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The harmonics of the

electron cyclotron frequency in plasmas.

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Abstract

The problem of the harmonics of the electron cyclotron frequency in plasmas is discussed.
The dispersion relation for propagation vectors nearly perpendicular to the external magnetic field is studied for some density, temperature and frequency ranges. The unperturbed distribution function is assumed to be Maxwellian. Longitudinal waves are also discussed in plasmas with electron streams and temperature anisotropies.

I. Introduction

Intense radiation at harmonics of the electron cyclotron frequency from a non-relativistic plasma immersed in a magnetic field was reported by G. Landauer[1] in 1961 and later by many others[2].

In previous papers[3], it has been shown that quasi-electrostatic waves propagate nearly perpendicular to the external magnetic field, $\vec{B}_0$, with a large index of refraction at the harmonics of the electron cyclotron frequency. These waves may explain the radiation observed perpendicularly to $\vec{B}_0$ by Landauer if a sufficiently large number of electrons with superthermal velocity perpendicular to $\vec{B}_0$ are present. The radiation emitted parallel to $\vec{B}_0$ could actually be explained by the same mechanism because the energy does not flow in the direction of the quasi-electrostatic waves.

The theory seems to apply to the other experiments also, and even in cases where it is very improbable that the electron distribution function departs from a Maxwellian in the superthermal range. In fact, in this case the condition $v_\perp / v_{phase} \gtrsim 1$ (where $v_\perp$ is the electron velocity perpendicular to $\vec{B}_0$), which is necessary in order to have radiation at the harmonics, can be fulfilled even in the thermal and subthermal velocity ranges by those waves which have been found in ref. 3 to have a phase velocity smaller than the thermal velocity. (The assumption of a non-Maxwellian distribution function is, of course, useful only for non-thermal radiation).

In ref. 3 we gave only a short and very rough discussion of the dispersion relation, the cyclotron radiation and the coupling between longitudinal and transverse waves. Here we study in detail the dispersion relation in the range of parameters which seems to be typical for the observed phenomena, namely: angular frequency ω of the same order of magnitude as the plasma frequency ω_p; collision frequency γ much smaller than ω and electron thermal velocity v_{th} two or three order of magnitude smaller than c.

For the longitudinal waves we take into account also deviations from thermal equilibrium.

Some other points in papers 3 require a through discussion. First, the radiation of longitudinal waves from an electron in a plasma with strong absorption. In deriving the energy loss of a spiralling electron, the absorption was neglected. However, numerical calculations in progress show that the given value is not far from the correct one.

Moreover the total power emitted by the plasma was estimated assuming complete incoherence between the radiating electrons. The energy loss from a charge being proportional to the square of the charge, any coherence effect should enhance the radiation.

Another very important point is the coupling between longitudinal and transverse waves, which in ref. 3 was calculated for the particularly simple and unrealistic case of a sharp edge density- and magnetic field profile.

One would think that this coupling may be very small in a more realistic model, because the characteristic lengths of the inhomogeneities in plasma are certainly much larger than the wave length of the longitudinal waves, λ. However, D. Pfirsch has suggested (private communication) that the coupling can still be large in the neighbourhood of the resonances of the index of refraction for longitudinal waves, N_1, even if the inhomogeneity in $\vec{B}_o$ (and/or density, mean velocity and collision frequency) is small, provided that the inequality

$$\frac{|\mathrm{grad}\ N_1|}{|N_1|} \gtrsim \frac{1}{\lambda} \qquad (1)$$

can be verified. (Then the WKB method cannot be applied.)

In this way a coupling between the longitudinal and the transverse waves, which go adiabatically into vacuum, can take place not only at the plasma-vacuum surface as it was assumed in ref. 3, but also in the interior of the plasma. The value of the coupling, however, seems to be not very different from that quoted in ref. 3 for the following reason. Local variations of the microscopic index of refraction in the neighbourhood of the harmonics of the cyclotron frequency that satisfy (1) may be similar in effect to the jump of N_1 quoted in ref. 3. Actually the jump of N_1 follows from that of $\vec{B}_o$ and density. However, it is possible to construct a plasma model in which a jump in $\vec{B}_o$ does not give a sensible change in N_1 and see that in this case the coupling is negligible[4]. Hence we comclude that the relation:

$$\text{Transmission Coefficient} \approx 2/|N_1|$$

is principally due to the rapid change in the index of refraction and that this may be critically dependent upon small variations in $\vec{B}_o$, density, etc..

Although the inhomogeneities of plasma and $\vec{B}_o$ are an essential feature of the problem, we take them into account only in order to evaluate the coupling and in the following we consider waves propagating in a homogeneous plasma. In this way we renounce a priori the description of several aspects of the problem like fine structure[5], tunnelling effect[6] and so on[7,8].

Here, as in ref. 3, we take account of Landau damping. The possible predominance of the collisions relative to Landau damping does not alter the form of the result, as will be briefly discussed at the end of section 2.

In sect. 2, the dispersion relation is given in its general form. In sect. 3 it is approximated and wave propagation at frequencies outside the harmonics is studied. Wave propagation in the neighbourhood of the harmonics is discussed in sect. 4 neglecting Landau damping and in sect. 5 including Landau damping. In sect. 6 the equation for longitudinal waves is considered for electron distribution functions departing from a Maxwellian to take into account streaming and temperature anisotropies. In App. 1 we approximate the series involved in the dispersion relation and give asymptotic expansions for Bessel functions of complex argument. For several ranges of the parameters, the dispersion relation may be reduced to a simpler class of equations. In App. 2, approximate solutions to these are given.

2. Dispersion relation

Let us consider a uniform and infinitely extended plasma with the following equilibrium electron distribution function

$$f_0(v) = \frac{d}{(\sqrt{2\pi}\, v_{th})^3} \exp\left(-v^2/(2 v_{th}^2)\right)$$

where d is the number of electrons per cm^3.

We shall look for waves of the form $\exp[i(\vec{K}\cdot\vec{r} - \omega t)]$, with an angular frequency ω high enough, so that ion motion can be neglected. Then in a Cartesian reference system ($\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$), such that the external magnetic field $\vec{B}_o$ is parallel to $\vec{e}_3$ and $\vec{K} = K_\perp \vec{e}_1 + K_\parallel \vec{e}_3$, the complex dielectric tensor has the following components[9]

$$\varepsilon_{ik} = \delta_{ik} + \left(\frac{\omega_p}{\omega}\right)^2 \cdot \frac{\gamma_{ik}}{\Delta} \quad ,$$

where:

$$\gamma_{11} = \frac{1}{\mu} \sum_{n=-\infty}^{+\infty} n^2 Z_n \cdot I_n e^{-\mu} \quad ;$$

$$\gamma_{12} = -\gamma_{21} = i \sum_{n=-\infty}^{+\infty} n Z_n \cdot (I_n e^{-\mu})' \quad ;$$

$$\gamma_{22} = \gamma_{11} - 2\mu \sum_{n=-\infty}^{+\infty} Z_n \cdot (I_n e^{-\mu})' \quad ;$$

$$\gamma_{13} = \gamma_{31} = \sqrt{\frac{2}{\mu}} \sum_{n=-\infty}^{+\infty} n \xi_n Z_n \cdot I_n e^{-\mu} \quad ;$$

$$\gamma_{23} = -\gamma_{32} = -i\sqrt{2\mu} \sum_{n=-\infty}^{+\infty} \xi_n Z_n \cdot (I_n e^{-\mu})' \quad ;$$

$$\gamma_{33} = 2\xi_o + 2 \sum_{n=-\infty}^{+\infty} \xi_n^2 Z_n \cdot I_n e^{-\mu} \quad ;$$

$$Z_n \equiv Z(\xi_n) \equiv \left(i\sqrt{\pi} \frac{K_\parallel}{|K_\parallel|} - 2 \int_0^{\xi_n} \exp(t^2)\, dt \right) \exp(-\xi_n^2) \; ;$$

$$\xi_n \equiv \frac{\omega + i\gamma - n\omega_e}{\omega \Delta} \; ; \quad \Delta \equiv \frac{\sqrt{2}\, v_{th} K_\parallel}{\omega} \; ; \quad \mu \equiv \frac{v_{th}^2 K_\perp^2}{\omega_e^2} \; .$$

$I_n \equiv I_n(\mu)$ is the modified Bessel function of the first kind,

$(I_{\bar{n}}\, e^{-\mu})'$ the derivative of $(I_{\bar{n}}\, e^{-\mu})$ with respect to μ, and ω_e the electron cyclotron frequency.

The dispersion relation

$$\left\| \left(\frac{\omega}{c}\right)^2 \varepsilon_{ik}(\omega, \vec{K}) + K_i K_k (1 - \delta_{ik}) \right\| = 0$$

can be written

$$A\, K^4 + B \left(\frac{\omega}{c}\right)^2 K^2 + C \left(\frac{\omega}{c}\right)^4 = 0 \qquad (2)$$

where
$$A = \varepsilon_{11} \sin^2\theta + 2\varepsilon_{13} \sin\theta\cos\theta + \varepsilon_{33}\cos^2\theta =$$

$$= 1 + \left(\frac{\omega_p}{\omega}\right)^2 \left(1 + \frac{1}{\Delta} \sum_{n=-\infty}^{+\infty} Z_n\, I_n e^{-\mu}\right) \Big/ \left(\left(\frac{\omega_e}{\omega}\right)^2 \mu + \frac{\Delta^2}{2}\right)$$

$$-B = \varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2 + (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2)\sin^2\theta + (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{23}^2)\cos^2\theta -$$

$$- 2(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{13})\sin\theta\cos\theta \,.$$

$$C = \|\varepsilon_{ik}\| = \varepsilon_{33}(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2) + \varepsilon_{11}\varepsilon_{23}^2 - \varepsilon_{22}\varepsilon_{13}^2 + 2\varepsilon_{12}\varepsilon_{13}\varepsilon_{23}$$

$$\sin\theta = \frac{K_\perp}{K} \quad ; \quad \cos\theta = \frac{K_\parallel}{K} \,.$$

Eq. (2) reduces to $A \approx 0$ in the case of quasi-longitudinal waves

In what follows we shall use eq. (2) to determine $K_\perp$ as a function of ω and $K_\parallel$, assuming ω and $K_\parallel$ real. In fact, although we consider here an infinitely extended plasma, we are actually concerned[3] with a plasma column parallel to $\vec{B_0}$, and with the problem of the radiation through the lateral surface of the plas ma. In analogy to Snell's law in Optics, which states that ω and $K_{\parallel}$ do not change through a surface of discontinuity, it seems reasonable to assume that also through the lateral surface of the plasma ω and $K_{\parallel}$ do not change.

This fact provides also an upper limit for $|K_{\parallel}|$, namely

$$|K_{\parallel}| = |K_{\parallel\,\mathrm{vacuum}}| \leqslant |K_{\mathrm{vacuum}}| = \frac{\omega}{c} .$$

A lower limit for $|K_{\parallel}|$ seems to be $1/L$, where L is the length of the plasma column.

Starting from eq. (2) and using the condition

$$\frac{1}{L} \leqslant |K_{\parallel}| \leqslant \frac{\omega}{c}$$

we want to show in the next sections that in the range of parameters ω_p/ω and v_{th}/c we are interested in, there are waves propagating nearly perpendicularly to $\vec{B}_0$ with a large index of refraction. In particular we shall see that there are longitudinal waves whose index of refraction has a line structure. The lines appear when ω/ω_e approaches the value $\bar{n}$, where $\bar{n}$ is an integer ranging from one to a number approximately equal to the inverse of the relative width of the lines $\omega/\Delta\omega$ (or $\omega/(\bar{n}\Delta\omega_e)$ if ω is fixed and $\vec{B}_0$ varies). Moreover there are waves whose index of refraction is large for all values of ω/ω_e up to about the same value as before. They can also give rise to radiation of lines, because of the line structure of the cyclotron radiation itself. For these waves, however, the ratio between absorption coefficient and wave length is larger than for the longitudinal waves and the transmission coefficient probably much smaller.

In Landauer's experiments the width of the lines, $\Delta\omega_e$, seems to be determined by Landau damping, because $\bar{n}\Delta\omega_e$ is about $10^{-2}\omega$ and γ can be estimated to be much smaller than $10^{-3}\omega$. The relative line width is therefore of the order of $(K_{\parallel} v_{th})/\omega$ i.e. of Δ . Accordingly we shall neglect collisions.

In the following we shall discuss the dispersion relation (2) not for arbitrary values of ω - this would be very complicated - but at points between every two consecutive harmonics such that $O(\omega - \bar{n}\omega_e) = O(\omega_e/2)$ and in the very neighbourhood of the harmonics $\omega \approx \bar{n}\omega_e$.

In the first case, $|\xi_n|$ is larger than 1 for all integer n, provided that

$$1/(\bar{n}\Delta) \gg 1. \qquad (3)$$

In the other case we have

$$|\xi_{\bar{n}}| \lessapprox 1,$$

if $\Delta \gtrapprox (\bar{n}\,\Delta\omega_e)/n$, and

$$|\xi_n|_{n\neq\bar{n}} \gtrapprox 2$$

if condition (3) is fulfilled.

Condition (3) is necessary in order to have large values of the index of refraction. A $|\Delta|$ of the order of 10^{-2} seems to be in agreement with the fact that Landauer has observed harmonics up to about the 45-th and measured for the relative width of the lines, $(\bar{n}\Delta\omega_e)/\omega$, an almost constant value of some percent.

Remembering that[9], when $|\xi_n| \gtrapprox 2$

$$Z_n = -\frac{1}{\xi_n}\left(1 + \frac{1}{2\xi_n^2} + \frac{3}{4\xi_n^4} + \dots\right) \qquad (4)$$

and that, when $|\xi_n| \ll 1$

$$Z_n \approx i\sqrt{\pi}\cdot\frac{K_\parallel}{|K_\parallel|}\, e^{-\zeta_n^2} - 2\zeta_n\left(1-\frac{2}{3}\zeta_n^2\right) \qquad (5)$$

we can give approximated expression for the quantities γ_{ik} valid for $O(\omega - \bar{n}\omega_e) = O(\omega_e/2)$ and for $\omega \approx \bar{n}\omega_e$ respectively.

Outside the harmonics, if we write $\omega = \nu\,\omega_e$ $(\nu \neq n)$, we get

$$\gamma_{11} = \frac{\nu\Delta}{\mu}\sum_{n=-\infty}^{+\infty}\frac{n^2}{n-\nu}\, I_n e^{-\mu} = \frac{\nu^2\Delta}{\mu}\left(1+\nu\sum_{n=-\infty}^{+\infty}\frac{1}{n-\nu}\cdot I_n e^{-\mu}\right)$$

$$\gamma_{12} = i\nu\Delta\sum_{n=-\infty}^{+\infty}\frac{n}{n-\nu}\left(I_n e^{-\mu}\right)' = i\nu^2\Delta\sum_{n=-\infty}^{+\infty}\frac{1}{n-\nu}\left(I_n e^{-\mu}\right)'$$

$$\gamma_{22} = \gamma_{11} - 2\nu\Delta\mu\sum_{n=-\infty}^{+\infty}\frac{1}{n-\nu}\left(I_n e^{-\mu}\right)' \qquad (6)$$

$$\gamma_{13} = -\frac{(\nu\Delta)^2}{\sqrt{2\mu}}\cdot\sum_{n=-\infty}^{+\infty}\left(\frac{1}{n-\nu}+\frac{\nu}{(n-\nu)^2}\right)\cdot I_n e^{-\mu}$$

$$\gamma_{23} = i(\nu\Delta)^2\sqrt{\frac{\mu}{2}}\cdot\sum_{n=-\infty}^{+\infty}\frac{1}{(n-\nu)^2}\cdot\left(I_n e^{-\mu}\right)'$$

$$\gamma_{33} = \nu\Delta\cdot\sum_{n=-\infty}^{+\infty}\frac{1}{n-\nu}\cdot I_n e^{-\mu}$$

and

$$A = 1+\left(\frac{\omega_p}{\omega}\right)^2\left(1+\nu\sum_{n=-\infty}^{+\infty}\frac{1}{n-\nu}\cdot I_n e^{-\mu}\right)\Big/\left(\frac{\mu}{\nu^2}+\frac{\Delta^2}{2}\right) .$$

These expressions hold approximately even if there are collisions, provided that $\gamma/\omega \ll 1$.

At the harmonics we get

$$\gamma_{11} = \frac{\bar{n}^2}{\mu} Z_{\bar{n}} \cdot I_{\bar{n}} e^{-\mu} + \frac{\bar{n}\Delta}{\mu} \sum_{n \neq \bar{n}} \frac{\bar{n}^2}{n-\bar{n}} \cdot I_n e^{-\mu}$$

$$= \frac{\bar{n}^2}{\mu} Z_{\bar{n}} \cdot I_{\bar{n}} e^{-\mu} + \frac{\Delta \bar{n}^2}{\mu} \left(1 - 2 I_{\bar{n}} e^{-\mu} + \bar{n} \sum_{n \neq \bar{n}} \frac{1}{n-\bar{n}} \cdot I_n e^{-\mu}\right)$$

$$\gamma_{12} = i\bar{n}\left(Z_{\bar{n}} - \Delta\right)\left(I_{\bar{n}} e^{-\mu}\right)' + i\Delta\bar{n}^2 \sum_{n \neq \bar{n}} \frac{1}{n-\bar{n}} \left(I_n e^{-\mu}\right)'$$

$$\gamma_{22} = \gamma_{11} - 2\mu\left[Z_{\bar{n}}\left(I_{\bar{n}} e^{-\mu}\right)' + \bar{n}\Delta \sum_{n \neq \bar{n}} \frac{1}{n-\bar{n}} \left(I_n e^{-\mu}\right)'\right]$$

$$\gamma_{13} = \bar{n}\sqrt{\frac{2}{\mu}}\, I_{\bar{n}} e^{-\mu} - \frac{(\bar{n}\Delta)^2}{\sqrt{2\mu}} \sum_{n \neq \bar{n}} \left(\frac{1}{n-\bar{n}} + \frac{\bar{n}}{(n-\bar{n})^2}\right) I_n e^{-\mu} \tag{7}$$

$$\gamma_{23} = -i\sqrt{2\mu}\left(I_{\bar{n}} e^{-\mu}\right)' + i\sqrt{\frac{\mu}{2}} \cdot (\bar{n}\Delta)^2 \sum_{n \neq \bar{n}} \frac{1}{(n-\bar{n})^2} \left(I_n e^{-\mu}\right)'$$

$$\gamma_{33} = 2\xi_{\bar{n}}^2 Z_{\bar{n}} \cdot I_{\bar{n}} e^{-\mu} + \bar{n}\Delta \cdot \sum_{n \neq \bar{n}} \frac{I_n e^{-\mu}}{n-\bar{n}}$$

and

$$A = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \left(1 + \bar{n} \sum_{n \neq \bar{n}} \frac{1}{n-\bar{n}} \cdot I_n e^{-\mu} + \frac{Z_{\bar{n}}}{\Delta} I_{\bar{n}} e^{-\mu}\right) \Big/ \left(\frac{\mu}{\bar{n}^2} + \frac{\Delta^2}{2}\right) \quad . \tag{8}$$

These expressions are still valid if there are collisions provided that $\gamma/\omega \ll \Delta$. If, however, the collision frequency is such that $1 \gg \gamma/\omega \gg \Delta$, then one has to substitute in the preceding formulae $Z_{\bar{n}}/\Delta$ with $i\omega/\gamma$.

3. Waves outside the harmonics of ω_e.

In this section we discuss the dispersion relation for values of ω/ω_e lying outside the intervals $\left(\bar{n} - \frac{\bar{n}\Delta\omega_e}{2\omega_e},\ \bar{n} + \frac{\bar{n}\Delta\omega_e}{2\omega_e}\right)$ if

ω is fixed or the intervals $\left(\bar{n}-\frac{\bar{n}\Delta\omega}{2\omega}\ ,\ \bar{n}+\frac{\bar{n}\Delta\omega}{2\omega}\right)$ if B_0 is fixed.

We suppose $|N_{\parallel}| \ll 1$ and look for propagation vectors nearly perpendicular to $\vec{B}_0$.

This last condition allows us to separate eq. (2) into the dispersion relation for ordinary waves $(\vec{E}_1 \parallel \vec{B}_0)$

$$N^2 = \varepsilon_{33} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \nu \sum_{n=-\infty}^{+\infty} \frac{1}{n-\nu} I_n e^{-\mu} \tag{9}$$

and the dispersion relation for extraordinary waves $(\vec{E}_1 \perp \vec{B}_0)$

$$N^2 = \varepsilon_{22} + \frac{\varepsilon_{12}^2}{\varepsilon_{11}} \tag{10}$$

where

$$\varepsilon_{11} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\nu^2}{\mu}\left(1 + \nu \sum_{n=-\infty}^{+\infty} \frac{1}{n-\nu} \cdot I_n e^{-\mu}\right)$$

$$\varepsilon_{12} = i\left(\frac{\omega_p}{\omega}\right)^2 \nu^2 \sum_{n=-\infty}^{+\infty} \frac{1}{n-\nu} \left(I_n e^{-\mu}\right)'$$

$$\varepsilon_{22} = \varepsilon_{11} - 2\left(\frac{\omega_p}{\omega}\right)^2 \nu\mu \sum_{n=-\infty}^{+\infty} \frac{1}{n-\nu} \left(I_n e^{-\mu}\right)'$$

We solve these equations in the regions fo the plane $\left(\frac{\omega}{\omega_e}, |\mu|\right)$ in which the ε_{ik}'s can be approximated by means of formulae found in App. 1.

Consider first ordinary waves. Using the power expansion for the series, a solution can be found which lies in the very neighbourhood of the"macroscopic" one,

$$N = \pm\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}\ .$$

The pure imaginary solution with $O(|\mu|)=O(\nu)=O(1)$, numerical ly found by Dnestrovskii and Kostomarov[10] can not be obtained with the expansions of App. 1.

All other solutions are complex, with $\mathrm{Re}(\mu)<0$, i.e. $|\mathrm{Re}(N)| < |\mathrm{Im}(N)|$. In fact, when $\mathrm{Re}(\mu)>0$,

$$\left| \nu \sum_{n=-\infty}^{+\infty} \frac{I_n e^{-\mu}}{n-\nu} \right| \leq \frac{\nu\pi}{|\sin(\nu\pi)|} \quad ,$$

so that there are no solutions, besides the macroscopic one, such that $|\mu| \gtrsim \nu$.

For $|\mu| < \nu$ and $\nu \gg 1$, eq. (9) becomes (see App. 1)

$$N^2 \approx -\left(\frac{\omega_p}{\omega}\right)^2 \frac{\nu\pi \cdot \cos(\nu\pi)}{\sin(\nu\pi)} I_\nu(\mu)\, e^{-\mu}$$

Since (see ref. 13 § 3.31)

$$\left| I_\nu(\mu)\, e^{-\mu} \right| \leq \frac{\left|\left(\frac{\mu}{2}\right)^\nu\right| \left|e^{-\mathrm{Re}(\mu)}\right| e^{\mathrm{Re}(\mu)}}{\sqrt{\frac{2\pi}{n+1}} \cdot \left(\frac{\nu+1}{e}\right)^{\nu+1}} = \left|\left(\frac{\mu e}{2(\nu+1)}\right)^\nu\right| \frac{e}{\sqrt{2\pi(\nu+1)}} \quad ,$$

there are no solutions in this range, besides the macroscopic one.

When $\mathrm{Re}(\mu) < 0$ and $|\mu| \gg \frac{\nu^2}{2}$ (the value of ν is arbitrary) eq. (9) can be written

$$(-2\mu)^{3/2} = \frac{(-1)^\nu\, 2\nu\sqrt{\pi}}{\sin(\nu\pi)} \left(\frac{\omega_p}{\omega}\, \nu \frac{v_{th}}{c}\right)^2 e^{-2\mu} \tag{11}$$

This equation is discussed in App. 2. It has an infinite number of solutions such that $|\mathrm{Re}(\mu)|$ and $|\mathrm{Im}(\mu)|$ go to infinity, with $|\mathrm{Re}(\mu)/\mathrm{Im}(\mu)| \to 0$.

When $\mathrm{Re}(\mu) < 0$ and $|\mu| \gg \nu \gg 1$, eq. (9) becomes

$$(-2\mu)^{3/2} = \frac{(-1)^\nu\, 2\nu\sqrt{\pi}}{\sin(\nu\pi)} \cdot \left(\frac{\omega_p}{\omega}\, \nu \frac{v_{th}}{c}\right)^2 \left[\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2} + \frac{\nu}{\mu}\right) e^{-\frac{\mu}{\nu}\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}+1\right)}\right]^\nu \tag{12}$$

Also this equation is discussed in App. 2. The solutions are similar to that of eq. (11).

Finally, it could be shown that there are no solutions such that $\mathrm{Re}(\mu) < 0$ and $1 \lesssim |\mu| \ll \nu$, $\nu \gg 1$.

For the extraordinary wave, the "macroscopic" index of refraction

$$N = \pm \left[1 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\nu^2}{1-\nu^2} - \frac{\left(\frac{\omega_p}{\omega}\right)^4 \frac{\nu^2}{(1-\nu^2)^2}}{1 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\nu^2}{1-\nu^2}} \right]^{1/2} = \pm \left[1 + \frac{\nu^2 \left(\frac{\omega_p}{\omega}\right)^2 \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)}{1-\nu^2+\nu^2\left(\frac{\omega_p}{\omega}\right)^2} \right]^{1/2} ,$$

holds approximately, as long as $N^2 \ll \left(\frac{c}{v_{th}}\right)^2 \frac{1}{\nu^2}$. Therefore there is no true resonance when ω approaches $\sqrt{\omega_p^2 + \omega_e^2}$ [8].

The pure imaginary index of refraction numerically found by Dnestrovskii and Kostomarov[11] in the range $O(|\mu|) = O(\nu)$ can not be recovored with the expansions of App. 1.

In the ranges in which the approximations of App. 1 are valid, there are no other solutions with $\mathrm{Re}(\mu) > 0$. In fact, for $\nu \approx \bar{n}+\frac{1}{2} \gg 1$ and for $|\mu| \gg \frac{\nu^2}{2}$ it is: $\varepsilon_{11} \approx 1$ and $|\varepsilon_{12}| \ll 1$.

The other solutions have $\mathrm{Re}(\mu) < 0$ and are complex. Rather tedious calculations show that they are

a) longitudinal waves with $|\mu| \gg \frac{\nu^2}{2}$ (ν arbitrary) given by

$$(-2\mu)^{3/2} = -2\left(\frac{\omega_p}{\omega}\right)^2 \frac{\nu^3 \pi}{\sqrt{\pi}\sin(\nu\pi)} e^{-2\mu} \; ; \qquad (13)$$

b) transverse waves with $1 \ll |\mu| \ll \frac{\nu^2}{2}$ and $\nu \gg 1$, given by

$$-\left(\frac{v_{th}}{c}\right)^2 = \left(\frac{\omega_p}{\omega}\right)^2 \frac{1}{\mu^2} \cdot \frac{\nu\pi}{\sin(\nu\pi)} \cdot \frac{\left(1 - \frac{1}{8\mu} - \frac{\nu^2}{4\mu^2}\right)}{\sqrt{-2\pi\mu}} (-1)^{\nu} \left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2} + \frac{\nu}{\mu}\right) e^{-\mu\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}+1\right)} \qquad (14)$$

Equations (13) and (14) are discussed in App. 2. The solutions have a structure quite similar to that already considered in this section.

4. Waves in the neighbourhood of $\bar{n}\omega_e$, neglecting Landau damping

When the ratio ω/ω_e tends to $\bar{n}$ ($\bar{n} = 1,2,..$) and $K_{\parallel}$ tends to zero at the same time, so that inequalities

$$|\xi_n| \gg 1 \quad \text{and} \quad \frac{1}{\pi\Delta} \gg 1$$

remain valid, Landau damping is negligible. Then the dispersion relation separates into equations (9) and (10) and can be discussed briefly. This case is not too important for the radiation problem discussed in sect. 1; it gives, however, information about the structure of the index of refraction.

Consider first ordinary waves. Their equation can be written

$$N^2 = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \left(\bar{n} \sum_{n \neq \bar{n}} \frac{I_n e^{-\mu}}{n - \bar{n}} - I_{\bar{n}} e^{-\mu} + \frac{\bar{n}\, I_{\bar{n}}\, e^{-\mu}}{\bar{n} - \nu} \right) \tag{15}$$

Being clear that no solution can reach a finite value, different from zero, in the limit $\omega/\omega_e = \bar{n}$, we look for μ's that approach zero and infinity

In the first case the series can be approximated by the zero order term of the expansion in powers of μ . Then eq.(15) becomes

$$0 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 (1 + \delta_{1\bar{n}}) + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}}{\bar{n} - \nu} \left(\frac{\mu}{2}\right)^{\bar{n}} \frac{1}{\bar{n}!}$$

and shows that N vanishes as $\left[(\omega_p^2 - \omega^2)(\bar{n} - \nu)\right]^{\frac{1}{2\bar{n}}}$

When $|\mu|$ goes to infinity, we distinguish the two possibilities $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(\mu) < 0$. When $\mathrm{Re}(\mu) > 0$, we get

$$\mu\sqrt{\mu} = \left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{v_{th}}{c}\right)^2 \frac{\bar{n}^3}{\bar{n} - \nu} \cdot \frac{1}{\sqrt{2\pi}} .$$

The right hand side being real, μ has to be real and positive. Hence the waves are possible only at the left side of the harmonics and N goes to infinity as $(\bar{n}-\nu)^{-1/3}$ When $\mathrm{Re}(\mu) < 0$, the dispersion relation becomes (see App 1)

$$-(-2\mu)^{3/2} = \left[\frac{2}{\sqrt{\pi}}\left(\frac{\omega_p}{\omega}\right)^2\left(\frac{v_{th}}{c}\right)^2\frac{(-1)^{\bar{n}}\bar{n}^3}{\bar{n}-\nu}\right]e^{-2\mu} \quad . \tag{16}$$

This equation is discussed in App 2. It has complex solutions with $|\mathrm{Im}(\mu)| > |\mathrm{Re}(\mu)|$; the order of infinity of $\mathrm{Im}(\mu)$ is larger than $(\bar{n}-\nu)^{-1/3}$ The waves exist on both sides of the harmonics.

Remembering the results of the preceding section we may describe the behaviour of N_{ord} as follows.
Outside the neighbourhood of the harmonics there is a real solution only when $\omega > \omega_p$, the zero temperature index of refraction. This index is zero at $\omega = \omega_p$, tends to infinity when the frequency approaches the harmonics from below and to zero when it approaches the harmonics from above.

Since N goes to zero as

$$\left[(\omega_p^2-\omega^2)(\bar{n}-\nu)\right]^{1/2\bar{n}} \tag{17}$$

when $\omega > \omega_p$ there is a pure imaginary N that vanishes below $\bar{n}\omega_e$ if $\bar{n} = 1,3,5,..$ and above $\bar{n}\omega_e$ if $\bar{n} = 2,4,...$ Its value for frequencies between the harmonics was computed in ref. lo.

When ω is smaller than ω_p , the macroscopic index is imaginary When the frequency approaches the harmonics $\bar{n}\omega_e$, it tends to zero from above if $\bar{n} = 1,3,.. < \omega_p/\omega$ and from below if $\bar{n} = 2,4,.. < \omega_p/\omega$. Moreover, in the neighbourhood of the harmonics there are two real indices, one going to zero and the other to infinity, both from below.

In addition to the pure real or pure imaginary vanishing indices, on both sides of the harmonics there are complex indices whose number grows with $\bar{n}$, according to (17). The complex waves found outside the hormonics tend to infinity with $\mathrm{Re}(\mu) < 0$, when the frequency approaches the harmonics from both sides.

The study of the extraordinary waves is analogous but a little more tedious . The equation is

$$\varepsilon_{11}\left(N^2 - \varepsilon_{22}\right) = \varepsilon_{12}^2 \qquad (18)$$

where

$$\varepsilon_{11} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{\nu^2}{\mu} + \frac{\nu^3}{\mu} \sum_{n \neq \bar{n}}' \frac{I_n e^{-\mu}}{n - \bar{n}} + \frac{\nu^3}{\mu} \frac{I_{\bar{n}} e^{-\mu}}{\bar{n} - \nu} \right) \quad ;$$

$$\varepsilon_{12} = i \left(\frac{\omega_p}{\omega}\right)^2 \left[\bar{n}^2 \sum_{n \neq \bar{n}}' \frac{\left(I_n e^{-\mu}\right)'}{n - \bar{n}} + 2\bar{n}\left(I_{\bar{n}} e^{-\mu}\right)' + \frac{\bar{n}^2 \left(I_{\bar{n}} e^{-\mu}\right)'}{\bar{n} - \nu} \right] \quad ;$$

$$\varepsilon_{22} = \varepsilon_{11} - \left(\frac{\omega_p}{\omega}\right)^2 2\mu\nu \left[\sum_{n \neq \bar{n}}' \frac{\left(I_n e^{-\mu}\right)'}{n - \bar{n}} + \frac{\left(I_{\bar{n}} e^{-\mu}\right)'}{\bar{n} - \nu} \right] .$$

In order to see that μ can not tend to a finite non-vanishing value when ν goes to $\bar{n}$ let us consider the term containing the factor $(\bar{n}-\nu)^{-2}$ in eq (18) The condition that this term vanishes leads to

$$I_{\bar{n}}'(\mu) = \pm \sqrt{1 + \frac{\bar{n}^2}{\mu^2}} \cdot I_{\bar{n}}(\mu)$$

which is not compatible with Bessel equation. There are , however, waves with μ going to 0. Consider first the fundamental frequency; then

$$\varepsilon_{11} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \nu^2 \left[1 + (1-\mu)\frac{\nu^2}{1-\nu^2} \right] \quad ;$$

$$\varepsilon_{12} = i\left(\frac{\omega_p}{\omega}\right)^2 \nu^2 (1-\mu)\left[\frac{2-\mu}{2\nu} + (1-\mu)\frac{\nu}{1-\nu^2}\right] \quad ;$$

$$\varepsilon_{22} = \varepsilon_{11}$$

and eq. (18) becomes

$$\varepsilon_{11} \pm i\,\varepsilon_{12} = 0 \tag{19}$$

which gives

$$\mu = (1-\nu)\left(1 - 2\left(\frac{\omega}{\omega_p}\right)^2\right) \quad . \tag{20}$$

At the harmonics we have

$$\varepsilon_{11} = \varepsilon_{22} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar n}{\mu}(\bar n - \nu) + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar n^2}{1-\bar n^2} + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar n^3}{\mu}\left(\frac{\mu}{2}\right)^{\bar n} \frac{1}{\bar n!(\bar n-\nu)} \quad ;$$

$$\varepsilon_{12} = i\left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar n}{1-\bar n^2} + i\left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar n^2}{\bar n-\nu}\left(\frac{\mu}{2}\right)^{\bar n} \frac{1}{\bar n!}\left(\frac{\bar n}{\mu} - 1\right)$$

The dispersion relation (19) gives then the solutions

$$\text{a)} \quad \mu = 2\left[\frac{\bar n!(\bar n-\nu)}{\bar n^3}\left(\frac{\bar n}{\bar n-1} - \left(\frac{\omega}{\omega_p}\right)^2\right)\right]^{\frac{1}{\bar n-1}} \quad ;$$

$$\text{b)} \quad \mu = 2\left[\frac{\bar n!(\bar n-\nu)}{\bar n^2}\left(\frac{\bar n}{\bar n+1} - \left(\frac{\omega}{\omega_p}\right)^2\right)\right]^{\frac{1}{\bar n}} \quad ; \tag{21}$$

$$\text{c)} \quad \mu = \bar n(\bar n-\nu)\Big/\left(\frac{\bar n}{\bar n\pm 1} - \left(\frac{\omega}{\omega_p}\right)^2\right)$$

The waves (2o) and (21) are neither longitudinal nor transerse. When $|\mu|$ goes to infinity and $\mathrm{Re}(\mu) > 0$, eq. (18) becomes

$$\mu\sqrt{\mu}\left(\frac{c}{\bar n v_{th}}\right)^2 - \sqrt{\mu} = -\left(\frac{\omega_p}{\omega}\right)^2\left(\frac{c}{v_{th}}\right)^2 \frac{\bar n}{\bar n-\nu}\cdot\frac{1}{\sqrt{2\pi}} \quad .$$

The solutions are:

a) a longitudinal wave existing only above the harmonics, which corresponds to the positive real value of μ given by

$$\mu^{3/2} = -\left(\frac{\omega_p}{\omega}\right)^2 \frac{n^3}{n-\nu} \cdot \frac{1}{\sqrt{2\pi}} \quad ;$$

b) a transverse wave existing only below the harmonics which has a positive real value of μ given by

$$\mu^{3/2} = \left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{v_{th}}{c}\right)^2 \frac{n^3}{n-\nu} \cdot \frac{1}{\sqrt{2\pi}} \quad ;$$

When $|\mu|$ goes to infinity and $Re(\mu) < 0$ the equation is complicate and has only solutions such that $|Re(\mu)|$ and $|Im(\mu)|$ tend to infinity at both sides of the harmonics.

These results taken toghether with those derived in the last section show how complicated the behaviour of the index of refraction and the polarisation are. A detailed description seems to be of academic interest because even few collisions or a small Landau damping alter the picture in an essential way. Therefore we give only a summary of the most typical properties.

A very large number of real and complex waves go to zero at the harmonics from both sides. A quasi-longitudinal wave tends to infinity from above at the harmonics and a quasi-transverse one from below. The last wave, however, exists only in the very neighbourhood of the harmonics and can easily be destroyd by collisions or Landau damping (see the next section). Unlike the ordinary waves, the extraordinary zero temperature solution is not valid in all intervals outside the harmonics.

In the neighbourhood of the frequency $\sqrt{\omega_p^2 + \omega^2}$, in fact, no real index exists and the complex ones do not exhibit a resonance.

5. Waves at the harmonics of ω_e, including Landau damping

When $N_\parallel$ does not vanish and $|\omega - \bar{n}\omega_e| < \omega\Delta$, the waves are no longer polarized exactely parallel or perpendicular to $\vec{B}_0$, so that the dispersion relation is much more involved. The effect of Landau damping is not only that of making $N_\perp$ finite at the resonances but also that of eliminating a number of waves, as will be clear in the following.

Since the components ε_{ik}'s are approximated by means of the formulae of App. 1, the dispersion relation will be solved not for arbitrary values of $\bar{n}$ and $|\mu|$, but in the following ranges: 1) $|\mu| \ll 1$ and $\bar{n}$ arbitrary; 2) $|\mu| \ll \bar{n}$ and $\bar{n} \gg 1$; 3) $\bar{n}^2 \gg |\mu| \gg \bar{n} \gg 1$ and finally 4) $|\mu| \gg \bar{n}^2/2$ where $\bar{n}$ is arbitrary.

1) $|\mu| \ll 1$ and $\bar{n}$ arbitrary.

It is useful to consider the fundamental frequency, ω_e, separately. The ε_{ik}'s are in this case

$$\varepsilon_{11} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{i\sqrt{\pi}}{2\Delta} - \frac{1}{4}\right) \qquad ;$$

$$\varepsilon_{12} = i\left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{i\sqrt{\pi}}{2\Delta} + \frac{1}{4}\right) \qquad ;$$

$$\varepsilon_{22} = \varepsilon_{11} - 2\left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{i\sqrt{\pi}\cdot\mu}{2\Delta} + 1\right) \qquad ;$$

$$\varepsilon_{13} = \left(\frac{\omega_p}{\omega}\right)^2 \sqrt{\frac{2}{\mu}} \left(\frac{\mu}{2\Delta} + \frac{\mu\Delta}{8}\right) \qquad ;$$

$$\varepsilon_{23} = -i\left(\frac{\omega_p}{\omega}\right)^2 \sqrt{\frac{\mu}{2}}\,\frac{1}{\Delta} \qquad ;$$

$$\varepsilon_{33} = 1 - \left(\frac{\omega_p}{\omega}\right)^2\left(1 - \xi_1^2\,\frac{i\sqrt{\pi}}{\Delta}\,\mu\right) \quad .$$

It may be proved that the dispersion relation does not have solutions such that $|\mu| \gtrsim 2\Delta\left(\frac{\omega}{\omega_p}\right)^2$, and that for $|\mu| \ll 2\Delta\left(\frac{\omega}{\omega_p}\right)^2$ it becomes approximately

$$\left(\frac{\omega_p}{\omega}\right)^2 \frac{Z_1}{2\Delta}\left[N_\perp^4 - N_\perp^2\left(3 - 2\left(\frac{\omega_p}{\omega}\right)^2\right) + \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)\left(2 - \left(\frac{\omega_p}{\omega}\right)^2\right)\right] = 0 \quad .$$

This equation has the solutions $N_\perp^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2$ and $N_\perp^2 = 2 - \left(\frac{\omega_p}{\omega}\right)^2$ which are the macroscopic ordinary and extraordinary waves, respectively. The Landau damping does not enter in the result.

At the harmonics $\omega = \bar{n}\omega_e$, the ε_{ik}'s become

$$\varepsilon_{11} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^2}{1-\bar{n}^2} + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^2}{\mu}\,\frac{Z_{\bar{n}}}{\Delta}\, I_{\bar{n}}(\mu) \qquad ;$$

$$\varepsilon_{12} = i\left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^2}{\bar{n}(1-\bar{n}^2)} + i\left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^2}{\mu}\left(\frac{Z_{\bar{n}}}{\Delta} - 1\right) I_{\bar{n}}(\mu) \;;$$

$$\varepsilon_{22} = \varepsilon_{11} + 2i\frac{\mu}{\bar{n}}\,\varepsilon_{12} - 2\mu\left(\frac{\omega_p}{\omega}\right)^2\left(\frac{\bar{n}}{\mu} - 1\right) I_{\bar{n}}(\mu) \;;$$

$$\varepsilon_{13} = \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}}{\Delta}\sqrt{\frac{2}{\mu}}\, I_{\bar{n}}(\mu) - \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}}{\Delta}\sqrt{\frac{2}{\mu}}\,\mu\,\frac{(\Delta\bar{n})^2}{(1-\bar{n}^2)^2} \quad ;$$

$$\varepsilon_{23} = -i\left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}}{\Delta}\sqrt{\frac{2}{\mu}}\left(I_{\bar{n}}(\mu) - \frac{\mu}{2\bar{n}}\,\Delta^2\,\frac{3\bar{n}^2-1}{(1-\bar{n}^2)^2}\right) \quad ;$$

$$\varepsilon_{33} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 + 2\,\xi_{\bar{n}}^2\,\frac{\mu}{\bar{n}^2}\left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^2}{\mu}\,\frac{Z_{\bar{n}}}{\Delta}\, I_{\bar{n}}(\mu) \quad ;$$

where $I_{\bar{n}}(\mu) = \left(\frac{\mu}{2}\right)^{\bar{n}} \frac{1}{\bar{n}!}$

The dispersion relation gives the macroscopic ordinary and extraordinary waves, whose index of refraction is that found in sect. 3, with $\nu = \bar{n}$. Moreover it has other solutions, corresponding to quasi-longitudinal waves given by

$$\frac{\mu}{2} = \left[\frac{2i\,\Delta\cdot\bar{n}!}{\sqrt{\pi}} \left(\left(\frac{\omega_e}{\omega_p}\right)^2 - \frac{1}{\bar{n}^2-1} \right) \right]^{\frac{1}{\bar{n}-1}} \qquad (22)$$

The index of refraction grows in magnitude with $\bar{n}$. Eq. (22) holds as long as $|\mu|$ is much smaller than 1, i.e. only at the first few harmonics.

2) $|\mu| \ll \bar{n}$ and $\bar{n} \gg 1$.

Suppose first that $\mathrm{Re}(\mu) > 0$. Using the approximations of App. 1 we can write

$$\varepsilon_{11} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^3}{\mu} \left(\frac{Z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right) \right) I_{\bar{n}}\, e^{-\mu} \quad ;$$

$$\varepsilon_{12} = -i\left(\frac{\omega_p}{\omega}\right)^2 \frac{1}{\bar{n}} \left(1 + \frac{6\mu}{\bar{n}^2}\right) + i\left(\frac{\omega_p}{\omega}\right)^2 \bar{n}^2 \left(\frac{\bar{n}}{\mu} - 1\right) \left(\frac{Z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right) \right) I_{\bar{n}}\, e^{-\mu} ;$$

$$\varepsilon_{22} = \varepsilon_{11} + \frac{2\mu}{\bar{n}}\, i\, \varepsilon_{12} - 2\mu \left(\frac{\omega_p}{\omega}\right)^2 \left(\frac{\bar{n}}{\mu} - 1\right) I_{\bar{n}}\, e^{-\mu} \quad ; \qquad (23)$$

$$\varepsilon_{13} = \left(\frac{\omega_p}{\omega}\right)^2 \sqrt{\frac{2}{\mu}}\, \frac{\bar{n}}{\Delta} \cdot I_{\bar{n}}\, e^{-\mu} - \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}\Delta}{\sqrt{2\mu}} \left(1 + \bar{n}^2 \left(\frac{\pi^2}{3} - \frac{1}{2} \ln^2\left(\frac{\mu}{2\bar{n}}\right) \right) I_{\bar{n}}\, e^{-\mu} \right) \quad ;$$

$$\varepsilon_{23} = -i\left(\frac{\omega_p}{\omega}\right)^2 \frac{\sqrt{2\mu}}{\Delta} \left(\frac{\bar{n}}{\mu} - 1\right) I_{\bar{n}}\, e^{-\mu} + i\left(\frac{\omega_p}{\omega}\right)^2 \Delta \sqrt{\frac{\mu}{2}} \left(\frac{3}{\bar{n}^2} + \bar{n}^2 \left(\frac{\bar{n}}{\mu} - 1\right) \left(\frac{\pi^2}{3} - \frac{1}{2} \ln^2\left(\frac{\mu}{2\bar{n}}\right) \right) I_{\bar{n}}\, e^{-\mu} \right) ;$$

$$\varepsilon_{33} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 - \left(\frac{\omega_p}{\omega}\right)^2 \bar{n} \left(\ln\left(\frac{\mu}{2\bar{n}}\right) - \frac{2 Z_{\bar{n}}}{(\bar{n}\Delta)}\, \xi_{\bar{n}}^2 \right) I_{\bar{n}}\, e^{-\mu} ;$$

where $I_{\bar{n}}(\mu) e^{-\mu} = \frac{1}{\sqrt{2\pi\bar{n}}} \left[\frac{\mu}{\bar{n}} \exp\left(-\frac{\mu}{\bar{n}} + \sqrt{1 + \left(\frac{\mu}{\bar{n}}\right)^2}\right) \Big/ \left(1 + \sqrt{1 + \left(\frac{\mu}{\bar{n}}\right)^2}\right) \right]^{\bar{n}}$.

A rather tedious analysis shows that the dispersion relation has the following solutions. First, longitudinal waves, whose equation

$$1-\left(\frac{\omega}{\omega_p}\right)^2 = \frac{\bar{n}^3}{\mu}\left(\frac{z_{\bar{n}}}{\Delta} - \ln'\left(\frac{\mu}{2\bar{n}}\right)\right) I_{\bar{n}}(\mu) e^{-\mu} ,$$

can be solved approximately by extracting the $\bar{n}$-th root of both sides. Because the logarithm is always small compared to $z_{\bar{n}}/\Delta$,

$$\left(\frac{\mu}{\bar{n}}\right)^{1-\frac{1}{\bar{n}}} = \frac{2}{e}\cdot\left[\left(1-\left(\frac{\omega}{\omega_p}\right)^2\right)\frac{\Delta\sqrt{2\pi\bar{n}}}{\bar{n}^2 z_{\bar{n}}}\right]^{1/\bar{n}} . \qquad (24)$$

This equation correspond to eq. (22) when $\bar{n}$ is large. There are also ordinary waves given by

$$N^2 = \left(\frac{\omega_p}{\omega}\right)^2 \bar{n}\left(2\frac{\xi_{\bar{n}}^2 z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right)\right) I_{\bar{n}}(\mu) e^{-\mu} , \qquad (25)$$

and extraordinary waves for which

$$N^2 = \varepsilon_{22} + \frac{\varepsilon_{12}^2}{\varepsilon_{11}} = \bar{n}\mu\left(\frac{z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right)\right) I_{\bar{n}}(\mu) e^{-\mu} . \qquad (26)$$

Taking the $\bar{n}$-th root of both sides, eq. (25) gives

$$\left(\frac{\mu}{\bar{n}}\right)^{1-\frac{1}{\bar{n}}} = \frac{2}{e}\left[\left(\frac{\omega}{\omega_p}\right)^2\left(\frac{c}{v_{th}}\right)^2 \frac{\sqrt{2\pi\bar{n}}}{\bar{n}^2}\cdot\frac{1}{\left(\frac{2\xi_{\bar{n}}^2 z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right)\right)}\right]^{\frac{1}{\bar{n}}} \qquad (27)$$

and eq. (26)

$$\frac{\mu}{\bar{n}} = \frac{2}{e}\left[\left(\frac{\omega}{\omega_p}\right)^2\left(\frac{c}{v_{th}}\right)^2 \frac{\Delta\sqrt{2\pi\bar{n}}}{\bar{n}^2 z_{\bar{n}}}\right]^{1/\bar{n}} \qquad (28)$$

These solutions satisfy the dispersion relation when the expressions in the brackets are much smaller than 1, because only then is $|\mu| \ll \bar{n}$. This happens when $\Delta\sqrt{2\bar{n}} < \left(\frac{\omega_p v_{th}\bar{n}}{\omega c}\right)^2 2\xi_{\bar{n}}^2$ in (27) and $\Delta\sqrt{2\bar{n}} < \left(\frac{\omega_p}{\omega}\frac{v_{th}\bar{n}}{c}\right)^2$ in (28) - that is, outside the range we are interested in.

The expressions for the ε_{ik}'s when $\mathrm{Re}(\mu) < 0$ can be obtained from the corresponding expressions (23) valid for $\mathrm{Re}(\mu) > 0$, with the following substitutions

$$I_{\bar{n}}(\mu) \to I_{\bar{n}}(-\mu)$$

$$\ln\left(\frac{\mu}{2\bar{n}}\right) \to \ln\left(-\frac{\mu}{2\bar{n}}\right)$$

and the expression $-\pi^2/3$ in ε_{13} and ε_{23} replaced by $\pi^2/6$. The resulting solutions are formally identical with (24), (27) and (28), where, of course, only the roots such that $\mathrm{Re}(\mu) < 0$ are to be considered.

3) $\bar{n}^2 > |\mu| \gg \bar{n} \gg 1.$

When $\mathrm{Re}(\mu) > 0$, one has

$$\varepsilon_{11} = 1 - \left(\frac{\omega_p}{\omega}\right)^2 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^3}{\mu}\left(\frac{Z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right)\right) I_{\bar{n}}\, e^{-\mu} \quad ;$$

$$\varepsilon_{12} = -\frac{i}{\bar{n}}\left(\frac{\omega_p}{\omega}\right)^2\left(1 + \frac{6\mu}{\bar{n}^2}\right) + i\left(\frac{\omega_p}{\omega}\right)^2 \bar{n}^2\left(\frac{Z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right)\right)\left(I_{\bar{n}}\, e^{-\mu}\right)' \; ;$$

$$\varepsilon_{22} = \varepsilon_{11} + 2i\frac{\mu}{\bar{n}}\varepsilon_{12}$$

$$\varepsilon_{13} = -\left(\frac{\omega_p}{\omega}\right)^2 \frac{\sqrt{2\mu}\cdot\Delta}{\bar{n}} + \left(\frac{\omega_p}{\omega}\right)^2 \sqrt{\frac{2}{\mu}}\,\frac{\bar{n}}{\Delta}\left(1 + \frac{\bar{n}\Delta^2}{2}\ln\left(\frac{\mu}{2\bar{n}}\right) - (n\Delta)^2\frac{\pi^2}{6}\right) I_{\bar{n}}\, e^{-\mu} ;$$

$$\varepsilon_{23} = -i\left(\frac{\omega_p}{\omega}\right)^2 \frac{\sqrt{2\mu}}{\Delta}\left(I_{\bar{n}}\, e^{-\mu}\right)' + i\left(\frac{\omega_p}{\omega}\right)^2 \Delta\sqrt{\frac{\mu}{2}}\left(\frac{3}{\bar{n}^2} + \frac{30\mu}{\bar{n}^4} + \bar{n}^2\left(\frac{\pi^2}{3} - \frac{\bar{n}^2}{2\mu^2}\right)\left(I_{\bar{n}}\, e^{-\mu}\right)\right) \; ;$$

$$\varepsilon_{33} = 1 - \left(\frac{\omega_p}{\omega}\right)^2\left(1 + \frac{\mu}{\bar{n}^2} - \bar{n}\left(2\frac{\xi_{\bar{n}}^2 Z_{\bar{n}}}{\bar{n}\Delta} - \ln\left(\frac{\mu}{2\bar{n}}\right)\right) I_{\bar{n}}\, e^{-\mu}\right) \quad ,$$

where (see App. 1)

$$I_{\bar{n}}(\mu)e^{-\mu} \simeq \frac{\left(1 + \frac{1}{8\mu} - \frac{\bar{n}^2}{4\mu^2}\right)}{\sqrt{2\pi\mu}}\left[\left(\sqrt{1+\left(\frac{\bar{n}}{\mu}\right)^2} - \frac{\bar{n}}{\mu}\right) e^{\frac{\mu}{\bar{n}}\left(\sqrt{1+\left(\frac{\bar{n}}{\mu}\right)^2} - 1\right)}\right]^{\bar{n}} ,$$

and

$$\left(I_{\bar{n}}(\mu)e^{-\mu}\right)' \simeq \frac{-\left(1 - \frac{\bar{n}^2}{\mu}\right)}{2\mu\sqrt{2\pi\mu}}\left[\left(\sqrt{1+\left(\frac{\bar{n}}{\mu}\right)^2} - \frac{\bar{n}}{\mu}\right) e^{\frac{\mu}{\bar{n}}\left(\sqrt{1+\left(\frac{\bar{n}}{\mu}\right)^2} - 1\right)}\right]^{\bar{n}} .$$

The order of magnitude of the different terms of the dispersion relation are such that only the quasi-longitudinal waves are possible. Their equation

$$1-\left(\frac{\omega_p}{\omega}\right)^2=\left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}^2\,\Sigma_{\bar{n}}}{\mu\sqrt{2\pi\mu}\,\Delta}\left(\sqrt{1+\left(\frac{\bar{n}}{\mu}\right)^2}-\frac{\bar{n}}{\mu}\right)^{\bar{n}}e^{\mu\left(\sqrt{1+\left(\frac{\bar{n}}{\mu}\right)^2}-1\right)} \tag{27}$$

is studied in App. 2. The solutions are

$$\mu=\ln\left(\left|\left(1-\left(\frac{\omega}{\omega_p}\right)^2\right)\frac{|K_{11}|}{K_{11}}\right|\frac{\Delta\bar{n}}{2}\left(\frac{\bar{n}^2}{(3/4\pm1/2+r)\pi}\right)^{3/2}\right)+i\,\frac{\bar{n}^2}{(3/4\pm1/2+r)\pi} \qquad (r=0,\ 1,\ 2,..)$$

$(1\ll\bar{n}\ll|\mu|\ll\bar{n}^2)$.

These indices of refraction, which were already mentioned in ref. 3, seem to play an important role in the observed emission of high harmonics.

When $Re(\mu)<0$, and $\bar{n}^2\gg|\mu|\gg\bar{n}$, the moduli of $I_{\bar{n}}e^{-\mu}$ and $\sum'_{n\neq\bar{n}}\frac{I_n e^{-\mu}}{n-\bar{n}}$ are larger than $|e^{-2\mu}|$. The components ε_{ik}'s are of the order of $e^{-2\mu}$ and no solutions to the dispersion relation exists.

4) $|\mu|\gg\bar{n}^2/2$ and $\bar{n}$ arbitrary.

Here we distinguish the cases: a) $Re(\mu)\gg1$ and $\bar{n}$ arbitrary or $Re(\mu)>0$ and $\bar{n}^2\gg1$ and b) $Re(\mu)\ll-1$ and $\bar{n}$ arbitrary or $Re(\mu)<0$ and $\bar{n}^2\gg1$.

a) The components of the dielectric tensor are:

$$\varepsilon_{11}=1+\left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}^2}{\mu}\left[1-\frac{\bar{n}^2}{\mu}+\frac{\Sigma_{\bar{n}}}{\Delta\sqrt{2\pi\mu}}\left(1-\frac{\bar{n}^2-1/4}{2\mu}\right)\right]$$

$$\varepsilon_{12}=-i\left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}}{2\mu}\left(1-\frac{3(\bar{n}^2-1/4)}{2\mu}\right)\left(\frac{\Sigma_{\bar{n}}}{\Delta}-1\right)\frac{1}{\sqrt{2\pi\mu}}+i\left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}^3}{\mu^2}\left(1-\frac{3}{2\sqrt{2\pi\mu}}-\frac{2\bar{n}^2}{3\mu}\right)$$

$$\varepsilon_{22}=\varepsilon_{11}+\frac{2\mu}{\bar{n}}\,i\,\varepsilon_{12}+\left(\frac{\omega_p}{\omega}\right)^2\frac{1}{\sqrt{2\pi\mu}}\left(1-\frac{3(\bar{n}^2-1/4)}{2\mu}\right)$$

$$\varepsilon_{13}=\left(\frac{\omega_p}{\omega}\right)^2\sqrt{\frac{2}{\mu}}\,\frac{\bar{n}}{\Delta\sqrt{2\pi\mu}}\left[1-\frac{\bar{n}^2-1/4}{2\mu}-(\bar{n}\Delta)^2\left(\frac{\pi^2}{6}\left(1-\frac{\bar{n}^2}{2\mu}\right)+\frac{1}{2\mu}+\sqrt{2\pi\mu}\left(\frac{1}{\mu}-\frac{2\bar{n}^2}{3\mu}\right)\right)\right]$$

$$\varepsilon_{23} = i\left(\frac{\omega_p}{\omega}\right)^2 \frac{1}{\sqrt{2\mu}}\,\frac{1}{\Delta\sqrt{2\pi\mu}}\left[1-\frac{3(\bar{n}^2-1/4)}{2\mu}-(\bar{n}\Delta)^2\left(\frac{\pi^2}{6}\left(1-\frac{3\bar{n}^2}{2\mu}\right)-\sqrt{2\pi\mu}\left(\frac{1}{\mu}-\frac{2\bar{n}^2}{\mu^2}\right)\right)\right]$$

$$\varepsilon_{33} = 1-\left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}^2}{\mu}\left(1-\frac{1}{\sqrt{2\pi\mu}}-\frac{\bar{n}^2}{3\mu}\right)+\left(\frac{\omega_p}{\omega}\right)^2\frac{2\,\xi_{\bar{n}}^2\,Z_{\bar{n}}}{\Delta\sqrt{2\pi\mu}}\left(1-\frac{\bar{n}^2-1/4}{2\mu}\right)$$

There is a longitudinal wave with $|\sqrt{2\pi\mu}| \ll 1/\Delta$ because, when

$$O\left(\left(\frac{\omega_p}{\omega}\right)^2\frac{Z_{\bar{n}}}{\Delta}\right) = O\left(-\frac{\mu}{\bar{n}^2}\right),$$

then B is approximately $\frac{2\mu}{\bar{n}^2 Z_{\bar{n}}^2}$, C approximately $\frac{2\mu^2}{\bar{n}^4 Z_{\bar{n}}^2}$ and the dispersion relation becomes

$$\varepsilon_{11} \approx 1+\left(\frac{\omega_p}{\omega}\right)^2\cdot\frac{\bar{n}^2}{\mu}\,\frac{Z_{\bar{n}}}{\Delta\sqrt{2\pi\mu}} = 0$$

The solution is

$$N = \pm\,\tfrac{1}{2}\left(\sqrt{3}\pm i\right)\left(\frac{\omega_p}{\omega}\right)^{2/3}\cdot\left(\frac{c}{v_{th}}\right)\frac{1}{(\bar{n}\Delta\sqrt{2})^{1/3}} \tag{30}$$

which has been already given in ref. 3.

With the value of Δ we have chosen, solution (30) is the only one possible. With much smaller values of Δ one get also ordinary and extraordinary waves; the first, when $\bar{n}\Delta \ll 2\xi_{\bar{n}}^2\left(\frac{\omega_p}{\omega}\frac{v_{th}}{c}\right)^2$ is given by

$$\left(\frac{c}{v_{th}}\right)^2 = 2\,\xi_{\bar{n}}^2\,\varepsilon_{11} \approx 2\,\xi_{\bar{n}}^2\left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}^2 Z_{\bar{n}}}{\Delta\mu\sqrt{2\pi\mu}}\,.$$

The second, when $\bar{n}\Delta \ll \left(\frac{\omega_p}{\omega}\frac{v_{th}}{c}\right)^2$ and is given by

$$\left(\frac{c}{v_{th}}\right)^2 = \varepsilon_{11} = \left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}^2 Z_{\bar{n}}}{\Delta\mu\sqrt{2\pi\mu}}\,.$$

b) The components of the dielectric tensor can be written

$$\varepsilon_{11} = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^2}{\mu}\left(\frac{Z_{\bar{n}}}{\Delta} - \frac{\bar{n}^2}{\mu}\right)\alpha$$

$$\varepsilon_{12} = -2i\left(\frac{\omega_p}{\omega}\right)^2 \bar{n}\left[\frac{Z_{\bar{n}}}{\Delta}\left(1+\frac{1}{4\mu}\right) - \frac{\bar{n}^2}{\mu}\right]\alpha$$

$$\varepsilon_{22} = \varepsilon_{11} + \frac{2i\mu}{\bar{n}}\varepsilon_{12} + 4\alpha\mu\left(\frac{\omega_p}{\omega}\right)^2\left(1+\frac{1}{4\mu}+\frac{\bar{n}^2}{4\mu^2}\right)$$

$$\varepsilon_{13} = \left(\frac{\omega_p}{\omega}\right)^2\sqrt{\frac{2}{\mu}}\,\frac{\bar{n}}{\Delta}\alpha + \left(\frac{\omega_p}{\omega}\right)^2\frac{\bar{n}\Delta}{\sqrt{2\mu}}\left\{\frac{\bar{n}^2}{\mu}+\frac{\pi^2\bar{n}^2}{6}\right\}\alpha$$

$$\varepsilon_{23} = 2i\left(\frac{\omega_p}{\omega}\right)^2\frac{\sqrt{2\mu}}{\Delta}\left(1+\frac{1}{4\mu}\right)\alpha + i\left(\frac{\omega_p}{\omega}\right)^2\sqrt{\frac{\mu}{2}}\,\Delta\,\frac{\pi^2\bar{n}^2}{3}\alpha$$

$$\varepsilon_{33} = 1 + \left(\frac{\omega_p}{\omega}\right)^2\left(\frac{2\,\zeta_{\bar{n}}^2 Z_{\bar{n}}}{\Delta} - \frac{\bar{n}^2}{\mu}\right)\alpha$$

where

$$\alpha = e^{-\mu} I_{\bar{n}} = (-1)^{\bar{n}}\frac{\left(1+\frac{\bar{n}^2-1/4}{2\mu}\right)}{\sqrt{-2\pi\mu}}e^{-2\mu} .$$

Calculations show that longitudinal waves are possible only as long as $|\mu| \ll \left(\frac{c}{v_{th}}\right)^2\frac{1}{4}$ and that they satisfy the equation

$$\left(\frac{\omega_p}{\omega}\right)^2 e^{-2\mu} = (-2\mu)^{3/2}\left(-\frac{i\Delta(-1)^{\bar{n}}}{2\bar{n}^2}\right) . \tag{31}$$

There are also transverse extraordinary waves such that $|\mu| \gg \left(\frac{c}{v_{th}}\right)^2$ corresponding to the equation

$$4\left(\frac{\omega_p}{\omega}\right)^2\frac{Z_{\bar{n}}(-1)^{\bar{n}}}{\sqrt{-2\pi\mu}}e^{-2\mu} = -\frac{\Delta}{\bar{n}^2}\left(\frac{c}{v_{th}}\right)^2 . \tag{32}$$

Equations (31) and (32) are solved in App. 2.

In conclusion, we may summarize in the following way what happens when the Landau damping is included:

1) The index of refraction is never zero or infinite at the harmonics. With the exception of the quasi-macroscopic waves, all possible oscillations have very short wave length and are rather strongly damped.

2) Using the results of sect. 3 one can see that when Δ is of the order of $\frac{v_{th}}{c}$ the Landau damping destroys ordinary waves, with the exception of the macroscopic one, which remains practically unchanged. The macroscopic extraordinary wave also remains unchanged.

3) Only waves with $\mathrm{Re}(\mu) > 0$, i.e. $|N_R| > |N_I|$, have a line structure: they are longitudinal waves. When $\mathrm{Re}(\mu) < 0$ the order of magnitude of N at the harmonics is the same as elsewhere.

6. Deviation from thermodynamical equilibrium

The results of the preceding sections have been derived starting from a Maxwellian electron distribution function. According to the model of ref. 3 and to the discussion of sect. 1, however, a deviation from thermodynamical equilibrium has to be assumed in order to explain the observed radiation.

Here we consider longitudinal waves only, propagating in a plasma described by the distribution function

$$f_o(\vec{v}) = \sum_{i=1}^{N} d_i F_{oi}(v_\perp, v_\parallel) = \sum_{i=1}^{N} \frac{d_i}{(2\pi)^{3/2} v_{th\perp i}^2 v_{th\parallel i}} \exp\left[-\frac{v_\perp^2}{2v_{th\perp i}^2} - \frac{(v_\parallel - v_{oi})^2}{2v_{th\parallel i}^2}\right]$$

which takes account of streams of particles and temperature anisotropy.

The dispersion relation is[8]

$$1+\sum_{i=1}^{N}\frac{\omega_{pi}^2}{K^2 v_{th\parallel i}^2}\left[1+\sum_{n=-\infty}^{+\infty}\frac{\omega-K_\parallel v_{oi}-n\omega_{ei}\left(1-T_{\parallel i}/T_{\perp i}\right)}{\sqrt{2}\,K_\parallel v_{th\parallel i}}\,Z_{ni}\,I_n\left(\frac{K_\perp^2 v_{th\perp i}^2}{\omega_{ei}^2}\right)e^{-\frac{K_\perp^2 v_{th\perp i}^2}{\omega_{ei}^2}}\right]=0$$

where
$$Z_{ni}=Z\left(\frac{\omega-K_\parallel v_{oi}-n\omega_{ei}}{\sqrt{2}\,K_\parallel v_{th\parallel i}}\right)$$

In what follows we shall consider only the case of small deviations from an isotropic, centered Maxwellian, in the sense that we shall suppose

$$d_1 \gg d_i \quad (i=2,3,\ldots,N)\ ; \qquad v_{th\parallel 1}=v_{th\perp 1} \quad ; \qquad v_{o1}=0 .$$

Moreover we shall assume $\omega \approx \bar{n}\omega_e$ $(\bar{n}=1,2,\ldots)$ and $|K_\perp^2| \gg K_\parallel^2$.

Consider first the isotropic case of two centered Maxwellians $(v_{o1}=v_{o2}=0)$ with very different temperatures: $v_{th\parallel 2}=$ $=v_{th\perp 2}\gg v_{th1}$ or $v_{th\parallel 2}=v_{th\perp 2}\ll v_{th1}$. In this way one obtains a distribution with relatively more fast or slow electrons than a Maxwellian, but without streams. If the condition $|\xi_{\bar{n}2}|\gtrsim 2$, i.e.

$$\bar{n}\,\frac{v_{th2}}{c}\lesssim\frac{1}{2^{3/2}|N_\parallel|}\ , \tag{33}$$

is not fulfilled, it is clear that the second distribution function does not contribute to the lines but modifies the background and that only sligthly as long as $d_2/v_{th2}\ll d_1/v_{th1}$
If condition (33) is fulfilled the dispersion relation is

$$K_\perp^2+\left(\frac{\omega_{p1}}{v_{th1}}\right)^2\left(1+\bar{n}\,e^{-\mu_1}\sum_{n\neq\bar{n}}\frac{I_n(\mu_1)}{n-\bar{n}}\right)+\left(\frac{\omega_{p2}}{v_{th2}}\right)^2\left(1+\bar{n}\,e^{-\mu_2}\sum_{n\neq\bar{n}}\frac{I_n(\mu_2)}{n-\bar{n}}\right)+$$
$$+\,i\sqrt{\pi}\left[\left(\frac{\omega_{p1}}{v_{th1}}\right)^2\frac{e^{-\mu_1}}{\Delta_1}I_{\bar{n}}(\mu_1)+\left(\frac{\omega_{p2}}{v_{th2}}\right)^2\frac{e^{-\mu_2}}{\Delta_2}I_{\bar{n}}(\mu_2)\right]=0 \tag{34}$$

This equation shows that the terms dependent on the index 2 are negligible compared to the other in the case $v_{th2}\gg v_{th1}$. The result is not too surprising beacuse we have introduced deviations from the Maxwellian in the distribution function in a velocity range far away from the phase velocity.

In the opposite case $v_{th2} \ll v_{th1}$, a modification arises when $(\omega_{p2}/v_{th2})^2 \gtrapprox (\omega_{p1}/v_{th1})^2$. Then the solutions of equation $A = 0$ are approximately obtained from those given in sect. 5 by labelling the quantities ω_p and v_{th} with the index 2.

An important change in the result could be expected in the case of a 2-humped isotropic distribution function having the maxima in the low velocity range, in the neighbourhood of the phase velocity. This function can be constructed by mean of two isotropic Maxwellians subtracted one from the other with the conditions $v_{th2} > v_{th1}$ and $(\omega_{p2}/v_{th2})^2 < \frac{v_{th2}}{v_{th1}}(\omega_{p1}/v_{th1})^2$. The dispersion relation is then given by eq. (34) where ω_{p2}^2 is replaced by $-\omega_{p2}^2$. However, because $(\omega_{p1}/v_{th1})^2$ is always larger than $(\omega_{p2}/v_{th2})^2$ the real and imaginary part of the index of refraction diminishes only slightly without a change in the order of magnitude.

Consider now the case of a plasma with a beam, with isotropic temperature. Then

$$A = A_{plasma} + \left(\frac{\omega_{p2}}{K_\perp v_{th2}}\right)^2 \left(1 + \frac{1 - \frac{K_\parallel v_0}{\omega}}{\Delta_2} e^{-\mu_2} \sum_{n=-\infty}^{\infty} I_n(\mu_2) Z_{n2}\right) = 0 .$$

If we assume $|N_\parallel| < 1$, it follows $K_\parallel v_0 \ll \omega$ and $K_\parallel v_{th2} \ll \omega$. When $v_0 \ll v_{th2}$, this case does not differ essentially from the preceding one. When $v_0 \gg v_{th2}$, the ζ_{n2}'s calculated at the harmonics are much larger than 1, so that the beam does not contribute to the lines centered about the harmonics. On the contrary, there can be lines centered on the Doppler shiftted frequency $\omega = \bar{n}\omega_e + K_{\parallel} v_o$. These lines are much smaller than the lines due to the plasma if $v_{th2}^2/d_2 \gg v_{th1}^2/d_1$, and higher if $v_{th2}^2/d_2 < v_{th1}^2/d_1$. Emission of Doppler shifted lines has been recently reported by G. Bekefi and E. B. Hooper, Jr.[2]

Another case is that of a centered anisotropic Maxwellian. Remembering that, at the harmonics,

$$|\xi_{n \neq \bar{n}}| \gg 1 \quad \text{and} \quad |\xi_{\bar{n}}| \ll 1$$

we get

$$A = 1 + \left(\frac{\omega_p}{\omega}\right)^2 \frac{\bar{n}^2}{\mu}\left(1 + \bar{n} \sum_{n \neq \bar{n}} \frac{I_n e^{-\mu}}{n - \bar{n}} + \left(\frac{z_{\bar{n}}}{\Delta} + \frac{T_\perp}{T_\parallel}\right) I_{\bar{n}} e^{-\mu}\right) .$$

This expression coincides with expression (8), when $z_{\bar{n}}/\Delta$ is substituted by $z_{\bar{n}}/\Delta + T_\perp/T_\parallel$. An appreciable change in wave propagation results only in the case of a very strong temperature anisotropy. Also then the relative width of the lines is of the order of Δ .

Finally, the case of a plsma with a beam $(v_o \gg v_{th})$ with anisotropic temperature can be deduced from the preceding ones.

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Appendix 1

A) Evaluation of the series involved in the tensor γ_{ik}

First we notice that the evaluation of the series contained in (6-8) can be reduced to the evaluation of $\sum_{n=-\infty}^{+\infty} \frac{I_n(\mu)}{n-\nu}$. In fact

$$\sum_{n\neq\bar{n}} \frac{I_n(\mu)}{n-\bar{n}} = \lim_{\nu\to\bar{n}} \left(\sum_{n=-\infty}^{+\infty} \frac{I_n(\mu)}{n-\nu} - \frac{I_{\bar{n}}(\mu)}{\bar{n}-\nu} \right)$$

$$\sum_{n\neq\bar{n}} \frac{I_n(\mu)}{(n-\bar{n})^2} = \lim_{\nu\to\bar{n}} \left(\frac{\partial}{\partial\nu} \sum_{n=-\infty}^{+\infty} \frac{I_n(\mu)}{n-\nu} - \frac{I_{\bar{n}}(\mu)}{(\bar{n}-\nu)^2} \right).$$

The series involving I'_n are the derivatives of the preceding ones with respect to μ.

The expansion in powers of μ, valid for $|\mu| \ll 1$, is

$$\sum_{n=-\infty}^{+\infty} \frac{I_n(\mu)}{n-\nu} = -\frac{1}{\nu} + \frac{\mu\nu}{1-\nu^2} + \dots + \left(\frac{\mu}{2}\right)^{\bar{n}} \frac{1}{\bar{n}!(\bar{n}-\nu)} + \dots$$

where the last term can be important for $\nu \approx \bar{n} > 1$.

Let us now substitute the integral representation of $I_n(\mu)$

$$I_n(\mu) = \frac{(-1)^n}{\pi} \int_0^\pi e^{-\mu\cos\varphi} \cos(n\varphi)\, d\varphi$$

in $\sum_{n=-\infty}^{+\infty} \frac{I_n}{n-\nu}$. Then, by means of the equation

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\varphi)}{\nu^2-n^2} = \frac{\pi}{2\nu} \frac{\cos(\nu\varphi)}{\sin(\nu\pi)} - \frac{1}{2\nu^2}$$

we get

$$\sum_{n=-\infty}^{+\infty} \frac{I_n}{n-\nu} = -\frac{1}{\sin(\nu\pi)} \int_0^\pi e^{-\mu\cos\varphi} \cos(\nu\varphi)\, d\varphi \qquad (35)$$

or, using formula[12]

$$\int_0^\pi e^{z\cos t} \cos(\nu t)\, dt = \pi I_\nu(z) + \sin(\nu\pi) \int_0^\infty e^{-z\cosh t - \nu t}\, dt \qquad \mathrm{Re}(z)>0$$

we get

$$\sum_{n=-\infty}^{+\infty} \frac{I_n}{n-\nu} = -\pi\frac{\cos(\nu\pi)}{\sin(\nu\pi)} I_\nu(\mu) - \int_0^\pi e^{\mu\cos t}\sin(\nu t)\,dt - \qquad (36)$$

$$- \cos(\nu\pi)\int_0^\infty e^{-\mu\cosh t - \nu t}\,dt\,, \qquad \mathrm{Re}(\mu) > 0\,;$$

$$\sum_{n=-\infty}^{+\infty} \frac{I_n}{n-\nu} = -\frac{\pi}{\sin(\nu\pi)} I_\nu(-\mu) - \int_0^\infty e^{\mu\cosh t - \nu t}\,dt\,, \qquad (37)$$

$$\mathrm{Re}(\mu) < 0\,.$$

Some approximations may be given for the integrals on the right of (35), (36) and (37), for which no simple exact expression is known (see ref. 13 p. 313).

When ν or $|\mathrm{Re}(\mu)|$ are much larger than 1 and $\mathrm{Re}(\mu) < 0$, we can write

$$\int_0^\infty e^{\mu\cosh t - \nu t}\,dt \approx \int_0^\infty e^{\frac{\mu}{2}t^2 - \nu t + \mu}\,dt = \frac{-i\,e^\mu}{\sqrt{-2\mu}}\, Z\left(\frac{i\nu}{\sqrt{-2\mu}}\right)$$

where Z(z) is the "plasma dispersion function" defined in sect. 2. With the expansions (4) and (5) we have:

when $\frac{\nu}{|\sqrt{-2\mu}|} \ll 1$,

$$\int_0^\infty e^{\mu\cosh t - \nu t}\,dt \approx \frac{\nu e^\mu}{\mu}\left(1 - \frac{\nu^2}{3\mu} + \frac{\nu^4}{15\mu^2}\right) \quad ;$$

when $\frac{\nu}{|\sqrt{-2\mu}|} \gtrsim 2$,

$$\int_0^\infty e^{\mu\cosh t - \nu t}\,dt \approx \frac{e^\mu}{\nu}\left(1 + \frac{\mu}{\nu^2} + \frac{3\mu^2}{\nu^4}\right) \quad .$$

Repeated integrations by parts of the integral

$$\int_0^\pi e^{\mu\cos t}\sin(\nu t)\,dt$$

give an expression valid when $|\mu| \ll \nu^2$, namely

$$\int_0^{\pi} e^{\mu\cos t}\sin(\nu t)\,dt \approx \frac{e^{\mu}}{\nu}\left(1+\frac{\mu}{\nu^2}+\frac{3\mu^2}{\nu^4}+\cdots\right)-\cos(\nu\pi)\frac{e^{-\mu}}{\nu}.$$

When $|\mu| \gg \nu^2$, the asymptotic method of Laplace gives

$$\int_0^{\pi} e^{\mu\cos t}\sin(\nu t)\,dt \approx \frac{\nu e^{\mu}}{\mu}\left(1-\frac{\nu^2}{3\mu}+\frac{\nu^4}{15\mu^2}+\cdots\right) .$$

In Table I approximations for $\nu\cdot\sum_{n=-\infty}^{+\infty}\frac{I_n e^{-\mu}}{n-\nu}$ and $\nu\sum_{n=-\infty}^{+\infty}\frac{(I_n e^{-\mu})'}{n-\nu}$ are quoted.

The series $\sum_{n=-\infty}^{\infty}\frac{I_n}{n-\nu}$ has been calculated for ν = n+1/2. The result

$$\sum_{n=-\infty}^{+\infty}\frac{I_n}{n-\bar{n}-1/2} = -e^{-\mu}\sum_{r=0}^{\bar{n}} 2^{\bar{n}-r}(-1)^r\binom{2\bar{n}-r}{r}\frac{\partial^{\bar{n}-r}}{\partial\mu^{\bar{n}-r}}\frac{\mathrm{Erfi}(\sqrt{2\mu})}{\sqrt{2\mu}},$$

is, however, too cumbersome to be used in the discussion of the dispersion relation.

The series $\sum_{n\neq\bar{n}}\frac{I_n}{n-\bar{n}}$ can be found to be

$$\sum_{r=0}^{<\frac{\bar{n}}{2}}(-1)^{\bar{n}+r}\frac{2^{\bar{n}-2r-1}}{\mu^{\bar{n}-2r}}\frac{(\bar{n}-r-1)!}{r!}\sum_{s=0}^{\bar{n}-2r-1}(-1)^s\frac{\mu^s}{s!}\left(e^{\mu}-I_0^{(s)}\right),$$

where $I_0^{(s)}$ is the s-th derivative of $I_0(\mu)$ with respect to the argument. However, for the discussion of the dispersion relation the approximations given in Table II are more useful. The corresponding values of $\bar{n}^2\sum_{n\neq\bar{n}}\frac{I_n e^{-\mu}}{(n-\bar{n})^2}$ and $\bar{n}^2\sum_{n\neq\bar{n}}\frac{(I_n e^{-\mu})'}{(n-\bar{n})^2}$ are given in Table III.

B) Asymptotic expansions of $I_n(\mu)$

Besides the well known expressions valid for $|\mu| \ll \frac{\sqrt{\nu}}{2}$ and $|\mu| \gg \frac{\nu^2}{2}$, we need two less common expressions valid for $|\mu| \ll \nu$, $\nu \gg 1$ and $|\mu| \gg \nu+1$, $|\mathrm{Re}(\mu)| \gg 1$. They can be derived from ref. 13 using the method developed in §§ 8.6 - 8-61. When $|\mu| \ll \nu$, $\nu \gg 1$, region 2 of fig. 21 has to be chosen if $\mathrm{Re}(\mu) < 0$ and region 3 if $\mathrm{Re}(\mu) > 0$. When $|\mu| \gg \nu+1$ the region 1 should be chosen.

The resulting expressions for $I_\nu(\mu)e^{-\mu}$ and $(I_\nu(\mu)e^{-\mu})'$ are given in Table IV.

Appendix 2

Equations (11),(13),(31) and (32) are of the form

$$e^{z} = A z^{p/2} , \tag{38}$$

where $\mathrm{Re}(z) > 0$, $|z| \gg 1$, p is a positive even number and A is complex. Let us write $z = \rho e^{i\theta}$, $|\theta| \le \frac{\pi}{2}$ and $A = \alpha e^{i(\varphi + 2k\pi)}$ $(k=0,\pm1,\pm2,..)$; taking then the logarithm of (38) and separating the real and imaginary parts, we get

$$\rho\cos\theta = \ln(\alpha\rho^{p/2}) \quad ,$$

$$\rho\sin\theta = \varphi + \tfrac{p}{2}\theta + 2k\pi \quad .$$

If $\ln(\alpha\rho^{p/2})/\rho$ is much smaller than 1, these equations have the approximate solutions

$$\begin{aligned} \theta &= \pm\tfrac{\pi}{2} \mp \varepsilon + 2m\pi \quad , \varepsilon > 0 && (m=0,\pm1,\pm2,..) \\ \rho &= \pm\varphi + \left(\tfrac{p}{4} + r\right)\pi && (r=0,\pm1,\pm2,..) \end{aligned} \tag{39}$$

where ε is equal to $\ln(\alpha\rho^{p/2})/\rho = \ln\left[\alpha\left(\pm\varphi + (\frac{p}{4}+r)\pi\right)^{p/2}\right] \Big/ \left[\pm\varphi + (\frac{p}{4}+r)\pi\right]$.

The following conditions are to be fulfilled

$$\begin{aligned} &\varepsilon > 0 , && \text{i.e.} \quad \alpha\rho^{p/2} > 1 \quad ; \\ &\varepsilon \ll 1 , && \text{i.e.} \quad |\ln(\alpha\rho^{p/2})|/\rho \ll 1 \; ; \\ &\rho \gg 1 , && \text{i.e.} \quad \pm\varphi + (p/4 + r)\pi \gg 1 \; . \end{aligned} \tag{40}$$

Solutions (39) are the only ones possible if condition, $\varepsilon \ll 1$, is satisfied for the given value of α and for all values of

ϱ in the range where eq. (38) is valid. Otherwise, there are also smaller solutions which are not of the kind of (39).

Because ϱ is much larger than 1 and p is 1 or 3, condition $\varepsilon \ll 1$ is always fulfilled if $\alpha \varrho^{p/2} > 1$ and $\alpha \lessapprox \varrho$. These last conditions are always satisfied in the case of eqs. (11), (13) and (32), which have therefore only the solutions (39). Eq. (31) has the solutions (39) when $\left(\frac{\omega_p^2}{\omega^2}\frac{2\bar{n}^2}{\Delta}\right)^{2/3} < \varrho < \left(\frac{c}{v_a}\right)^2$; in the interval $2\bar{n}^2 \ll \varrho < \left(\frac{\omega_p^2}{\omega^2}\frac{2\bar{n}^2}{\Delta}\right)^{2/3}$ it has no solutions with $\mathrm{Re}(z) > 0$.

It is interesting to note that $|\mathrm{Re}(z)|/|\mathrm{Im}(z)|$ is equal to ε and goes to zero as $|z|$ goes to infinity.

Remembering that $\lim_{\delta \to 0}(1+\delta)^{z/\delta} = e^z$ we can write eqs. (12), (14) and (27) in the form

$$e^{\frac{\nu^3}{2z^2} + \frac{z}{\sqrt{1+(\nu/z)^2}} \pm z} = B\, z^{q/2} \,, \tag{41}$$

where $\mathrm{Re}(z) > 0$, $\left|\frac{\nu}{z}\right| \ll 1$, q is a positive even number and B is complex. (The form with the minus sign corresponds to eq. (27).)

If we write

$$\frac{\nu^3}{2z^2} + \frac{z}{\sqrt{1+(\nu/z)^2}} \pm z = \gamma e^{i\omega}$$

and $B = \beta e^{i(\psi + 2K\pi)}$ we get, analogously to eqs. (39)

$$\omega = \pm\frac{\pi}{2} \mp \varepsilon + 2m\pi$$

$$\gamma = \pm\left(\psi + q/2\,\theta + 2K\pi\right) \,,$$

where $$\varepsilon = \ln\left(\beta \varrho^{q/2}\right)/\gamma \quad , |\varepsilon| \ll 1 .$$

That is $$\mathrm{Re}\left(\frac{\nu^3}{2z^2} + \frac{z}{\sqrt{1+(\nu/z)^2}} \pm z\right) = \pm\left(\psi + \frac{q}{2}\theta + 2K\pi\right)\varepsilon \,,$$

$$\operatorname{Im}\left(\frac{\nu^3}{2z^2} + \frac{z}{\sqrt{1+(\nu/z)^2}} \pm z \right) = \psi + \tfrac{9}{2}\theta + 2K\pi$$

When the positive sign on the left hand side is taken, these equations can be approximately written

$$\operatorname{Re}(2z) = \pm(\psi + 9/2\,\theta + 2K\pi)\varepsilon \quad ,$$

$$\operatorname{Im}(2z) = \psi + 9/2\,\theta + 2K\pi \quad ,$$

and therefore one obtains $\operatorname{tg}\theta = \pm\varepsilon$ which gives

$$\theta = \pm\frac{\pi}{2} \mp \varepsilon + 2m\pi \quad , \quad \varepsilon > 0 \quad , \qquad (42)$$

$$\rho = \frac{\pm\psi + (9/4 + r)\pi}{2} \; .$$

The conditions to be satisfied are quite similar to that for eqs. (39).

Eq. (13) has only the solutions (42). Eq. (14) has solutions with Re(z) > 0 only in the interval $\frac{\nu^2}{2} \gg \rho > \left(\left(\frac{\omega}{\omega_p}\right)^2 \frac{\nu\pi}{\sin(\nu\pi)} \cdot \frac{1}{\sqrt{2\pi}} \left(\frac{c}{v_{th}}\right)^2 \right)^{2/5}$ and then they are given by (42).

With the negative sign on left hand side of eq. (41), one has analogously $-\operatorname{ctg}\theta = \pm\varepsilon$ which gives

$$\theta = \pm\frac{\pi}{2} \pm \varepsilon + 2m\pi \quad ; \qquad (43)$$

$$\frac{\nu^2}{2\rho} = \frac{\pm\psi + (9/4 + r)\pi}{2} \; .$$

Therefore ε has to be negative and the conditions to be fulfilled are $\beta\rho^{9/2} < 1 \quad ; \quad \left(\ln|\beta\rho^{9/2}|\right)/(\nu^2/2\rho) \ll 1$

$$1 \ll \pm\psi + (9/4 + r)\pi \ll \nu^2 \; .$$

In the case of eq. (27) they are always satisfied. Expressions (43) are therefore the only possible solutions.

One could have developed the exponent in (41) in powers before writing it in the form $\gamma e^{i\omega}$ but it is easier to see that the error made in determining ϱ and θ is of the order of $\left|\frac{\nu}{z}\right|$ by the method we have used.

TABLE I

$\nu \cdot \sum_{n=-\infty}^{\infty} \frac{I_n(\mu)\, e^{-\mu}}{n-\nu}$ $\nu \neq 0, \pm 1, \dots$	$\nu \cdot \sum_{n=-\infty}^{\infty} \frac{\left(I_n(\mu)\, e^{-\mu}\right)'}{n-\nu}$ $\nu \neq 0, \pm 1, \dots$	Validity region
$-1 + \frac{\mu}{1-\nu^2}$	$\frac{1}{1-\nu^2}$	$\lvert\mu\rvert \ll 1$, sufficiently far from $0, \pm 1, \dots$.
$-1 - \frac{\mu}{\nu^2} - \frac{\pi\nu}{tg(\nu\pi)} I_\nu(\mu)\, e^{-\mu}$	$-\frac{1}{\nu^2} - \frac{6\mu}{\nu^4} - \frac{\nu\pi}{tg(\nu\pi)} \left(I_\nu(\mu)\, e^{-\mu}\right)'$	$\lvert\mu\rvert \ll \nu^2$; $\nu \gg 1$; $Re(\mu) > 0$.
$-1 - \frac{\mu}{\nu^2} - (-1)^\nu \frac{\nu\pi}{\sin(\nu\pi)} I_\nu(\mu)\, e^{-\mu}$	$-\frac{1}{\nu^2} - \frac{6\mu}{\nu^4} - (-1)^\nu \frac{\nu\pi}{\sin(\nu\pi)} \left(I_\nu(\mu)\, e^{-\mu}\right)'$	$\lvert\mu\rvert \ll \nu^2$; $\nu \gg 1$; $Re(\mu) < 0$.
$-\frac{\nu^2}{\mu}\left(1 - \frac{\nu^2}{3\mu}\right) - \frac{\nu\pi}{tg(\nu\pi)} \frac{\left(1 - \frac{\nu^2 - 1/4}{2\mu}\right)}{\sqrt{2\pi\mu}}$	$\frac{\nu^2}{\mu^2} - \frac{2\nu^4}{3\mu^3} + \frac{\nu\pi}{tg(\nu\pi)} \cdot \frac{1}{2\mu} \cdot \frac{\left(1 - \frac{3(\nu^2 - 1/4)}{2\mu}\right)}{\sqrt{2\pi\mu}}$	$\lvert\mu\rvert \gg \frac{\nu^2}{2}$; $Re(\mu) \gg 0$; $\nu \gg 1$ or $Re(\mu) \gg 1$, ν arbitrary.
$-\frac{\nu^2}{\mu} - \frac{\nu\pi}{\sin(\nu\pi)} \cdot \left(1 + \frac{\nu^2 - 1/4}{2\mu}\right) \frac{e^{-2\mu}}{\sqrt{-2\pi\mu}}$	$2 \frac{\nu\pi}{\sin(\nu\pi)} \left(1 + \frac{\nu^2 + 1/4}{2\mu}\right) \cdot \frac{e^{-2\mu}}{\sqrt{-2\pi\mu}}$	$\lvert\mu\rvert \gg \frac{\nu^2}{2}$; $Re(\mu) \ll 0$; $\nu \gg 1$ or $Re(\mu) \ll -1$, ν arbitrary .

TABLE II

$\bar{n}\cdot\sum'_{n\neq\bar{n}} \frac{I_n(\mu)e^{-\mu}}{n-\bar{n}}$	$\bar{n}\cdot\sum'_{n\neq\bar{n}} \frac{\left(I_n(\mu)e^{-\mu}\right)'}{n-\bar{n}}$	Validity region
$-1+\frac{\mu}{1-\bar{n}^2}$	$\frac{1}{1-\bar{n}^2}$	$\lvert\mu\rvert\ll 1$; $\bar{n}\neq 1$.
$-1+\frac{3}{4}\mu$	$\frac{3}{4}$	$\lvert\mu\rvert\ll 1$; $\bar{n}=1$.
$-1-\frac{\mu}{\bar{n}^2}-\bar{n}\cdot\ln\left(\frac{\mu}{2\bar{n}}\right)\cdot I_{\bar{n}}(\mu)e^{-\mu}$	$-\frac{1}{\bar{n}^2}-\frac{6\mu}{\bar{n}^4}-\bar{n}\ln\left(\frac{\mu}{2\bar{n}}\right)\left(I_{\bar{n}}(\mu)e^{-\mu}\right)'-$ $-\frac{\bar{n}}{\mu}I_{\bar{n}}(\mu)e^{-\mu}$	$\lvert\mu\rvert\ll\bar{n}^2$; $\bar{n}\gg 1$; $\mathrm{Re}(\mu)>0$.
$-1-\frac{\mu}{\bar{n}^2}-\bar{n}\cdot\ln\left(\frac{-\mu}{2\bar{n}}\right)\cdot I_{\bar{n}}(-\mu)e^{-\mu}$	$-\frac{1}{\bar{n}^2}-\frac{6\mu}{\bar{n}^4}-\bar{n}\ln\left(\frac{-\mu}{2\bar{n}}\right)\left(I_{\bar{n}}(-\mu)e^{-\mu}\right)'-$ $-\frac{\bar{n}}{\mu}I_{\bar{n}}(-\mu)e^{-\mu}$	$\lvert\mu\rvert\ll\bar{n}^2$; $\bar{n}\gg 1$; $\mathrm{Re}(\mu)<0$.
$-\frac{\bar{n}^2}{\mu}\left(1-\frac{1}{\sqrt{2\pi\mu}}-\frac{\bar{n}^2}{3\mu}\right)$	$\frac{\bar{n}^2}{\mu^2}\left(1-\frac{3}{2\sqrt{2\pi\mu}}-\frac{2\bar{n}^2}{3\mu}\right)$	$\lvert\mu\rvert\gg\frac{\bar{n}^2}{2}$; $\begin{cases}\mathrm{Re}(\mu)\gg 1 \text{ ; } \bar{n} \text{ arbitrary;}\\ \text{or } \frac{\bar{n}^2}{2}\gg 1 \text{ ; } \mathrm{Re}(\mu)>0.\end{cases}$
$-\frac{\bar{n}^2}{\mu}(-1)^{\bar{n}}\frac{\left(1-\frac{\bar{n}^2-1/4}{2\mu}\right)}{\sqrt{-2\pi\mu}}e^{-2\mu}$	$\frac{2\bar{n}^2}{\mu}(-1)^{\bar{n}}\frac{\left(1-\frac{\bar{n}^2-7/4}{2\mu}\right)}{\sqrt{-2\pi\mu}}e^{-2\mu}$	$\lvert\mu\rvert\gg\frac{\bar{n}^2}{2}$; $\begin{cases}\mathrm{Re}(\mu)\ll -1 \text{ , } \bar{n} \text{ arbitrary;}\\ \text{or } \mathrm{Re}(\mu)<0 \text{ , } \frac{\bar{n}^2}{2}\gg 1.\end{cases}$

TABLE III

$\bar{n}^2 \cdot \sum\limits_{n \neq \bar{n}} \dfrac{I_n(\mu)\, e^{-\mu}}{(n - \bar{n})^2}$	$\bar{n}^2 \cdot \sum\limits_{n \neq \bar{n}} \dfrac{\left(I_n(\mu)\, e^{-\mu}\right)'}{(n - \bar{n})^2}$	Validity region
$1 + \mu \dfrac{3\bar{n}^2 - 1}{(1 - \bar{n}^2)^2}$	$\dfrac{3\bar{n}^2 - 1}{(1 - \bar{n}^2)^2}$	$\lvert\mu\rvert \ll 1$; $\bar{n} \neq 1$.
$1 - \frac{7}{8}\mu$	$-\frac{7}{8}$	$\lvert\mu\rvert \ll 1$; $\bar{n} = 1$.
$1 + \frac{3\mu}{\bar{n}^2} + \bar{n}^2\left(\frac{\pi^2}{3} - \frac{1}{2}\ln^2\left(\frac{\mu}{2\bar{n}}\right)\right) I_{\bar{n}}(\mu) e^{-\mu}$	$\frac{3}{\bar{n}^2} + \frac{30\mu}{\bar{n}^4} + \bar{n}^2\left(\frac{\pi^2}{3} - \frac{1}{2}\ln^2\left(\frac{\mu}{2\bar{n}}\right)\right)\left(I_{\bar{n}}(\mu) e^{-\mu}\right)' -$ $- \frac{\bar{n}^2}{\mu}\ln\left(\frac{\mu}{2\bar{n}}\right) I_{\bar{n}}(\mu) e^{-\mu}$	$\lvert\mu\rvert \ll \bar{n}$; $\bar{n} \gg 1$; $Re(\mu) > 0$.
$1 + \frac{3\mu}{\bar{n}^2} - \bar{n}^2\left(\frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{-\mu}{2\bar{n}}\right)\right) \cdot I_{\bar{n}}(-\mu) e^{-\mu}$	$\frac{3}{\bar{n}^2} + \frac{30\mu}{\bar{n}^4} - \bar{n}^2\left(\frac{\pi^2}{6} + \frac{1}{2}\ln^2\left(\frac{-\mu}{2\bar{n}}\right)\right)\left(I_{\bar{n}}(-\mu) e^{-\mu}\right)' -$ $- \frac{\bar{n}^2}{\mu}\ln\left(\frac{-\mu}{2\bar{n}}\right) \cdot I_{\bar{n}}(-\mu)\, e^{-\mu}$	$\lvert\mu\rvert \ll \bar{n}$; $\bar{n} \gg 1$; $Re(\mu) < 0$.
$1 + \frac{3\mu}{\bar{n}^2} + \bar{n}^2\left(\frac{\pi^2}{3} - \frac{\bar{n}^2}{2\mu^2}\right) I_{\bar{n}}(\mu)\, e^{-\mu}$	$\frac{3}{\bar{n}^2} + \frac{30\mu}{\bar{n}^4} + \bar{n}^2\left(\frac{\pi^2}{3} - \frac{\bar{n}^2}{2\mu^2}\right)\left(I_{\bar{n}}(\mu) e^{-\mu}\right)' +$ $+ \frac{\bar{n}^4}{\mu^3} I_{\bar{n}}(\mu) e^{-\mu}$	$\bar{n}^2 \gg \lvert\mu\rvert \gg \bar{n} \gg 1$; $Re(\mu) > 0$.
$1 + \frac{3\mu}{\bar{n}^2} - \bar{n}^2\left(\frac{\pi^2}{6} + \frac{\bar{n}^2}{2\mu^2}\right) I_{\bar{n}}(-\mu)\, e^{-\mu}$	$\frac{3}{\bar{n}^2} + \frac{30\mu}{\bar{n}^4} - \bar{n}^2\left(\frac{\pi^2}{6} + \frac{\bar{n}^2}{2\mu^2}\right)\left(I_{\bar{n}}(-\mu) e^{-\mu}\right)' +$ $+ \frac{\bar{n}^4}{\mu^3} I_{\bar{n}}(-\mu) e^{-\mu}$	$\bar{n}^2 \gg \lvert\mu\rvert \gg \bar{n} \gg 1$; $Re(\mu) < 0$.
$\frac{\bar{n}^2}{\mu} + \frac{\bar{n}^4}{\mu^2} + \frac{\pi^2 \bar{n}^2}{3\sqrt{2\pi\mu}}\left(1 - \frac{\bar{n}^2 - 1/4}{2\mu}\right)$	$\frac{\bar{n}^2}{\mu^2} - \frac{2\bar{n}^4}{\mu^3} - \frac{\pi^2 \bar{n}^2}{6\mu\sqrt{2\pi\mu}}\left(1 - \frac{3(\bar{n}^2 - 1/4)}{2\mu}\right)$	$\lvert\mu\rvert \gg \frac{\bar{n}^2}{2}$; $\begin{cases} Re(\mu) \gg 1, \bar{n} \text{ arbitrary} \\ \text{or } Re(\mu) > 0, \bar{n} \gg 1. \end{cases}$
$(-1)^{\bar{n}} \frac{\pi^2 \bar{n}^2}{6\sqrt{-2\pi\mu}}\left(1 + \frac{\bar{n}^2 - 1/4 - 6/\pi^2}{2\mu}\right) \cdot e^{-2\mu}$	$(-1)^{\bar{n}} \frac{\pi^2 \bar{n}^2 e^{-2\mu}}{3\sqrt{-2\pi\mu}}\left(1 + \frac{\bar{n}^2 + 1/4 - \frac{6}{\pi^2}}{2\mu} + \frac{3\left(\bar{n}^2 - 1/4 - \frac{6}{\pi^2}\right)}{(2\mu)^2}\right)$	$\lvert\mu\rvert \gg \frac{\bar{n}^2}{2}$; $\begin{cases} Re(\mu) \ll -1, \bar{n} \text{ arbitrary} ; \\ \text{or } Re(\mu) < 0, \bar{n} \gg 1 . \end{cases}$

TABLE IV

$I_\nu(\mu)\,e^{-\mu}$ ν real positive	$\left(I_\nu(\mu)\,e^{-\mu}\right)'$ ν real positive	Validity region
$\dfrac{\left(1+\left(\frac{\mu}{2}\right)^2\frac{1}{\nu+1}\right)}{\Gamma(\nu+1)}\left(\frac{\mu}{2}\right)^\nu e^{-\mu}$	$\dfrac{\left(\frac{\nu}{\mu}-1+\frac{\mu}{4}\cdot\frac{\nu+2}{\nu+1}\right)}{\Gamma(\nu+1)}\left(\frac{\mu}{2}\right)^\nu e^{-\mu}$	$\left\|\left(\frac{\mu}{2}\right)^2\right\| \ll \nu+1$
$\dfrac{\left(1-\frac{1}{12\nu}-\frac{\mu^2}{2\nu^2}\right)}{\sqrt{2\pi\nu}}\left(\frac{\mu}{\nu}\cdot\dfrac{e^{-\frac{\mu}{\nu}+\sqrt{1+\left(\frac{\mu}{\nu}\right)^2}}}{1+\sqrt{1+\left(\frac{\mu}{\nu}\right)^2}}\right)^\nu$	$\dfrac{\left(\frac{\nu}{\mu}-1\right)\left(1-\frac{1}{12\nu}-\frac{\mu^2}{4\nu^2}\right)}{\sqrt{2\pi\nu}}\left(\frac{\mu}{\nu}\cdot\dfrac{e^{-\frac{\mu}{\nu}+\sqrt{1+\left(\frac{\mu}{\nu}\right)^2}}}{1+\sqrt{1+\left(\frac{\mu}{\nu}\right)^2}}\right)^\nu$	$\|\mu\| \ll \nu$; $\nu \gg 1$.
$\dfrac{\left(1+\frac{1}{8\mu}-\frac{\nu^2}{4\mu^2}\right)}{\sqrt{2\pi\mu}}\left[\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}-\frac{\nu}{\mu}\right)e^{\frac{\mu}{\nu}\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}-1\right)}\right]^\nu$	$-\dfrac{\left(1-\frac{\nu^2}{\mu}\right)}{2\mu\sqrt{2\pi\mu}}\left[\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}-\frac{\nu}{\mu}\right)e^{\frac{\mu}{\nu}\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}-1\right)}\right]^\nu$	$\|\mu\| \gg \nu+1$; $Re(\mu) \gg 1$.
$\dfrac{\left(1-\frac{1}{8\mu}-\frac{\nu^2}{4\mu^2}\right)}{\sqrt{-2\pi\mu}}\left[(-1)\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}+\frac{\nu}{\mu}\right)e^{-\frac{\mu}{\nu}\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}+1\right)}\right]^\nu$	$\dfrac{-2\left(1+\frac{1}{8\mu}\right)}{\sqrt{-2\pi\mu}}\left[(-1)\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}+\frac{\nu}{\mu}\right)e^{-\frac{\mu}{\nu}\left(\sqrt{1+\left(\frac{\nu}{\mu}\right)^2}+1\right)}\right]^\nu$	$\|\mu\| \gg \nu+1$; $Re(\mu) \ll -1$.
$\dfrac{1}{\sqrt{2\pi\mu}}\left(1-\dfrac{\nu^2-1/4}{2\mu}\right)$	$-\dfrac{1}{2\mu}\cdot\dfrac{1}{\sqrt{2\pi\mu}}\left(1-\dfrac{3(\nu^2-1/4)}{2\mu}\right)$	$\|\mu\| \gg \frac{\nu^2}{2}$ $\begin{cases} Re(\mu) \gg 0, \ \nu \gg 1 \\ \text{or } Re(\mu) \gg 1, \ \nu \text{ arbitrary.} \end{cases}$
$(-1)^\nu\dfrac{\left(1+\frac{\nu^2-1/4}{2\mu}\right)}{\sqrt{-2\pi\mu}}\,e^{-2\mu}$	$-2(-1)^\nu\dfrac{\left(1+\frac{\nu^2+1/4}{2\mu}\right)}{\sqrt{-2\pi\mu}}\,e^{-2\mu}$	$\|\mu\| \gg \frac{\nu^2}{2}$ $\begin{cases} Re(\mu) \ll 0, \ \nu \gg 1, \\ \text{or } Re(\mu) \ll -1, \ \nu \text{ arbitrary.} \end{cases}$

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