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"On the Dynamics of a Toroidal Θ-Pinch"

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ABSTRACT: The equation of motion of the meridional centre of mass of a toroidal Θ-pinch is derived. It is shown that the equation $d^2 S/dt^2 = 2c^2/S$ (S = distance of the meridional centre of mass from the axis; c = Newtonian speed of sound) is a good approximation. It gives an outward acceleration independent of the ratio p/B^2 . The right hand side contains contributions of about equal magnitude due to the gas pressure and due to the diamagnetic behaviour of the plasma. Corrections to this relation due to "geometrical" effects are of second order in the aspect ratio A ($A \ll 1$) while dynamical corrections due to compression or expansion of the plasma are of first order in A . Both are of minor importance in practical applications.

1) If we introduce a system of cylindrical coordinates (s, φ, z), a toroidal Θ-pinch can be defined as a magnetohydrodynamic configuration where all quantities are independent of φ and where the magnetic field is everywhere toroidal:

$$\underset{\sim}{B} = (0, B(s,z), 0) \tag{1}$$

The plasma shall nowhere touch the walls of the discharge vessel. There then exists in the meridian plane a ring-like region where the gas pressure $p \equiv 0$. In this region the magnetic field is necessarily of the form

$$B(s,z) = \frac{s_o}{s} B_o \quad \text{where} \quad p(s,z) \equiv 0 \tag{2}$$

$s_o B_o$ is a function of time only which we shall not specify (it is, of course, proportional to the total current through the external coil).

If the motion is at some time purely meridional it will remain so. We assume this to be the case. The equation of motion is then:

$$\rho \frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{f} \qquad \text{with} \qquad \underset{\sim}{f} = (f_s, 0, f_z) \tag{3}$$

and

$$f_s = -\frac{1}{2\mu_o s^2} \frac{\partial(B^2 s^2)}{\partial s} - \frac{\partial p}{\partial s} \tag{4}$$

$$f_z = -\frac{1}{2\mu_o} \frac{\partial B^2}{\partial z} - \frac{\partial p}{\partial z} \tag{5}$$

As a consequence of eq. (2)

$$\underset{\sim}{f} \equiv 0 \qquad \text{where} \qquad p \equiv 0 \tag{6}$$

For further reference we note that from eqs. (4) and (5) with eq. (2), the following theorem of the virial type holds:

$$\iint\limits_{p \neq 0} ds\, dz\, \underset{\sim}{f} \cdot \underset{\sim}{r} = 2 \iint\limits_{p \neq 0} ds\, dz\, p \tag{7}$$

where $\underset{\sim}{r} = (s, o, z)$ is the radius vector and the integrals are to be extended over that portion of the meridional plane where p is different from zero. For the simple proof see the appendix.

2) A complete analysis of the dynamic behaviour of the toroidal Θ-pinch implies solution of the equation of motion, the equation of continuity, an energy equation and the equation describing $B(s, z; t)$. We shall, however, show that some interesting conclusions can be drawn on the gross behaviour with much less knowledge. Such conclusions seem warranted since it is known that a toroidal Θ-pinch can under no circumstances be in a static equilibrium [1] .

3) We define M as the mass per unit angle of φ by

$$M = \iint\limits_{\rho \neq 0} ds\, dz\, s\, \rho\, (s, z), \tag{8}$$

so that $2\pi M$ is the total mass in the pinch.

We define further averages of functions $F(s,z)$ by

$$\langle F \rangle = M^{-1} \cdot \iint\limits_{\rho \neq 0} ds\, dz\, s\, F(s, z)\, \rho\, (s, z)\, . \tag{9}$$

Of particular interest among these averages is $\langle \underset{\sim}{r} \rangle$, which in a certain sense may be called the meridional centre of mass.

For further reference we consider a simple example

$$\rho = \begin{cases} \rho_c = \text{const. for } (s - S_c)^2 + z^2 < R^2 \\ 0 \qquad \text{elsewhere} \end{cases} \tag{10}$$

We assume the aspect ratio $A = R / S_c$ of this homogeneous ring of circular cross section to be small compared to one, and evaluate the average of some powers of s and z up to order A^2

$$\begin{aligned}
\langle s^2 \rangle &= S_c^2 \left(1 + \tfrac{3}{4} A^2\right) \qquad \langle z^2 \rangle = \tfrac{1}{4} R^2 \\
\langle s \rangle &= S_c \left(1 + \tfrac{1}{4} A^2\right) \qquad \langle z \rangle = 0 \\
\langle s^0 \rangle &= 1 \\
\langle s^{-1} \rangle &= S_c^{-1} \\
\langle s^{-2} \rangle &= S_c^{-2} \left(1 + \tfrac{1}{4} A^2 + \dots\right) \\
\langle s^{-3} \rangle &= S_c^{-3} \left(1 + \tfrac{3}{4} A^2 + \dots\right)
\end{aligned} \qquad (11)$$

It should be noted that the meridional centre of mass does not coincide with the centre of the ring.

In making use of these relations we shall effectively reverse the logic. If we consider configurations in which the plasma cross section is essentially circular, such that

$$2\left(\langle s^2 \rangle - \langle s \rangle^2\right)^{1/2} \approx 2\left(\langle z^2 \rangle - \langle z \rangle\right)^{1/2},$$

we call that quantity the (minor) radius R of the plasma. The aspect ratio A we then define by

$$A = R \cdot \langle s^{-1} \rangle \text{ or: } A = R \cdot \langle s \rangle^{-1}$$

The definitions of A may be considered equivalent, since we are only interested in cases where $A \ll 1$ and A occurs only in correction terms. We then assume that the internal relations between the different quantities of eq. (11) are approximately true to second order in A for all quantities occuring in eq. (11), e.g.:

$$\langle s \rangle \cdot \langle s^{-1} \rangle = 1 + \tfrac{1}{4} A^2$$

4) From our definitions it is (with the aid of the equation of continuity) easily proved (see appendix) that

$$\frac{d}{dt}\langle \underset{\sim}{r} \rangle = \langle \underset{\sim}{v} \rangle \tag{12}$$

$$\frac{d}{dt}\langle \underset{\sim}{v} \rangle = \langle \frac{d\underset{\sim}{v}}{dt} \rangle = \langle \frac{\underset{\sim}{f}}{\rho} \rangle .$$

Therefore:

$$M \frac{d^2}{dt^2}\langle s \rangle = \iint ds\, dz\, f_s\, s \tag{13}$$

$$= 2 \iint ds\, dz\, p - \iint ds\, dz\, f_z\, z \tag{14}$$

In passing from eq. (13) to eq. (14) the theorem (7) has been used.

Eq. (14) is more useful than eq. (13) since under all ordinary circumstances the integral containing f_z is small in magnitude compared to the integral containing f_s . This is plausible on geometrical grounds, if the configuration possesses a plane of symmetry, say $z = 0$ (the integral $\iint ds\, dz\, f_z\, z$ is invariant under a translation along z). In addition, it is plausible that f_z is smaller in magnitude then f_s when the plasma is not rapidly being compressed, since equilibrium in the z-direction is possible while it is not in the s-direction.

We shall return to this point and neglect the f_z-integral in eq. (14):

$$\frac{d^2}{dt^2}\langle s \rangle = \frac{2}{M} \iint ds\, dz\, p \tag{15}$$

$$= 2 \langle p / \rho s \rangle$$

$$= 2 c^2 \langle s^{-1} \rangle$$

In the last expression the Newtonian speed of sound c has been assumed uniform.

If we approximate $\langle s^{-1} \rangle$ by $\langle s \rangle^{-1}$ we arrive at the result derived under more restricted assumptions by E. Remy [2].

It is particularly noteworthy that the acceleration of the meridional centre of mass is independent of the local values of the ratio $p : B^2/2\mu$, but depends only on the geometry and (for a given composition of the plasma) on the temperature. If we assume that at some time the meridional centre of mass is at rest at a distance S from the major axis of the torus (i.e. $\langle s \rangle = S,\ dS/dt = 0$), then it takes a time τ to move to the slightly larger value $S + \delta S$, where

$$\tau = \frac{1}{c} (S \cdot \delta S)^{1/2} \quad , \quad S \gg \delta S > 0 \,. \tag{16}$$

In the present approximation, the results do not even depend on the aspect ratio. Its influence can be estimated by combining the model of eq. (10) with eq. (15), resulting in

$$\frac{d^2}{dt^2} \langle s \rangle = \frac{2c^2}{\langle s \rangle} \left(1 + \frac{1}{4} A^2\right) \quad ; \; A \ll 1 \tag{17}$$

or

$$\frac{d^2}{dt^2} \left\{ S_c \left(1 + \frac{1}{4} A^2\right) \right\} = 2c^2 S_c^{-1} \quad ; \; A \ll 1 \tag{18}$$

Though the factor 1/4 in front of A^2 can not be considered to be precise, it is obvious that the aspect ratio has indeed generally only a very minor influence on the outward acceleration of the pinch.

5) In the preceding section we have assumed

$$|\langle f_z \, z/\rho s \rangle| \ll \langle p/\rho s \rangle$$

and therefore neglected the terms containing f_z. This assumption is especially questionable when the plasma undergoes a fast compression or expansion; more precisely it is the deviation from equilibrium and therefore the acceleration of the compression (or expansion) which matters. We discuss this effect by considering a model of sufficient adaptability. We assume that the spatial variations of p and B are, with two time dependent parameters ($s_0 B_0$ and $s_0^2(1+\alpha)$) related by

$$\frac{1}{2}\frac{s^2}{s_0^2} B^2 + (1+\alpha)\, p = \frac{1}{2} B_0^2 \; . \tag{19}$$

Consequently:

$$\underset{\sim}{f} = -\left(1 - (1+\alpha)\frac{s_0^2}{s_2}\right) \underset{\sim}{\nabla} p \tag{20}$$

If s_0 is arbitrarily chosen to be close to $\langle s \rangle$, then the choice $\alpha = 0$ gives as nearly an approximation to static equilibrium as is possible. This is evident on inspection of eq. (20); the force vanishes on the cylinder $s = s_0$, it is parallel to $\underset{\sim}{\nabla} p$, if $s < s_0$; and antiparallel if $s > s_0$. $\alpha > 0$ corresponds to an accelerated compression, $\alpha < 0$ to a decelerated compression (or to an accelerated expansion), for $\alpha \leq -1$ the plasma is not confined at all by the magnetic field.

The "ansatz" (19) or (20) should be sufficiently versatile (besides the two independent parameters, one of the functions p or B can be freely chosen) to approximate sufficiently most configurations of actual interest.

To render these considerations more precise, we define appropriate averages for the acceleration of compression in axial direction (a_z) and in lateral direction (a_s) by

$$a_z = \frac{\langle (z - \langle z \rangle) \cdot \dot{v}_z \rangle}{(\langle z^2 \rangle - \langle z \rangle^2)^{1/2}} \qquad ; \; \dot{\underset{\sim}{v}} = \frac{d\underset{\sim}{v}}{dt} \qquad a_s = \frac{\langle (s - \langle s \rangle) \cdot \dot{v}_s \rangle}{(\langle s^2 \rangle - \langle s \rangle^2)^{1/2}} \tag{21}$$

Assuming eq. (19), $p/\rho = c^2$ (c uniform), $\langle z \rangle = 0$, and choosing (without loss of generality)

$$s_0^2 = \langle s^{-2} \rangle^{-1} \tag{22}$$

we obtain

$$a_z = - c^2 \alpha \langle z^2 \rangle^{-1/2}$$

(23)

$$a_s = c^2 \frac{2 - \langle s \rangle (\langle s^{-1} \rangle + (1+\alpha) \langle s^{-2} \rangle^{-1} \cdot \langle s^{-3} \rangle)}{(\langle s^2 \rangle - \langle s \rangle^2)^{1/2}}$$

We use the simple model of eq. (10) to define the (minor) radius of the plasma R and the aspect ratio A and express the different averages by these quantities up to the order A^2 :

$$a_z = - \frac{2c^2}{R} \alpha$$

(24)

$$a_s = - \frac{2c^2}{R} (A^2 + \alpha (A^2 + \alpha (1 + \frac{3}{4} A^2))) \; ; A \ll 1$$

In our model - and presumably in reality as well under normal circumstances - a sufficiently fast compression ($\alpha \gg A^2$) will be essentially isotropic, while near equilibrium ($|\alpha| \lesssim A^2$) the lateral compression due to the outward drift becomes noticeable. During a fast pinch discharge, α will be first larger than 1 (but not by orders of magnitude), then it will through values $0 > \alpha > -1$ approach values $|\alpha| \ll 1$, possibly with superimposed oscillations.

It remains to derive the equation of motion of the meridional centre of mass for our present model. Using eq. (13) or (14) and eq. (19) with (22), one obtains

$$\frac{d^2}{dt^2} \langle s \rangle = c^2 \left\{ \langle s^{-1} \rangle + (1 + \alpha) \langle s^{-2} \rangle^{-1} \langle s^{-3} \rangle \right\} . \tag{25}$$

or introducing A in the same fashion as before:

$$\frac{d^2}{dt^2} \langle s \rangle = \frac{2c^2}{\langle s \rangle} \left(1 + \frac{1}{2} A^2 + \alpha \left(\frac{1}{2} + \frac{3}{8} A^2\right)\right) ; \; A \ll 1 \tag{26}$$

Of course, both coefficients of A^2 cannot be considered to be reliable, least of all $\frac{3}{8}\alpha$. We therefore omit this correction term. Employing the approximate relation $a_z = \frac{1}{2}\ddot{R}$ of the appendix, we re-write eq. (26) as ($S = \langle s \rangle$)

$$\begin{aligned} \frac{d^2}{dt^2} \left(S + \frac{1}{4} R\right) &= \frac{2c^2}{S} \left(1 + \frac{1}{2} A^2\right) \\ &= \frac{d^2}{dt^2} S \left(1 + \frac{1}{4} A\right) \end{aligned} \tag{27}$$

This form shows clearly that the correction to eq. (15) stemming from the compression is small though larger than the "geometrical" corrections considered in eq. (17) (since it is of relative order A^1).

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Appendix

Proof of eq. (7)

We introduce $q(s,z)$ by

$$2q = B^2 s^2 - B_o^2 s_o^2 .$$

Then $q \equiv 0$ where $p \equiv 0$ (eq. (2)), and

$$\underset{\sim}{f} = -\frac{1}{\mu_o s^2} \underline{\nabla} q - \underline{\nabla} p .$$

Integration by parts yields

$$\iint\limits_{p \not\equiv 0} ds\, dz\, f_s \cdot s = \iint\limits_{p \not\equiv 0} ds\, dz\, (p - q/s^2 \mu_o)$$

$$\iint ds\, dz\, f_z \cdot z = \iint ds\, dz\, (p + q/s^2 \mu_o)$$

This proves the assertion.

In our present notation, the model of eq. (19) is equivalent to the assumption:

$$q = -s_o^2 (1+\alpha) p$$

In contrast, eq. (15) would certainly be precise if

$$q = -s^2 p .$$

Here the contribution to the acceleration of the centre of mass due to the "diamagnetic" effects (represented by q) and that directly due to the pressure are exactly the same.

Proof of eq. (12)

All integrals are to be extended over the region where $\rho \neq 0$

$$\frac{d}{dt} \langle \underline{r} \rangle = \frac{d}{dt} \frac{1}{M} \iint ds\, dz\, s\, \underline{r}\, \rho$$

$$= \frac{1}{M} \iint ds\, dz\, s\, \underset{\sim}{r} \frac{\partial \rho}{\partial t}$$

(since $dM/dt = 0$ by conservation of mass)

$$= -\frac{1}{M} \iint ds\,dz\, s\, \underset{\sim}{r} \left(\frac{\partial}{\partial z} \rho v_z + \frac{1}{s} \frac{\partial}{\partial s} \rho s v_s \right)$$

(equation of continuity)

$$= \frac{1}{M} \iint ds\, dz\, s\, \rho\, \underset{\sim}{v}$$

(integration by parts)

$$= \langle \underset{\sim}{v} \rangle .$$

Similarly:

$$\frac{d}{dt} \langle \underset{\sim}{v} \rangle = \frac{d}{dt} \frac{1}{M} \iint ds\, dz\, s\, \underset{\sim}{v}\, \rho$$

$$= \frac{1}{M} \iint ds\, dz\, s\, \frac{\partial}{\partial t} (\rho\, \underset{\sim}{v})$$

$$= \frac{1}{M} \iint ds\, dz\, s \left\{ \rho \frac{d\underset{\sim}{v}}{dt} - \underset{\sim}{v} (\nabla \cdot \rho \underset{\sim}{v}) - \rho (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{v} \right\}$$

$$= \frac{1}{M} \iint ds\, dz\, s\, \frac{d\underset{\sim}{v}}{dt}\, \rho$$

$$= \left\langle \frac{d\underset{\sim}{v}}{dt} \right\rangle$$

<u>Remark regarding eq. (21)</u>

It may seem more natural to take instead of eq. (21) the following measure of acceleration

$$\frac{d^2}{dt^2} \langle \delta z^2 \rangle^{1/2}$$

$$= a_z + \langle \delta z^2 \rangle^{-3/2} \cdot \left\{ \langle \delta z^2 \rangle \langle v_z^2 \rangle - \langle \delta z \cdot v_z \rangle^2 \right\}$$

($\delta z = z - \langle z \rangle$), and correspondingly in the s -direction. This measure implies some knowledge of the spatial distribution of $\underset{\sim}{v}$ (not only of $\dot{\underset{\sim}{v}}$). In the simplest possible case in which $v_z \propto \delta z$ it reduces however to a_z . This is, for instance, the case when the ring model of eq. (10) is to hold identically in time; here the relation

$$a_z = \frac{1}{2} \ddot{R} \qquad \text{holds, or}$$

$$\alpha = -\frac{1}{4} \frac{R \ddot{R}}{c^2}$$

[1] L. Biermann and A. Schlüter, Z. Naturf. 12a, 805 (1957)

[2] E. Remy, MPI-PA 62/1