad (2.13)$$

$$H_2 Z^- = 2\pi i u_p^2 \left[ \frac{2}{n} \left\{ F_1^- \frac{m}{M} \frac{\partial F^i}{\partial u} - F^i \frac{\partial F_2^-}{\partial u} \right\} + \frac{m}{M} \frac{\partial F^i}{\partial u} H^- \right] \quad (2.14)$$

$Z^-$ is the dielectric constant. We have

$$Z^\pm = 1 \mp 2\pi i u_p^2 \frac{\partial F_2^\pm}{\partial u} \quad (2.15)$$

The sum of eqs. (2.13) and (2.14) gives

$$H = 2\pi i u_p^2 \left[ \frac{1}{n} \left\{ F_1^- \frac{\partial F_2}{\partial u} - F_1 \frac{\partial F_2^-}{\partial u} \right\} + \frac{\partial F_2}{\partial u} H^- - \frac{\partial F_2^-}{\partial u} H \right] \quad (2.16)$$

This is essentially the integral equation of Lenard. Exactly as in his paper it can be proven that $H$ is real.

The solution of eq. (2.16) (see Ref. 10 and 17) is given in Appendix A. When $H$ is known $H_{1,2}$ follow from eqs. (2.13), (2.14) and $H_{ee}$ etc. from eqs. (2.9) up to (2.12).

This procedure and the results are given in Appendix B.

We find for instance

$$n H_{ee} = 2\pi i u_p^2 \left[ \left( \frac{F^e}{Z^+} - \frac{\partial F^e}{\partial u} 2\pi i u_p^2 A^+ \right)^- \frac{\partial F^e}{\partial u} - \left( \frac{F^e}{Z^-} + \frac{\partial F^e}{\partial u} 2\pi i u_p^2 A^- \right) \frac{\partial F^{e-}}{\partial u} \right] \quad (2.17)$$

where

$$A = \frac{F_1}{Z^+ Z^-} = \frac{F_1}{2\pi i u_p^2 \partial F_2 / \partial u} \left( \frac{1}{Z^+} - \frac{1}{Z^-} \right) \quad (2.18)$$

The Fourier transform of the radial distribution function $\hat{W}_{ee}(\vec{k})$ is given by

$$\hat{W}_{ee}(\vec{k}) = \int d^3v_1 d^3v_2 \, \hat{g}_{ee}(\vec{k}, \vec{v}_1, \vec{v}_2) = \int_{-\infty}^{+\infty} H_{ee}(\vec{k}, u) \, du \quad (2.19)$$

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17) R. Balescu and H.S. Taylor, Phys. Fluids 4, 85 (1961)

Substituting eq. (2.17) into eq. (2.19) and interchanging the $u$-integration with the one implied by the minus operation in the first term of the right hand side of eq. (2.17) we obtain

$$n\hat{W}_{ee} = 2\pi i u_p^2 \int_{-\infty}^{+\infty} \left[ \left( \frac{F^e}{Z^+} - 2\pi i u_p^2 \frac{\partial F^e}{\partial u} A^+ \right) \frac{\partial F^{e+}}{\partial u} - \left( \frac{F^e}{Z^-} + 2\pi i u_p^2 \frac{\partial F^e}{\partial u} A^- \right) \frac{\partial F^{e-}}{\partial u} \right] du \quad (2.20)$$

After some rearrangements described in Appendix C we arrive at

$$n\hat{W}_{ee} = -1 + \int_{-\infty}^{+\infty} \left[ \frac{F^e}{\partial F^e / \partial u} \left\{ \frac{\partial F^{e+}}{\partial u} \frac{1}{Z^+} \left( 1 - 2\pi i u_p^2 \cdot \frac{mz}{M} \frac{\partial F^{i+}}{\partial u} \right) + \text{complex conjugate} \right\} + \frac{(2\pi i u_p^2)^2}{Z^- Z^+} \cdot \frac{\partial F^{e+}}{\partial u} \frac{\partial F^{e-}}{\partial u} z \left( \frac{\partial F^e}{\partial u} \right)^{-1} \left( F^e \frac{m}{M} \frac{\partial F^i}{\partial u} - F^i \frac{\partial F^e}{\partial u} \right) \right] du \quad (2.21)$$

or

$$n\hat{W}_{ee} = -1 + \int_{-\infty}^{+\infty} \frac{1}{Z^- Z^+} \left[ F^e \left| 1 - 2\pi i u_p^2 \frac{mz}{M} \frac{\partial F^{i+}}{\partial u} \right|^2 - z F^i (2\pi i u_p^2)^2 \frac{\partial F^{e+}}{\partial u} \frac{\partial F^{e-}}{\partial u} \right] du \quad (2.22)$$

The same expression may be obtained in a quite different way. Considering the Vlasov equation as an exact equation for the microscopic density in phase space of the particles in an actual system (sum of delta functions) and averaging over the initial conditions by means of the zeroth order distribution functions one can derive an expression for the mean squared four-dimensional Fourier transform $\langle |\varrho(\vec{k},\omega)|^2 \rangle$ This quantity is proportional to the differential cross-section for light scattering. It was obtained by Salpeter [6] by means of the outlined procedure for the case of two Maxwellians.

Integration over $\omega$ leads to eq. (2.22) by virtue of eqs. (3.1), (3.3).

In exactly the same way as eq. (2.22) we derive

$$n\hat{W}_{ii} = -z + \int_{-\infty}^{+\infty} \frac{1}{z^- z^+} \left[ z F^i \left| 1 - 2\pi i u_p^2 \frac{\partial F^{e+}}{\partial u} \right|^2 - F^e \left( 2\pi i u_p^2 \frac{mz}{M} \right)^2 \frac{\partial F^{i+}}{\partial u} \frac{\partial F^{i-}}{\partial u} \right] du \tag{2.23}$$

$$n\hat{W}_{ei} = 2\pi i u_p^2 z \int_{-\infty}^{+\infty} \frac{1}{z^- z^+} \left[ F^e \left( 1 - 2\pi i u_p^2 \frac{mz}{M} \frac{\partial F^{i+}}{\partial u} \right) \cdot \frac{m}{M} \frac{\partial F^{i-}}{\partial u} - F^i \left( 1 + 2\pi i u_p^2 \frac{\partial F^{e-}}{\partial u} \right) \frac{\partial F^{e+}}{\partial u} \right] du \tag{2.24}$$

and

$$\hat{W}_{ie} = \hat{W}_{ei}^* \tag{2.25}$$

The functions $\hat{W}_{ee}$ and $\hat{W}_{ii}$ are real, but $\hat{W}_{ie}$ and $\hat{W}_{ei}$ are complex in general. The relation (2.25) is identical with the purely logical requirement in physical space

$$W_{ie}(\vec{z}) = W_{ei}(-\vec{z})$$

If $F_i(\vec{v})$ and $F_e(\vec{v})$ are isotropic, then

$$\hat{W}_{ie} = \hat{W}_{ei} \tag{2.26}$$

and both are real because now

$$\mathrm{Im}\,\hat{W}_{ei} = \pi u_p^2 z \int_{-\infty}^{+\infty} \frac{1}{z^- z^+} \left( F^e \frac{m}{M} \frac{\partial F^i}{\partial u} - F^i \frac{\partial F^e}{\partial u} \right) du = 0$$

## 3. Connection of the radial distribution functions with the test particle problem.

Density fluctuation spectra may be defined as ensemble averages in the following way

$$P_{ab}(\vec{k}) = \frac{1}{V} \langle \varrho_a(\vec{k}) \varrho_b^*(\vec{k}) \rangle \tag{3.1}$$

where the subscripts a and b denote species (electrons or ions), $V$ is the volume of the system and $\varrho_{a,b}$ are Fourier transforms of the microscopic

deviations from the average density, i.e.

$$n(\vec{x}) - n = \frac{1}{8\pi^3} \int \rho(\vec{k}) \exp(i\vec{k}\cdot\vec{x})\, d^3k$$

$$n(\vec{x}) = \sum_{i=1}^{N} \delta(\vec{x} - \vec{X}_i(t)) \tag{3.2}$$

Then we have [1,2,9]

$$P_{ab}(\vec{k}) = n_a \{ \delta_{a,b} + n_b \hat{w}_{ab}(\vec{k}) \} \tag{3.3}$$

$\delta_{a,b}$ being unity for equal species a and b and zero for unequal species.
We note that eq. (2.25) follows immediately from eqs. (3.1) and (3.3).

The quantity $P_{ab}(\vec{k})$ may be calculated by means of the concept of "dressed particles" as introduced by Rosenbluth and Rostocker [12]. We consider a homogeneous many species plasma in which a test particle of type a with the coordinates $\vec{X} = \vec{X}_a + \vec{v}_a t$ is moving with the constant velocity $\vec{v}_a$. In the reference system of the test particle the linearized Vlasov equation for the particles of type b is (cf.Ref. 12 and 9):

$$\left(\frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{x}} + \vec{v}_a \frac{\partial}{\partial \vec{X}}\right) f_{ab}(\vec{x} - \vec{X}, \vec{v}, \vec{v}_a, t)$$

$$= \frac{e_b n_b}{m_b} \frac{\partial F_b(\vec{v})}{\partial \vec{v}} \frac{\partial \phi(\vec{x} - \vec{X}, \vec{v}_a, t)}{\partial \vec{x}} \tag{3.4}$$

and the Poisson equation has the form

$$\frac{\partial}{\partial \vec{X}} \frac{\partial \phi(\vec{x} - \vec{X}, \vec{v}_a, t)}{\partial \vec{x}}$$

$$= -4\pi \left[ e_a \delta(x - \vec{X}) + \sum_b e_b \int f_{ab}(\vec{x} - \vec{X}, \vec{v}, \vec{v}_a, t)\, d^3v \right] \tag{3.5}$$

where $F_b(\vec{v})$ is the equilibrium distribution and $f_{ab}$ is its linear perturbation caused by the test particle.

By means of a Fourier Laplace transformation one deduces from (3.4) and (3.5) the following asymptotic expression for $t \to \infty$ :

$$\hat{n}_{ab}(\vec{k}, u_a) = -\frac{4\pi e_a e_b n_b}{k^2 m_b} \frac{2\pi i \left.\frac{\partial F^{b-}(u)}{\partial u}\right|_{u=u_a}}{Z^-(u_a)} \tag{3.6}$$

where $F^b(u)$ is defined as in eq. (2.7) and $\hat{n}_{a,b}$ is the Fourier transform of *)

$$n_{ab}(\vec{x}-\vec{X}, \vec{v}_a, t) = \int d^3v \, f_{ab}(\vec{x}-\vec{X}, \vec{v}, \vec{v}_a, t) \tag{3.7}$$

and

$$Z^-(u_a) = 1 + \sum_c u_{pc}^2 \, 2\pi i \left.\frac{\partial F^{c-}(u)}{\partial u}\right|_{u=u_a} \tag{3.8}$$

$$u_{pc}^2 = \frac{4\pi e_c^2 n_c}{m_c k^2} \tag{3.9}$$

In calculating (3.6) from (3.4) and (3.5) it was assumed that the initial perturbations as functions of $\vec{x} - \vec{X}_a$ have finite range.

In the special case of a two component plasma we obtain from (3.6):

$$n_{ee}(\vec{k}, u_e) = -2\pi i \, u_p^2 \frac{\left.\frac{\partial F^{e-}}{\partial u}\right|_{u=u_e}}{Z^-(u_e)}$$

$$n_{ie}(\vec{k}, u_i) = 2\pi i \, u_p^2 Z \frac{\left.\frac{\partial F^{e-}}{\partial u}\right|_{u=u_i}}{Z^-(u_i)} \tag{3.10}$$

($Z^-(u)$ and $u_p^2$ are now defined as in eqs. (2.15) and $u_e$ and $u_i$ are the velocities of the test particle under consideration).

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*) $\hat{n}_{ab}(\vec{k}, \vec{v}_a)$ does not depend on the components of $\vec{v}_a$ perpendicular to $\vec{k}$ and is therefore denoted by $\hat{n}_{ab}(\vec{k}, u_a)$.

The quantity $P_{ab}(\vec{k})$ now may be obtained from (3.6) by considering the plasma to consist of dressed test particles in the sense that their fluctuation spectra are additive. With this assumption the following expression follows from (3.6):

$$P_{ab}(\vec{k}) = \sum_c n_c \int \{\delta_{a,c} + n_{ca}(\vec{k},u)\} \cdot \{\delta_{b,c} + n_{cb}(\vec{k},u)\} F^c(u)\, du \tag{3.11}$$

From (3.3) and (3.11) follows for the two component plasma with $n_e = n_i/z \equiv n$ :

$$n \hat{w}_{ee}(\vec{k}) = -1 + \int \{|1 + n_{ee}(\vec{k},u)|^2 F^e(u) + \frac{1}{z}|n_{ie}(\vec{k},u)|^2 F^i(u)\}\, du \tag{3.12}$$

It should be remarked that eqs. (3.11) and (3.12) follow also from a general relation derived by Rostoker [18] between the correlation function and the test particle problem (cf. section 4).

If we now substitute (3.10) into (3.12) we obtain again eq. (2.22). In the same way eqs. (2.23) and (2.24) may be deduced.

We conclude this section with the remark that Renau [4,5] did not write eq. (3.4) in the reference system of the test particle and therefore omitted the term $\vec{v}_a \cdot \frac{\partial f_{ab}}{\partial X}$. In order to investigate the asymptotic solution he put $\partial/\partial t = 0$. By this procedure he only considers test particles which are at rest relative to the plasma. Renau also does not calculate $\hat{w}_{ee}$ by means of (3.12) but by putting

$$n \hat{w}_{ee}(\vec{k}) = u_{ee}(\vec{k},0)$$

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18) N. Rostoker, Phys. Fluids 7, 491 (1964)

## 4. Discussion of the special case of two Maxwellians.

The energy exchange between electrons and ions is slow compared to the thermalisation time of each species. It is therefore useful to study a plasma in which electrons and ions have Maxwellian distributions with different temperatures, i.e., putting $m_e = m$, $m_i = M$ for a moment,

$$F^{e,i}(u) = \left(\frac{m_{e,i}}{2\pi\,\Theta_{e,i}}\right)^{1/2} \exp\left(-\frac{m_{e,i}\,u^2}{2\,\Theta_{e,i}}\right) \equiv F_M^{e,i}(u) \qquad (4.1)$$

Substituting this into eq. (2.21), performing a contour integration over the first term of the integrand and rearranging the second part we obtain

$$n\,\hat{W}_{ee}(\vec{K}) = -\frac{D_e^{-2}}{K^2 + D^{-2}}\left[1 + \frac{\Theta_e - \Theta_i}{\Theta_e} I\right] \qquad (4.2)$$

where $D_{e,i}$ and $D$ are Debye lengths defined in eq. (1.3) and

$$I = \mathrm{Re}\,\frac{1}{\pi i} P\int_{-\infty}^{+\infty} \frac{dw}{w} \frac{\left|1 + 2\pi i\,\varepsilon\,\pi^{-1/2} w \{\exp(-\varepsilon^2 w^2)\}^+\right|^2}{1 + \varepsilon\,\frac{\Theta_i}{2\Theta_e}\exp\{(1-\varepsilon)w^2\}} \cdot \qquad (4.3)$$

$$\cdot \frac{1}{1 + 2\pi i\,(K^2 + D^{-2})\,\pi^{-1/2} w\left[D_i^{-2}\{\exp(-w^2)\}^+ + \varepsilon D_e^{-2}\{\exp(-\varepsilon^2 w^2)\}^+\right]}$$

with

$$\varepsilon = \left(\frac{m}{\Theta_e}\,\frac{\Theta_i}{M}\right)^{1/2} \qquad (4.4)$$

In the limit $\varepsilon \to 0$ contour integration is again possible and leads to the result

$$\lim_{\varepsilon \to 0} n\,\hat{W}_{ee}(\vec{K}) = -\frac{1}{D_e^2}\left[\frac{\Theta_i}{\Theta_e}\,\frac{1}{K^2 + D^{-2}} + \frac{\Theta_e - \Theta_i}{\Theta_e}\,\frac{1}{K^2 + D_e^{-2}}\right] \qquad (4.5)$$

and Fourier transformation gives eq. (1.2). Details about the calculations leading to eqs. (4.2), (4.3) are given in Appendix D.

It is also proven in Appendix D that corrections due to the finite ratio of thermal velocities are of order $\varepsilon$.
After some calculations one obtains

$$w_{ee}(r) = -\frac{e^2}{\Theta_e r}\left[\frac{\Theta_i}{\Theta_e}\exp\left(-\frac{r}{D}\right) + \frac{\Theta_e-\Theta_i}{\Theta_e}\cdot\right. \tag{4.6}$$

$$\left.\cdot\left\{\left(1-\frac{\varepsilon\,\Theta_i}{2\,\Theta_e}-\frac{\varepsilon\, r}{4\,D_e}\right)\exp\left(-\frac{r}{D_e}\right)+\frac{\varepsilon\,\Theta_i}{2\,\Theta_e}\exp\left(-\frac{r}{D}\right)\right\}\right] + O(\varepsilon^2)$$

In a hydrogen plasma with the usual condition that $\Theta_i$ is not considerably larger than $\Theta_e$ the correction found above is not more than a few percent. We therefore return to eq. (1.2), i.e. eq. (4.6) with $\varepsilon = 0$.

We observe that, if $\Theta_e \geq \Theta_i$, $w_{ee}(r)$ is always negative. If, however, $\Theta_e < \Theta_i$, $w_{ee}(r)$ changes sign at a distance $r_0$ given by

$$r_0 = D_e\,\frac{\ln\left(1-\Theta_e/\Theta_i\right)}{1-\left(1+2\,\Theta_e/\Theta_i\right)^{1/2}} \tag{4.7}$$

For $\Theta_e \ll \Theta_i$ we have $r_0 \approx 2\,D_e/2$

The correlation has the "wrong sign" for $r > r_0$.
This phenomenon may be called "anti-shielding".
Also the average charge density around an electron described by $w_1(r) = w_{ee}(r) - w_{ei}(r)$ shows such a behaviour.
The radial distribution functions $w_{ei}$ etc. may be found in a similar way as eq. (4.6). In the limit $\varepsilon \to 0$ one has

$$w_{ei}(r) = w_{ie}(r) = \frac{Z\,e^2}{\Theta_e\, r}\exp\left(-\frac{r}{D}\right) \tag{4.8}$$

$$w_{ii}(r) = -\frac{Z^2 e^2}{\Theta_i\, r}\exp\left(-\frac{r}{D}\right)$$

It can be seen easily that the average charge density around an electron changes sign at the distance

$$r_0' = D_e \frac{\ln\left\{1 + \frac{(z+1)\,\Theta_e}{(1+z\,\Theta_e/\Theta_i)^{1/2} - 1}\right\}}{\left(1 + \frac{z\,\Theta_e}{\Theta_i}\right)^{1/2} - 1} \qquad (4.9)$$

If $\Theta_e \ll \Theta_i$ then $r_0' \simeq \frac{1+z}{\lambda z} D_e$ .

Anti-shielding is also possible in a one component plasma, i.e. electron gas and continuous neutralizing background. This case is obtained from our formalism in the limit $z \to 0$ .
Wolff [19] showed already in this case that for the (one-dimensional) distribution function

$$F(u) = \frac{2\,\alpha^{1/4}}{\Gamma(1/4)} \exp(-\alpha u^4) \qquad (4.10)$$

one finds

$$w(r) = -\frac{1}{4\pi n D_0^2}\left(\frac{2}{r} - \frac{1}{2D_0}\right) \exp\left(-\frac{r}{D_0}\right) \qquad (4.11)$$

with

$$D_0^{-2} = \omega_p^2\, \alpha^{1/2}\, 4\, \frac{\Gamma(3/4)}{\Gamma(1/4)} \qquad (4.12)$$

It follows immediately from eq. (4.11) that $w(r)$ changes sign at $r = 4 D_0$ .
The analogy between the example of Wolff and the anti-shielding of eq. (1.2) lies in the fact that in both cases there are less electrons in the tail of the distribution than in an equilibrium with the same average energy per particle.

We want to point out that anti-shielding is not surprising in the light of the connection between the correlation function and the test particle problem as given by Rostoker [18] which may be

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19) P.A. Wolff, Phys. Fluids 5, 316 (1962)

$$r_0' = D_e \frac{\ln\left\{1 + \frac{(z+1)\,\Theta_e}{(1+z\,\Theta_e/\Theta_i)^{1/2} - 1}\right\}}{\left(1 + \frac{z\,\Theta_e}{\Theta_i}\right)^{1/2} - 1} \qquad (4.9)$$

If $\Theta_e \ll \Theta_i$ then $r_0' \simeq \frac{1+z}{2z} D_e$ .

Anti-shielding is also possible in a one component plasma, i.e. electron gas and continuous neutralizing background. This case is obtained from our formalism in the limit $z \to 0$ .
Wolff [19] showed already in this case that for the (one-dimensional) distribution function

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The analogy between the example of Wolff and the anti-shielding of eq. (1.2) lies in the fact that in both cases there are less electrons in the tail of the distribution than in an equilibrium with the same average energy per particle.

We want to point out that anti-shielding is not surprising in the light of the connection between the correlation function and the test particle problem as given by Rostoker [18] which may be

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19) P.A. Wolff, Phys. Fluids 5, 316 (1962)

expressed as

$$g_{ab}(\vec{r}, \vec{v}_a, \vec{v}_b) = F_a(\vec{v}_a)\, f_{ab}(\vec{r}, \vec{v}_a, \vec{v}_b) + F_b(\vec{v}_b)\, f_{ab}(-\vec{r}, \vec{v}_b, \vec{v}_a) + \sum_c n_c \int F_c(\vec{v}_c)\, f_{ca}(\vec{R}, \vec{v}_c, \vec{v}_a)\, f_{cb}(\vec{R}+\vec{r}, \vec{v}_c, \vec{v}_b)\, d^3R\, d^3v_c \tag{4.13}$$

where the notation is analogous to that in section 3.

If a and b represent electrons then eq. (3.12) follows by integration over $\vec{v}_a$, $\vec{v}_b$ and Fourier transformation with respect to $\vec{r}$. The first two terms in the right hand side of eq. (3.12) may be interpreted in the way that one of the electrons under consideration is a test particle and the other one a field particle or that both are in the cloud of a third electron, the last term on the other hand describes the two electrons being in the cloud of an ion. Analyzing the derivation of eq. (1.2) we conclude that if

$$w_{ee}(r) = w_{ee}^{I}(r) + w_{ee}^{II}(r)$$

$$w_{ee}^{I}(r) = -\frac{e^2}{\Theta_e r} \exp\left(-\frac{r}{D_e}\right) \tag{4.14}$$

$$w_{ee}^{II}(r) = -\frac{e^2}{\Theta_e r} \frac{\Theta_i}{\Theta_e} \left[\exp\left(-\frac{r}{D}\right) - \exp\left(-\frac{r}{D_e}\right)\right]$$

then $w_{ee}^{II}(r)$ stands for the term in which an ion plays the role of the test particle and $w_{ee}^{I}(r)$ represents all other terms. We see that $w_{ee}^{I}(r)$ is negative and $w_{ee}^{II}(r)$ positive for all $r$. The leading terms for large $r$ are those with the factor $\exp(-\frac{r}{D_e})$ because $D_e > D$. In equilibrium, however, these terms cancel. If $\Theta_e < \Theta_i$ their sum is positive and this means anti-shielding.

Anti-shielding arises from the enhanced tendency of two electrons to approach both an ion in the case that the electron temperature is lower than the equilibrium value $\Theta = \Theta_i$ .
It is tempting to assume in a similar way that the antishielding in the case (4.11) of a one-component plasma is due to an enhanced tendency of two electrons to avoid the neighbourhood of a third one as a consequence of the relative lack of fast electrons in the distribution (4.10).

## 5. The influence of an external magnetic field.

If an external magnetic field $\vec{B}$ is present we have to add expressions (a, b represent electrons or ions)

$$-\frac{e}{mc}\left[(\vec{v}_a \times \vec{B})\cdot\frac{\partial}{\partial \vec{v}_a} + (\vec{v}_b \times \vec{B})\cdot\frac{\partial}{\partial \vec{v}_b}\right] g_{ab}(\vec{x}_a, \vec{x}_b, \vec{v}_a, \vec{v}_b, t) \quad (5.1)$$

to the left hand sides of equations of the type (2.1). The general kinetic theory as described in section 2 then becomes very complicated. In the situation of section 4, however, where we have two Maxwellians, and in the limit $\varepsilon \to 0$ (cf. eq. (4.4)) the system of equations of the type (2.1) may be solved by means of the ansatz ( $F_{Ma,b}$ are Maxwellian distributions)

$$\frac{\partial}{\partial t} = 0 \; ; \; g_{ab}(\vec{x}_a, \vec{x}_b, \vec{v}_a, \vec{v}_b) = F_{Ma}(\vec{v}_a) F_{Mb}(\vec{v}_b) \chi(\vec{x}_a - \vec{x}_b) \quad (5.2)$$

if one neglects all scalar products containing the velocity of an ion as a factor. This neglect corresponds to the limit $\varepsilon \to 0$ . It is seen immediately that under these conditions all expressions of the type (5.1) vanish and it is possible to recover eq. (1.2) and eq. (4.8) [7].
Consequently a magnetic field does not change the results of section 4 in zeroth order of $\varepsilon$ .

Appendix A.

Solution of the integral equation (2.16)

Writing $H = H^+ + H^-$ in eq. (2.16) and using eq. (2.15) we have

$$Z^- H^+ + Z^+ H^- = \frac{2\pi i u_p^2}{n} \left\{ F_1^- \frac{\partial F_2}{\partial u} - F_1 \frac{\partial F_2^-}{\partial u} \right\}$$

Dividing by $Z^- Z^+$ one can separate positive and negative frequency parts.

$$n H^+ = 2\pi i u_p^2 Z^+ \left\{ \frac{F_1^- \frac{\partial F_2}{\partial u} - F_1 \frac{\partial F_2^-}{\partial u}}{Z^- Z^+} \right\}^+$$

and a similar equation for $H^-$.

This decomposition is unique because, if we split up zero in positive and negative frequency parts

$$0 = H_0^+ + H_0^-$$

then $H_0^+$ and $H_0^-$ are required to be regular in the entire complex plane and to vanish at infinity. This implies $H_0^+ \equiv H_0^- \equiv 0$

A condition for our procedure to be valid is that $Z^+$ and $Z^-$ have no zeros in the upper- and lower half plane respectively. It is shown in Ref. 10 that this is true if $F_2(u)$ obeys the Penrose criterion [20] for electrostatic stability.

We express $\frac{\partial F_2}{\partial u}$ and $\frac{\partial F_2^-}{\partial u}$ in terms of $Z^\pm$ by means of eq. (2.15) and obtain

$$n H^+ = Z^+ \left\{ \frac{F_1^- F_1^- Z^+ - F_1^+ Z^-}{Z^+ Z^-} \right\}^+ = - F_1^+ + \left( \frac{F_1}{Z^+ Z^-} \right)^+ Z^+$$

We arrive at

$$n H^\pm = - F_1^\pm + A^\pm Z^\pm \qquad \text{(A.1)}$$

where A is given in eq.(2.18) .

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20) O. Penrose, Phys. Fluids 3, 258 (1960)

## Appendix B.

## Evaluation of $H_{ee}$ etc.

Substituting eq. (A.1) into eqs. (2.13) and (2.14) and eliminating $\partial F_2^- / \partial u$ by means of eq. (2.15) one derives

$$n H_1 + F^e = \frac{F^e}{Z^-} + 2\pi i u_p^2 \frac{\partial F^e}{\partial u} A^-$$
$$n H_2 + Z F^i = \frac{Z F^i}{Z^-} + 2\pi i u_p^2 \frac{mz}{M} \frac{\partial F^i}{\partial u} A^- \quad \text{(B.1)}$$

Substituting again eq. (B.1) into eqs. (2.9), (2.10), (2.11) and (2.12) and remembering that for real $u$ positive and negative part of a function are complex conjugate we arrive at eq. (2.17) and

$$n H_{ei} = 2\pi i u_p^2 \left[ -\left( \frac{Z F^i}{Z^+} - \frac{mz}{M} \frac{\partial F^i}{\partial u} 2\pi i u_p^2 A^+ \right)^- \frac{\partial F^e}{\partial u} \right. \quad \text{(B.2)}$$
$$\left. + \left( \frac{F^e}{Z^-} + \frac{\partial F^e}{\partial u} 2\pi i u_p^2 A^- \right) \frac{mz}{M} \frac{\partial F^i}{\partial u} \right]$$

$H_{ii}$ is obtained by interchanging $F^e$ with $Z F^i$ and $\frac{\partial F^e}{\partial u}$ with $\frac{mz}{M} \frac{\partial F^i}{\partial u}$ in eq. (2.17) and $H_{ie}$ by doing the same in eq. (B.2).

Integrating eq. (2.17) over $u$ and applying the rule

$$\int_{-\infty}^{+\infty} \psi^-(u) \chi(u) \, du = \int_{-\infty}^{+\infty} \psi(u) \chi^+(u) \, du \quad \text{(B.3)}$$

which follows directly from eq. (2.6), we find eq. (2.20). In the same way one gets

$$n \hat{w}_{ei} = 2\pi i u_p^2 \int_{-\infty}^{+\infty} \left[ -\left( \frac{Z F^i}{Z^+} - 2\pi i u_p^2 \frac{mz}{M} \frac{\partial F^i}{\partial u} A^+ \right) \frac{\partial F^{e+}}{\partial u} \right. \quad \text{(B.4)}$$
$$\left. + \left( \frac{F^e}{Z^-} + 2\pi i u_p^2 \frac{\partial F^e}{\partial u} A^- \right) \frac{mz}{M} \frac{\partial F^{i-}}{\partial u} \right] du$$

The corresponding formula for $\hat{w}_{ii}$ is obtained from eq. (2.20) by interchanging $F^e$ with $z F^i$ and $\partial F^e/\partial u$ with $\frac{mz}{M}\frac{\partial F^i}{\partial u}$. By the same procedure $\hat{w}_{ie}$ is derived from eq. (B.4).

Appendix C.

Derivation of the formulae (2.21) up to (2.25).

Substitution of A from eq. (2.18) and of the identities

$$\frac{F^e}{Z^\pm} \equiv \frac{\partial F^e}{\partial u}\left[\left(\frac{F^e}{\frac{\partial F^e}{\partial u} Z^\pm}\right)^+ + \left(\frac{F^e}{\frac{\partial F^e}{\partial u} Z^\pm}\right)^-\right]$$

into eq. (2.20) gives

$$n\hat{w}_{ee} = 2\pi i u_p^2 \int_{-\infty}^{+\infty}\left[\left\{\left(\frac{F^e}{\frac{\partial F^e}{\partial u} Z^+} - \frac{F_1}{\frac{\partial F_2}{\partial u} Z^+}\right)^+ + \left(\frac{F^e}{\frac{\partial F^e}{\partial u} Z^+}\right)^- + \left(\frac{F_1}{\frac{\partial F_2}{\partial u} Z^-}\right)^+\right\}\frac{\partial F^e}{\partial u}\frac{\partial F^{e+}}{\partial u} - \{\text{compl. conj.}\}\right] du$$

Omitting purely positive or negative frequency functions from the integrand, because these do not contribute to the integral, one obtains

$$n\hat{w}_{ee} = 2\pi i u_p^2 \int_{-\infty}^{+\infty}\left[\left(\frac{\partial F^{e+}}{\partial u}\right)^2\left(\frac{F^e}{\frac{\partial F^e}{\partial u} Z^+}\right)^- - \left(\frac{\partial F^{e-}}{\partial u}\right)^2\left(\frac{F^e}{\frac{\partial F^e}{\partial u} Z^-}\right)^+ + \frac{\partial F^{e+}}{\partial u}\frac{\partial F^{e-}}{\partial u}\left(\frac{1}{Z^+} - \frac{1}{Z^-}\right)\left(\frac{F^e}{\frac{\partial F^e}{\partial u}} - \frac{F_1}{\frac{\partial F_2}{\partial u}}\right)\right] du$$

Now we apply the rule (B.3) in order to remove the $\pm$ operations from $\left(\frac{F^e}{\frac{\partial F^e}{\partial u} Z^\pm}\right)^\pm$. We carry out the subtraction in the last factor of the integrand with the aid of the definitions for $F_1$ and $F_2$ given in eq. (2.8).

Finally we subtract unity from the entire expression and add it again in the form

$$\int_{-\infty}^{+\infty} \frac{F^e}{\partial F^e/\partial u} \left\{ \left( \frac{\partial F^e}{\partial u} \right)^+ + \left( \frac{\partial F^e}{\partial u} \right)^- \right\} du$$

using eqs. (2.8) and (2.15) in order to simplify the sum. The result is given in eq. (2.21). Eq. (2.22) follows directly by taking the common denominator $Z^- Z^+$

Defining

$$\hat{w}_1 = \int H_1 \, du = \hat{w}_{ee} - \hat{w}_{ei} \qquad (C.1)$$

we find from eqs. (B.1), (2.15), (2.18) and (2.8)

$$n \hat{w}_1 = -1 + \int_{-\infty}^{+\infty} \left( \frac{F^e}{Z^-} + 2\pi i u_p^2 A \frac{\partial F^{e+}}{\partial u} \right) du$$

$$= -1 + \int_{-\infty}^{+\infty} \frac{1}{Z^- Z^+} \left[ F^e \left( 1 - 2\pi i u_p^2 \frac{\partial F_i^+}{\partial u} \right) + 2\pi i u_p^2 F_i \frac{\partial F^{e+}}{\partial u} \right] du$$

$$= -1 + \int_{-\infty}^{+\infty} \frac{du}{Z^- Z^+} \left[ F^e \left( 1 - 2\pi i u_p^2 \frac{m_i}{M} \frac{\partial F^{i+}}{\partial u} \right) + 2\pi i u_p^2 Z F^i \frac{\partial F^{e+}}{\partial u} \right]$$

The last equality of eq. (C.1) together with eq. (2.21) now leads directly to eq. (2.24).

Eqs. (2.23) and (2.25) follow from eqs. (2.22) and (2.24) respectively by virtue of the rule (B.5).

## Appendix D.

## Derivation of eqs. (4.2), (4.3), (4.6).

The first term between the curly brackets in eq. (2.21) is a positive frequency function, say $\Lambda^+$ and therefore regular in the upper half plane. The factor $F^e (\partial F^e/\partial u)^{-1}$ creates a pole at $u = 0$ for the terms between the curly brackets

separately but not for their sum. Therefore

$$\int_{-\infty}^{+\infty} \frac{F^e}{\partial F^e/\partial u} \{ \cdots\cdots \} du = 2\pi i \,(\text{Residue in } u=0 \text{ of } \Lambda^+)$$

(It is easily verified that the contour at infinity does not contribute)

Substitution of the Maxwell distributions (4.1) and elementary calculations using $\left(u F^e\right)^{\pm}_{u=0} = \left(u F^i\right)^{\pm}_{u=0} = \pm \frac{1}{2\pi i}$ yield eq. (4.2).

The integral $I$ is obtained in the form of eq. (4.3) by means of the substitutions

$$w = \left(\frac{M}{2\theta_i}\right)^{1/2} u \qquad \left(u F^{e,i}\right)^{\pm} = \pm \frac{1}{2\pi i} + u \left(F^{e,i}\right)^{\pm}$$

$$\frac{1}{Z^- Z^+} = \frac{1}{Z^- - Z^+} \left( \frac{1}{Z^+} - \frac{1}{Z^-} \right)$$

By virtue of the last identity one obtains

$$I = -\alpha \pi^{-1/2} \int_{-\infty}^{+\infty} \exp(-w^2)\, \ell^2(\varepsilon w) \left| 1 + 2\pi^{1/2} \alpha i w G^+(w) \right|^{-2} dw \qquad (D.1)$$

where

$$\alpha = D_i^{-2} \left(k^2 + D^{-2}\right)^{-1} \qquad p = D_i^2 / D_e^2 = \theta_i / 2\theta_e$$

$$\ell(\varepsilon w) = \left| 1 + 2\pi^{1/2} i \varepsilon w \left\{ \exp(-\varepsilon^2 w^2) \right\}^+ \right| \qquad (D.2)$$

$$G(w) = \exp(-w^2) + \varepsilon p \exp(-\varepsilon^2 w^2)$$

We investigate the order of magnitude of $I_R$ i.e. the contribution to $I$ from the region $w > R$
Because the integrand is an even function of $w$

$I_R = I_{-R}$ if $I_{-R}$ is contributed by $w < -R$

From the Nyquist diagrams of the type described by e.g. Penrose [20] it follows that $\ell(\varepsilon w) \leq 1$

Furthermore

$$\left|1+2\pi^{1/2}\alpha i w\, G^{+}(w)\right|^2 = \left(1+\alpha\pi^{-1/2} w\, P\int_{-\infty}^{+\infty}\frac{G(\xi)}{\xi-w}\,d\xi\right)^2 +$$

$$+\pi\alpha^2 G^2(w) w^2 > \pi\alpha^2 w^2 G^2(w) > \pi\alpha^2\rho^2\varepsilon^2 w^2 \exp(-2\varepsilon^2 w^2)$$

Therefore

$$|I_R| < \frac{1}{\alpha\rho^2\pi^{3/2}}\,\frac{1}{\varepsilon^2}\int_R^\infty \frac{1}{w^2}\exp\{-(1-2\varepsilon^2)w^2\}\,dw$$

If $R>1$ then $w^2>w$ for $w\geq R$ and

$$\int_R^\infty \frac{1}{w^2}\exp(-\beta w^2)dw < \frac{1}{R^2}\int_R^\infty \exp(-\beta w)\,dw = \frac{1}{R^2\beta}\exp(-\beta R)$$

Consequently

$$I_R < \frac{1}{\alpha\rho^2\pi^{3/2}}\,\frac{1}{\varepsilon^2}\,\frac{\exp\{-(1-2\varepsilon^2)R\}}{1-2\varepsilon^2} \qquad (D.3)$$

i.e. $I_R$ is exponentially small in $R$. We now choose $R=\varepsilon^{-1/2}$ and return to eq. (4.3) in which we replace the limits of integration by $-R$ and $+R$

$$I = Re\;\frac{1}{\pi i}P\int_{-R}^{+R}\frac{dw}{w}\,\frac{\ell^2(\varepsilon w)}{\left[1+\varepsilon\rho\exp\{(1-\varepsilon^2)w^2\}\right]\left[1+2\pi^{1/2}\alpha i w G^{+}(w)\right]} \qquad (D.4)$$

By contour integration we have

$$I = \frac{1}{1+\varepsilon\rho} + Re\left[2\sum \text{Residues} - \frac{1}{\pi i}I_\phi\right] \qquad (D.5)$$

The residues belong to zeros $w_n$ of $f(w^2)=$
$= 1+\varepsilon\rho\exp\{(1-\varepsilon^2)w^2\}$ in the upper half plane with $|w_n|<R$ and $I_\phi$ is an integral along a semi circle with radius $R$ in the upper half plane.

It is easily seen that

$$z_n \equiv w_n^2 = \frac{1}{1-\varepsilon^2}\left\{\ln\frac{1}{\varepsilon\rho}+(2n+1)\pi i\right\},\quad n=0,\pm1,\pm2,\cdots$$

The transformation $z = w^2$ maps the poles of the upper half $w$-plane into the poles of the entire $z$-plane.

$$2\sum \text{Residues} = \sum_{|z_n|<R^2} \frac{1}{(1-\alpha) z_n f'(z_n)} \left[ 1 + \frac{\alpha}{(1-\alpha)^2 z_n} + O\left(\frac{1}{z_n^2}, \varepsilon^2 z_n\right) \right] \quad \text{(D.6)}$$

Terms of order $\varepsilon^2 z_n$ are due to Taylor expansions of $f^2(\varepsilon w)$ and $\{\exp(-\varepsilon^2 w^2)\}^+$. For the other exponential in $G^+(w)$ i.e. $\{\exp(-w^2)\}^+$ an asymptotic expansion has been used. This is possible because $|z_n| \gg 1 \gg \varepsilon^2 |z_n|$

In the integrand of $I_\phi$ the same expansions show that the integrand is a function of $w^2$. The semi circle can then be transformed into a full circle in the $z$-plane.

$$I_\phi = \frac{1}{2} \oint_{|z|=R^2} \frac{dz}{(1-\alpha) z f(z)} \left[ 1 + \frac{\alpha}{(1-\alpha) 2 z} + O\left(\frac{1}{z^2}, \varepsilon^2 z\right) \right] \quad \text{(D.7)}$$

All the terms in $I_\phi$ may now be calculated by the residue method again. For the residue at $z = 0$ one needs an expansion of $f^{-1}(z)$ in powers of $z$

$$\frac{1}{\pi i} I_\phi = \frac{1}{(1-\alpha)(1+\varepsilon p)} \left\{ 1 - \frac{1}{2} \frac{\varepsilon p \alpha}{1-\alpha} \right\} + O(\varepsilon^2) + \quad \text{(D.8)}$$

$$+ \frac{1}{1-\alpha} \sum_{|z_n|<R^2} \frac{1}{z_n f'(z_n)} \left[ 1 + \frac{\alpha}{(1-\alpha) 2 z_n} + O\left(\frac{1}{z_n^2}, \varepsilon^2 z_n\right) \right]$$

The first two terms in the right hand side arise from the pole at $z = 0$.

A crucial point is that the terms $O\left(\frac{1}{z_n^2}, \varepsilon^2 z_n\right)$ are identical with the corresponding terms in eq. (D.6).

From eqs. (D.5), (D.6) and (D.8) then follows that the residues of the poles at $z_n$ cancel and therefore

$$I = -\frac{\alpha}{(1-\alpha)(1+\varepsilon p)} + \frac{1}{2} \frac{\varepsilon p \alpha}{(1-\alpha)^2} + O(\varepsilon^2)$$

or, with the aid of eqs. (D.2) and (4.2),

$$\eta \hat{W}_{ee} = -D_e^{-2}\left[\frac{\theta_i}{\theta_e}\frac{1}{k^2+D^{-2}} + \frac{\theta_e-\theta_i}{\theta_e}\left\{\left(1-\frac{\varepsilon\theta_i}{2\theta_e}\right)\frac{1}{k^2+D_e^{-2}} + \right.\right.$$

$$\left.\left. + \frac{\varepsilon\theta_i}{2\theta_e}\frac{1}{k^2+D^{-2}} - \frac{1}{2}\varepsilon\frac{D_e^{-2}}{(k^2+D_e^{-2})^2}\right\}\right] + O(\varepsilon^2)$$

Fourier transformation now yields eq. (4.6).