ION WAVES IN AN INHOMOGENEOUS PLASMA

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IPP 2/33 December 1963

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ION WAVES IN AN INHOMOGENEOUS PLASMA

N. D'Angelo

I. Introduction

Low-frequency oscillations (10 to 30 kc/sec) have been observed [1] in cesium and potassium thermal plasmas, produced by surface ionization of neutral atoms on a hot tungsten plate. The oscillations were interpreted [2] as low-frequency ion waves propagating azimuthally in the plasma column, perpendicularly to both the magnetic lines and the existing density gradients. The phase velocity of the waves has the same magnitude of the ion (or electron) macroscopic velocity, associated with the diamagnetic current, which must flow in the azimuthal direction for the plasma to be confined.

The calculations of ref. (2) considered only waves with a propagation vector exactly perpendicular to the magnetic lines. Furthermore, the effect of ion or electron drifts along the magnetic lines was not included.

In the present paper we extend those calculations to waves propagating at an arbitrary angle to the direction of the magnetic field $\underline{B}_o$ which, again, is perpendicular to the density gradient existing in the plasma. The effect of ion and electron drifts along the magnetic lines is also considered.

II. Theory

We shall use the Boltzmann moment equations for the ions and the electrons and make use of a Cartesian geometry. The uniform and constant magnetic field $\underline{B}_o$ is in the direction of the positive z axis. For the zero-order density distribution of ions and electrons we shall assume $n_o(x) = \bar{n}_o\, e^{-\lambda x}$, with $\bar{n}_o$ and λ constant. We shall consider waves of frequency small enough that charge quasi-neutrality is preserved and neglect throughout Poisson's equation. The ß of the plasma (ratio of material to magnetic pressure) is assumed to be so small that the approximation of a uniform and time-constant magnetic field can be made.

We shall consider first the equations for the ions and then obtain those for the electrons simply by changing the sign of the electric charge and the value of the mass.

The basic equations are:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\, \underline{v}_i) = 0$$
$$n\, m_i \frac{\partial \underline{v}_i}{\partial t} + n\, m_i\, \underline{v}_i \cdot \nabla \underline{v}_i + \varkappa T \cdot \nabla n + q\, n\, \nabla \varphi - q\, n\, \underline{v}_i \times \underline{B} = 0. \qquad (1)$$

For the zero-order, steady state solution, one obtains:

$$n_o \nabla \cdot \underline{v}_{oi} + \underline{v}_{oi} \cdot \nabla n_o = 0$$
$$\underline{v}_{oi} \cdot \nabla \underline{v}_{oi} + c_i^2 \frac{\nabla n_o}{n_o} + \frac{q}{m_i} \nabla \varphi_o - \omega_{ci}\, \underline{v}_{oi} \times \hat{z} = 0 \quad , \qquad (2)$$

where $c_i^2 = \frac{\varkappa T_i}{m_i}$, $\omega_{ci} = \frac{qB_o}{m_i}$, and $\hat{z}$ is the unit vector in the positive z direction. We shall also call $\hat{x}$ and $\hat{y}$ the unit vectors in the positive x and y directions, respectively. For the zero-order ion macroscopic velocity we write $\underline{v}_{oi} = \mu_i \hat{x} + \eta_i \hat{y} + \zeta_i \hat{z}$ and assume that μ_i, η_i, and ζ_i are constant. Furthermore, the zero-order electric field is taken equal to zero everywhere.

The zero-order eqs. (2) give:

$$\mu_i = 0$$
$$\lambda c_i^2 + \omega_{ci}\, \eta_i = 0 \quad . \qquad (3)$$

Eq. (3_1) means that there is no ion macroscopic velocity in the direction of the density gradient. Eq. (3_2) states that, in the absence of any zero-order electric field, the ion pressure is entirely balanced by the ion diamagnetic current. For the first-order quantities (perturbation) we assume:

- 4 -

$$n_1 = \bar{n}_1 \, e^{-\lambda x} \, e^{i(K_\perp y + K_\parallel z - \omega t)}$$

$$\underline{v}_{i1} = e^{i(K_\perp y + K_\parallel z - \omega t)} \, \overline{\underline{v}_{i1}} \tag{4}$$

$$\varphi_1 = \bar{\varphi}_1 \, e^{i(K_\perp y + K_\parallel z - \omega t)}$$

with n_1 and $\bar{\varphi}_1$ constant. $K_\perp$ and $K_\parallel$ are the (real) components of the propagation vector in the y and z directions, respectively. For $\overline{\underline{v}_{i1}}$ we shall write $\bar{\underline{v}}_{i1} = \bar{v}_{i1x}\,\hat{x} + \bar{v}_{i1y}\,\hat{y} + \bar{v}_{i1z}\,\hat{z}$, with $\bar{v}_{i1x}$, $\bar{v}_{i1y}$, and $\bar{v}_{i1z}$ constant.

Inserting (4) into the first-order linearized equations obtained from (1), and making use of (3), one obtains four equations in five unknown, $\bar{n}_1$, $\bar{\varphi}_1$, $\bar{v}_{i1x}$, $\bar{v}_{i1y}$, and $\bar{v}_{i1z}$. By eliminating the three components of the velocity, one is finally reduced, with some algebra, to

$$\left[K_\parallel^2 \, c_i^2 - \Omega_i^2 + \left(\lambda - \frac{K_\perp}{\omega_{ci}} \Omega_i\right) \frac{\Omega_i K_\perp c_i^2 \omega_{ci}}{\omega_{ci}^2 - \Omega_i^2} \right] \nu \, +$$

$$+ \left[K_\parallel^2 \, c_i^2 + \left(\lambda - \frac{K_\perp}{\omega_{ci}} \Omega_i\right) \frac{\Omega_i K_\perp c_i^2 \omega_{ci}}{\omega_{ci}^2 - \Omega_i^2} \right] \bar{\psi}_{i1} = 0 \;, \tag{5}$$

where $\Omega_i = \omega - K_\perp \eta_i - K_\parallel \zeta_i$, $\nu = \dfrac{\overline{n_1}}{\overline{n_o}}$, and $\bar{\psi}_{i1} = \dfrac{q}{m_i c_i^2} \bar{\varphi}_1$.

By defining

$$A_i^2 = K_\parallel^2 \, c_i^2 + \left(\lambda - \frac{K_\perp}{\omega_{ci}} \Omega_i\right) \frac{\Omega_i K_\perp c_i^2 \omega_{ci}}{\omega_{ci}^2 - \Omega_i^2} \tag{6}$$

one obtains from eq. (5):

$$(A_i^2 - \Omega_i^2)\, \nu + A_i^2 \, \bar{\psi}_{i1} = 0 \;. \tag{7}$$

A similar equation holds for the electrons. With $\alpha = T_e/T_i$, it is $\bar{\psi}_{e1} = -1/\alpha \, \bar{\psi}_{i1}$. Therefore, one finally arrives at a system of two equations in two unknown, ν, $\bar{\psi}_{i1}$.

$$(A_i^2 - \Omega_i^2)\,\nu + A_i^2\,\bar{\psi}_{i1} = 0$$

$$(A_e^2 - \Omega_e^2)\,\nu + \frac{1}{\alpha}\,A_e^2\,\bar{\psi}_{e1} = 0 \quad . \tag{8}$$

The dispersion relation is obtained by setting equal to zero the determinant of the coefficients:

$$\begin{vmatrix} A_i^2 - \Omega_i^2 & A_i^2 \\ A_e^2 - \Omega_e^2 & -\frac{1}{\alpha}\,A_e^2 \end{vmatrix} = 0 \quad . \tag{9}$$

In the following we shall consider only the case $\alpha = 1$ ($T_i = T_e$) and take $\zeta_i = -\zeta_e$. It is convenient to use dimensionless quantities and measure frequencies in terms of ω_{ci}, lengths in terms of the ion gyroradius ρ_i, and velocities in terms of c_i. We shall set:

$$\begin{aligned} x &= \omega_r/\omega_{ci} \\ y &= \omega_i/\omega_{ci} \\ \alpha_\perp &= K_\perp\,\rho_i \\ \alpha_\parallel &= K_\parallel\,\rho_i \\ \Lambda &= \lambda\,\rho_i \\ p &= \zeta_i/c_i \end{aligned} \tag{10}$$

The dispersion relation (9) becomes:

$$N_e\,(N_i - \xi_i^2) + N_i\,(N_e - \xi_e^2) = 0 \tag{11}$$

with

$$\begin{aligned} \xi_i &= (x + \alpha_\perp\Lambda - \alpha_\parallel p) + i\,y \\ \xi_e &= (x - \alpha_\perp\Lambda + \alpha_\parallel p) + i\,y \\ N_i &= \alpha_\parallel^2 + (\Lambda - \alpha_\perp\,\xi_i)\,\frac{\alpha_\perp\,\xi_i}{1-\xi_i^2} \\ N_e &= \alpha_\parallel^2\,\gamma - (\Lambda + \alpha_\perp\,\xi_e\,\frac{1}{\gamma})\,\frac{\alpha_\perp\,\xi_e}{1-\xi_e^2\,\frac{1}{\gamma^2}} \end{aligned} \tag{12}$$

and $\gamma = m_i/m_e$.

Eq. (11) has been solved numerically for a number of cases. The results are given in Figs. 1 to 23.

The quantity $x = \omega_r/\omega_{ci}$ is plotted versus $\alpha_\perp$, both for $\alpha_\perp$ positive and negative. For $y = 0$, the solution is given as a full curve. For complex conjugate solutions the result is plotted as a dotted line and the values of y indicated at a few points.

Only two values of γ ($\gamma = 1.8 \cdot 10^3$ and $\gamma = 2.3 \cdot 10^5$) have been considered, corresponding to hydrogen and cesium plasma, respectively. As a check on the numerical calculations, the case $\Lambda = 0$ (homogeneous plasma) has also been included. It will be noticed (Fig. 1) that one recovers the electrostatic cyclotron oscillations [3] and the ordinary ion waves propagating along the magnetic lines. For $\Lambda \neq 0$ and $\alpha_\parallel = 0$ one recovers the two types of waves considered in ref. (2), with dispersion relation $x \approx -\alpha_\perp\Lambda$ and $x \approx 1/\Lambda\,\alpha_\perp$ (Figs. 4 and 5). Another root also appears (Fig. 4) which corresponds to frequencies quite below those of the wave with $x \approx -\alpha_\perp\Lambda$. For the case of thermal cesium plasmas, this has frequencies below 1 kc/sec, if $|\alpha_\perp| < \sim 1$.

It is also interesting to note that, for $\alpha_\parallel = 0$ (propagation exactly perpendicular to the magnetic lines) the wave [2] with $x \approx -\alpha_\perp\Lambda$ exists for $\alpha_\perp < 0$ (wave moving "with the ions"). However, if $\alpha_\parallel \neq 0$, also a wave with $x \approx \alpha_\perp\Lambda$ and $\alpha_\perp > 0$ can exist (wave moving "with the electrons").

Examination of Figs. 1 to 23 will make clear a number of other interesting points, particularly when ion and electron drifts along the magnetic lines are included ($p \neq 0$). Figs. 1, 2, and 3 are for the case of a homogeneous plasma.

From Fig. 4 to Fig. 11 no drift of ions and electrons along the magnetic lines is assumed ($p = 0$). The effect of increasing $\alpha_\parallel$ from zero (exact perpendicular propagation) to $\alpha_\parallel = 10^{-3}$ is apparent.

In Figs. 12 to 23 the value of p is increased from a minimum value $p = 10^{-3}$ to a maximum value $p = 100$. Information is obtained as to the effect of ion and electron drifts along the magnetic lines. One must keep in mind, however, that the Boltzmann moment equations are not likely to give very reliable results in those cases in which the interaction between a wave and particles travelling near the phase velocity is of primary importance.

I wish to thank Mr. A. Jelic for performing the numerical calculations.

- 8 -
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Fig. 1

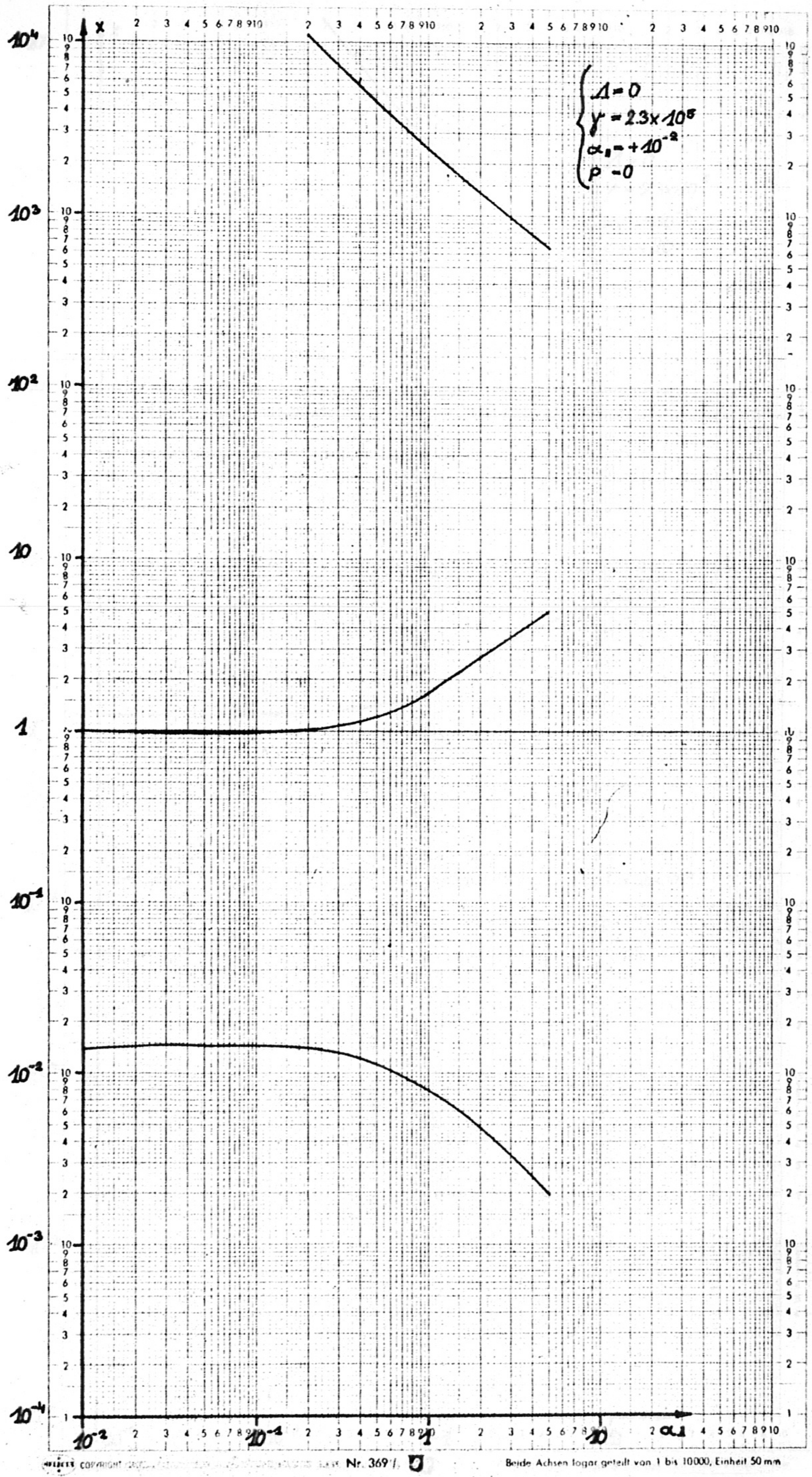

x
10^4
10^3
10^2
10
1
10^-1
10^-2
10^-3
10^-4
10^-2
10^-1
1
10
α1
Δ = 0
γ = 2.3 x 10^5
α2 = + 10^-2
p = 0
Nr. 369
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 2

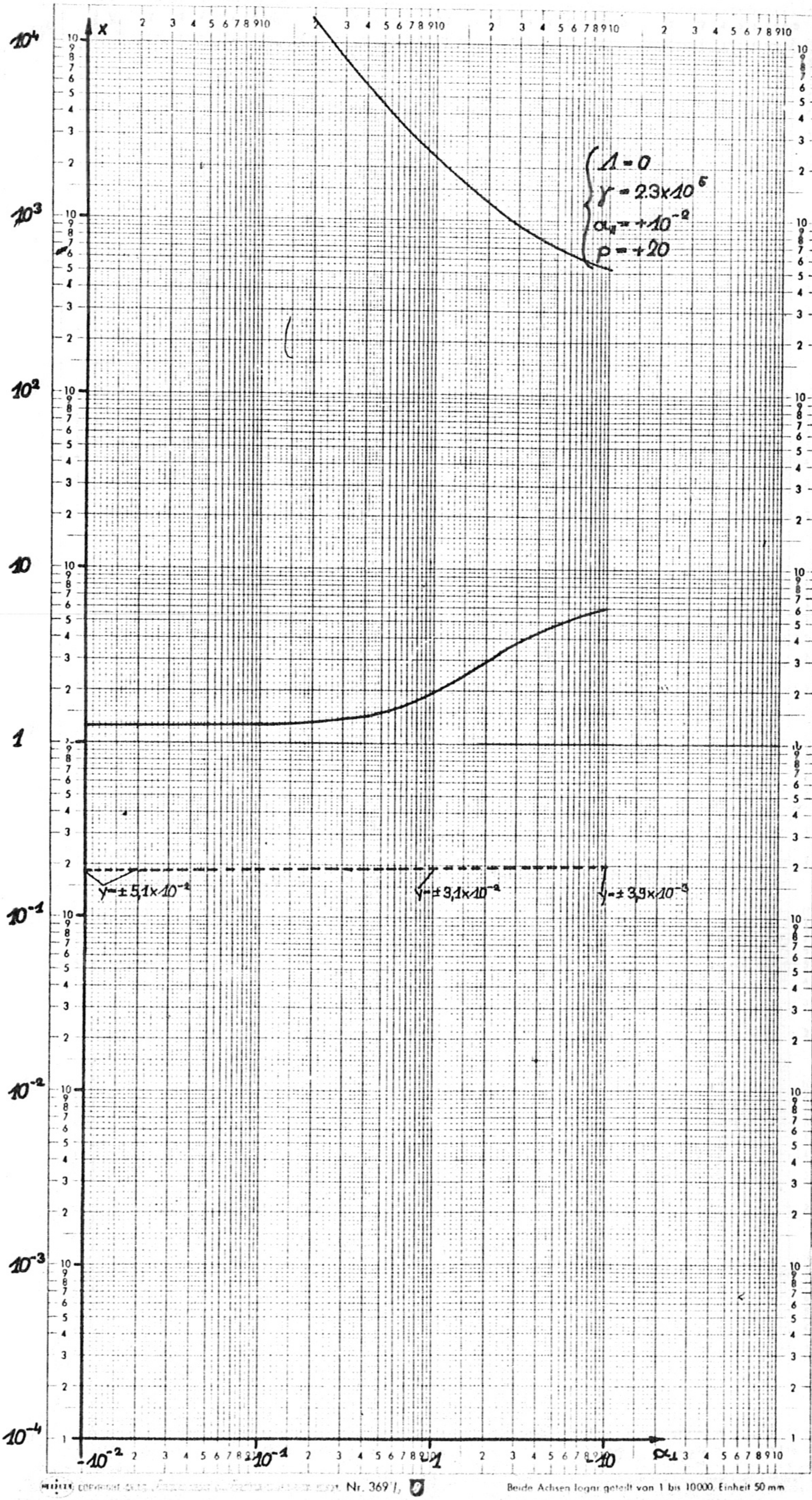

x
10⁴
10³
10²
10
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
-10⁻²
10⁻¹
1
10
α⊥
Λ = 0
γ = 2.3×10⁶
α∥ = +10⁻²
p = +20
γ = ±5,1×10⁻²
γ = ±3,1×10⁻²
γ = ±3,9×10⁻³
Nr. 369 ½
Beide Achsen logar. geteilt von 1 bis 10000. Einheit 50 mm

Fig. 3

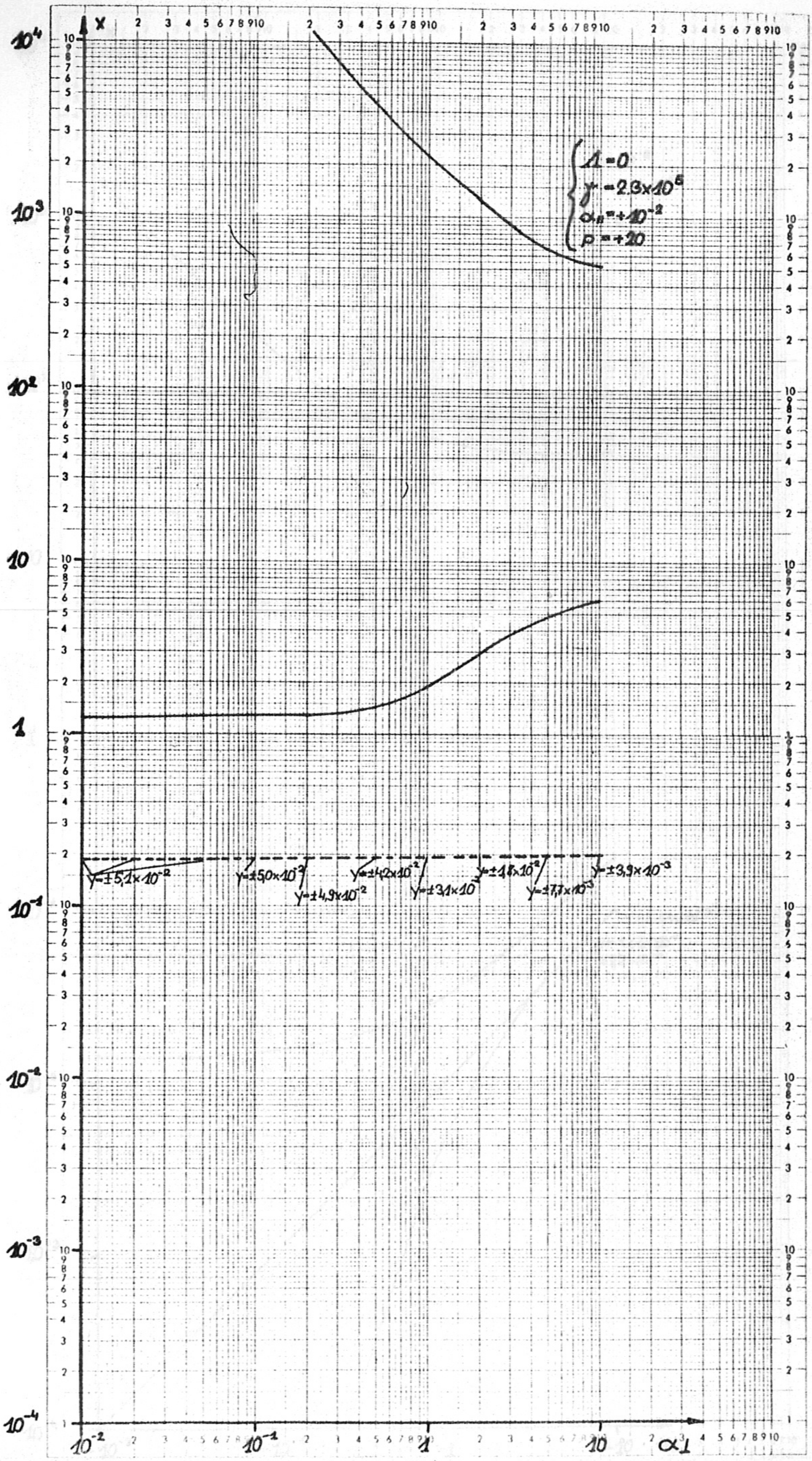
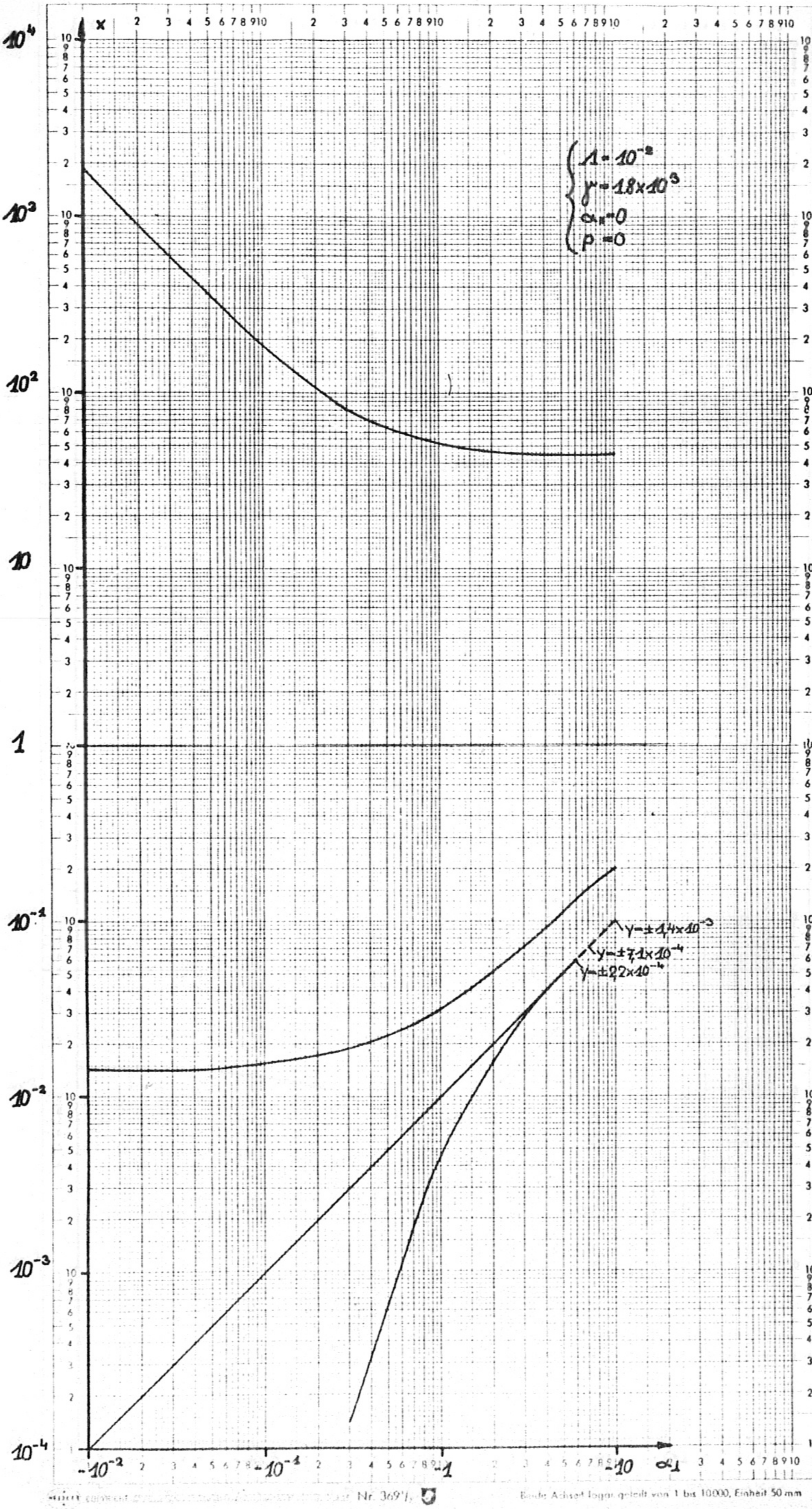
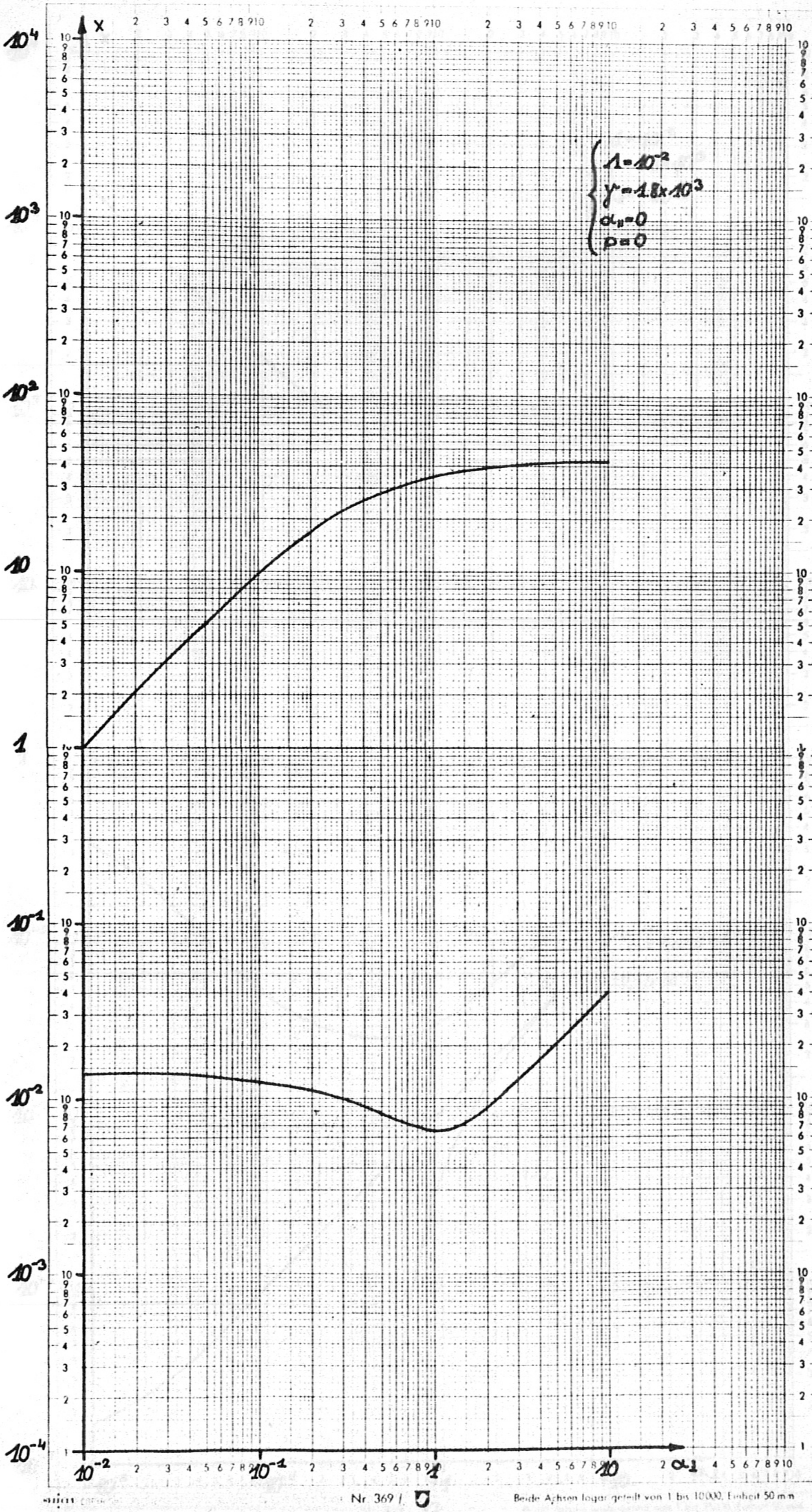

x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
1
10
$\alpha_\perp$
$\Lambda = 0$
$\gamma = 2.3 \times 10^5$
$\alpha_\parallel = +10^{-2}$
$p = +20$
$\gamma = \pm 5{,}1 \times 10^{-2}$
$\gamma = \pm 5{,}0 \times 10^{-2}$
$\gamma = \pm 4{,}9 \times 10^{-2}$
$\gamma = \pm 4{,}2 \times 10^{-2}$
$\gamma = \pm 3{,}1 \times 10^{-2}$
$\gamma = \pm 1{,}8 \times 10^{-2}$
$\gamma = \pm 7{,}7 \times 10^{-3}$
$\gamma = \pm 3{,}9 \times 10^{-3}$

Fig. 4



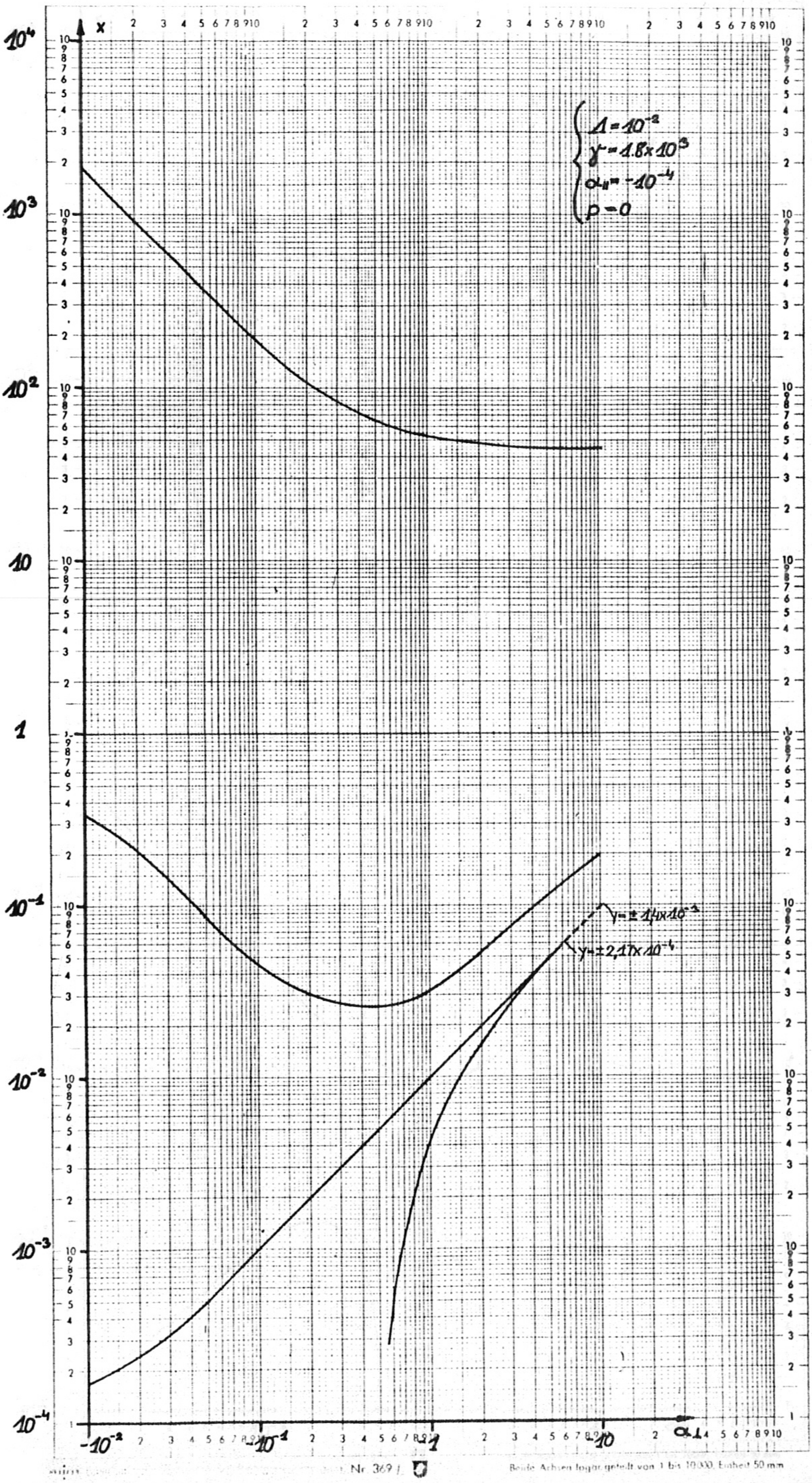
x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_x = 0$
$p = 0$
$y = \pm 1.4 \times 10^{-3}$
$y = \pm 7.1 \times 10^{-4}$
$y = \pm 2.2 \times 10^{-4}$
10^{-2}
10^{-1}
1
10
α_1
Nr. 369½
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 5



x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_{\parallel} = 0$
$p = 0$
10^{-2}
10^{-1}
1
10
$\alpha_{\perp}$

Fig. 6



x
10^4
10^3
10^2
10
1
10^-1
10^-2
10^-3
10^-4
Δ = 10^-2
γ = 1.8x10^3
α_∥ = -10^-4
p = 0
y = ± 1,4x10^-3
y = ± 2,17x10^-4
-10^-2
-10^-1
-1
-10
α_⊥
Nr. 369 L
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 7

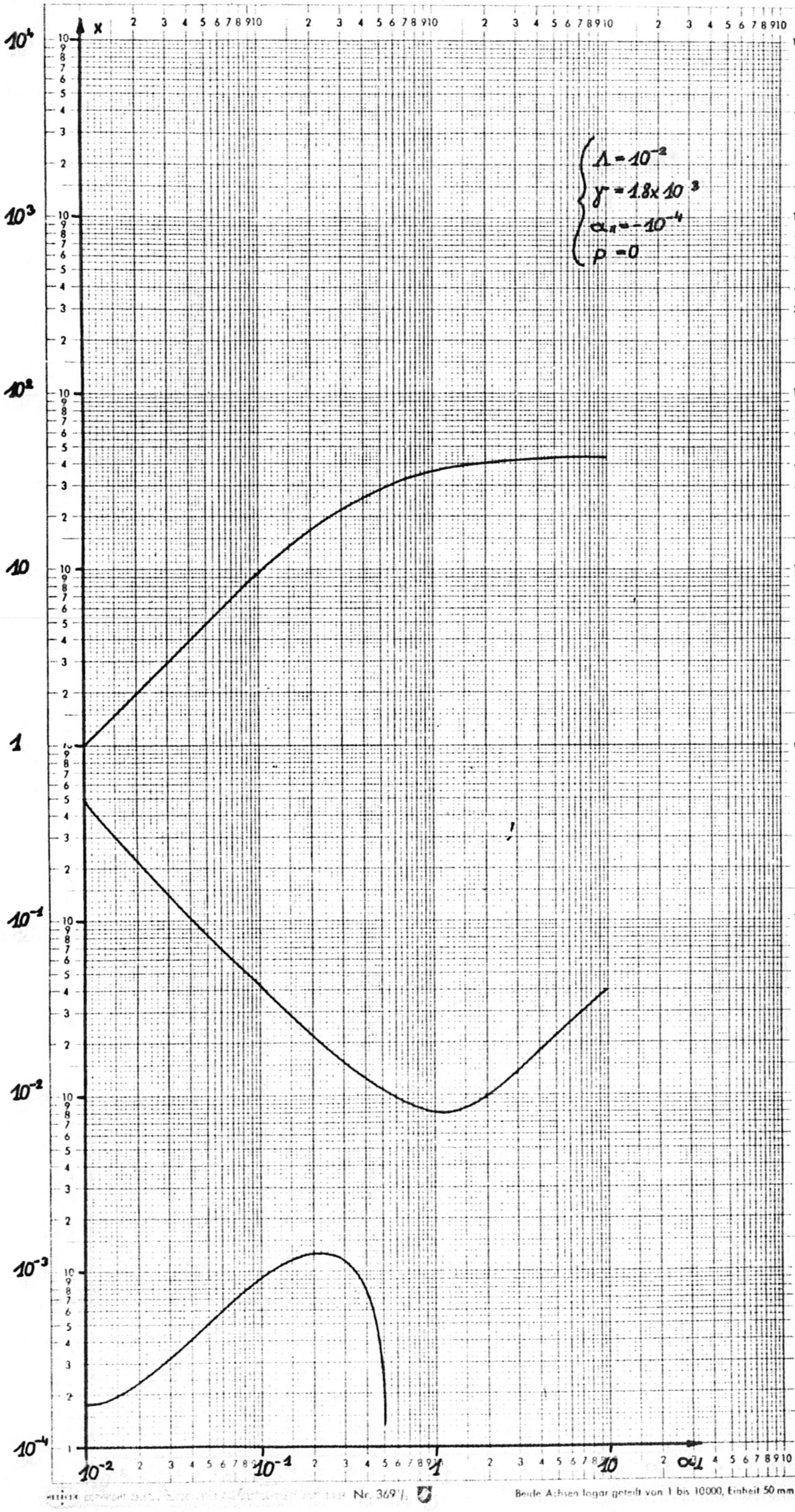
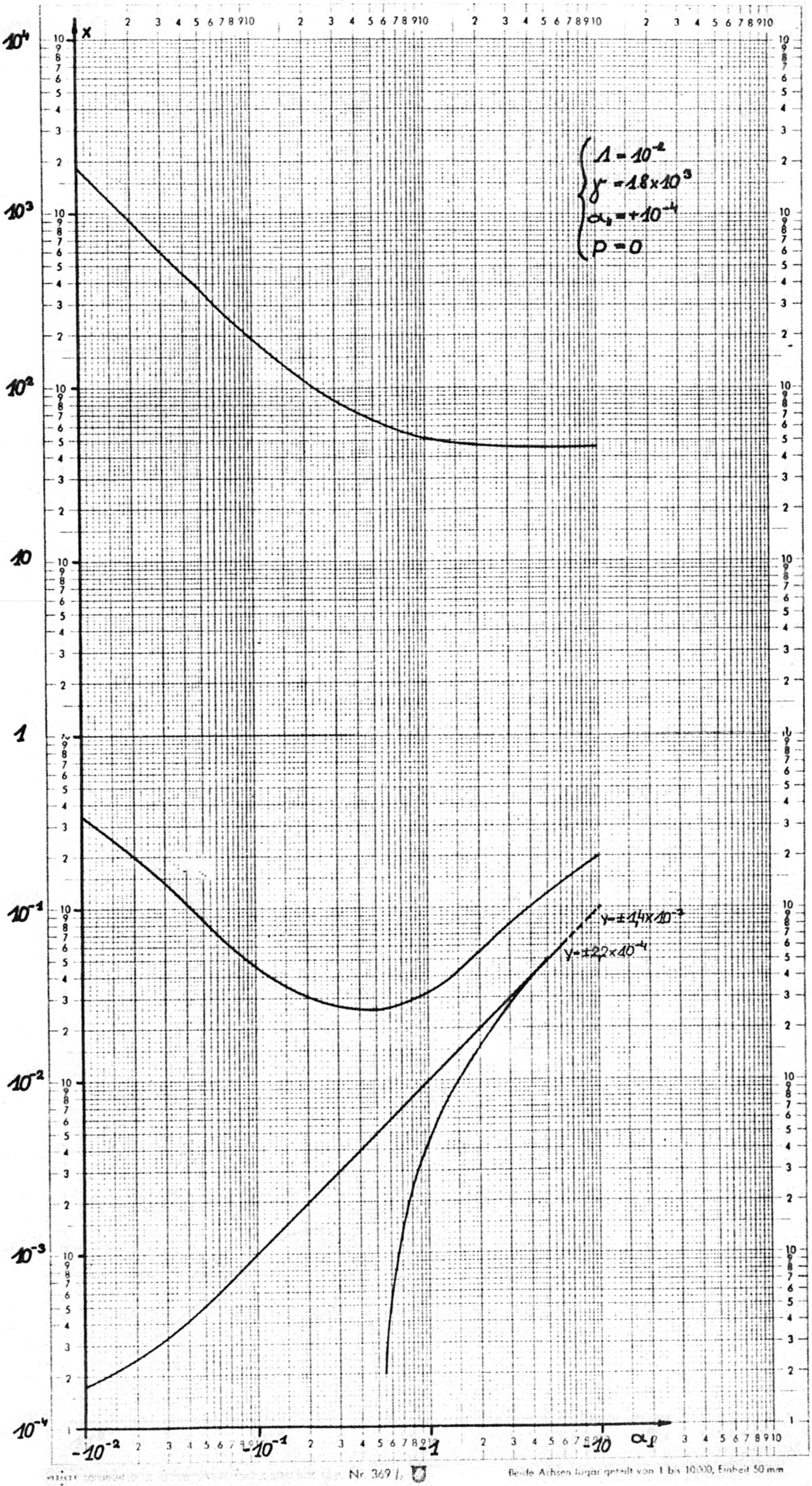
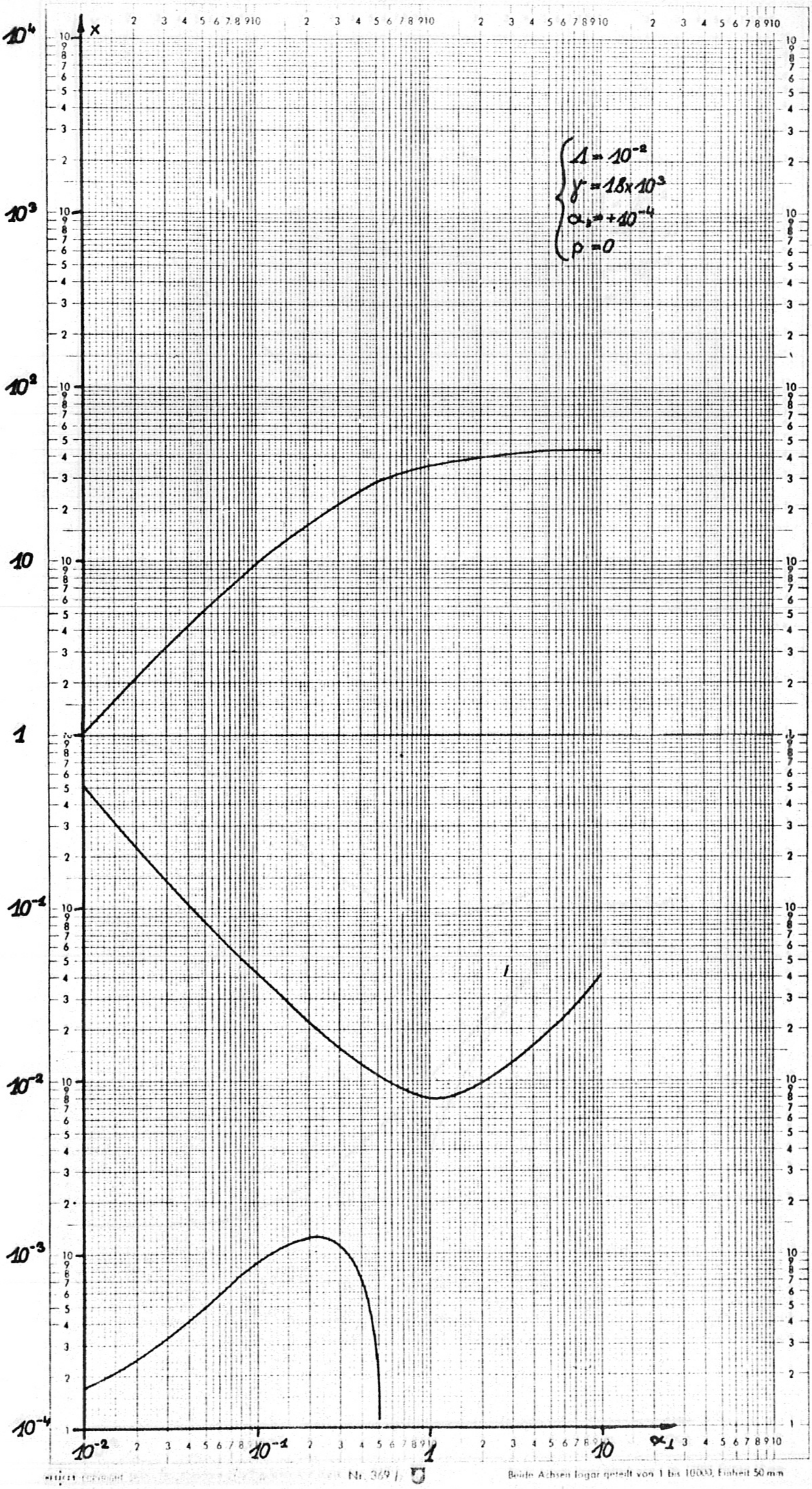

x
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^{3}$
$\alpha_1 = -10^{-4}$
$\rho = 0$
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
1
10
ω_L
Nr. 369½
Beide Achsen logar geteilt von 1 bis 10000, Einheit 50 mm

Fig. 8



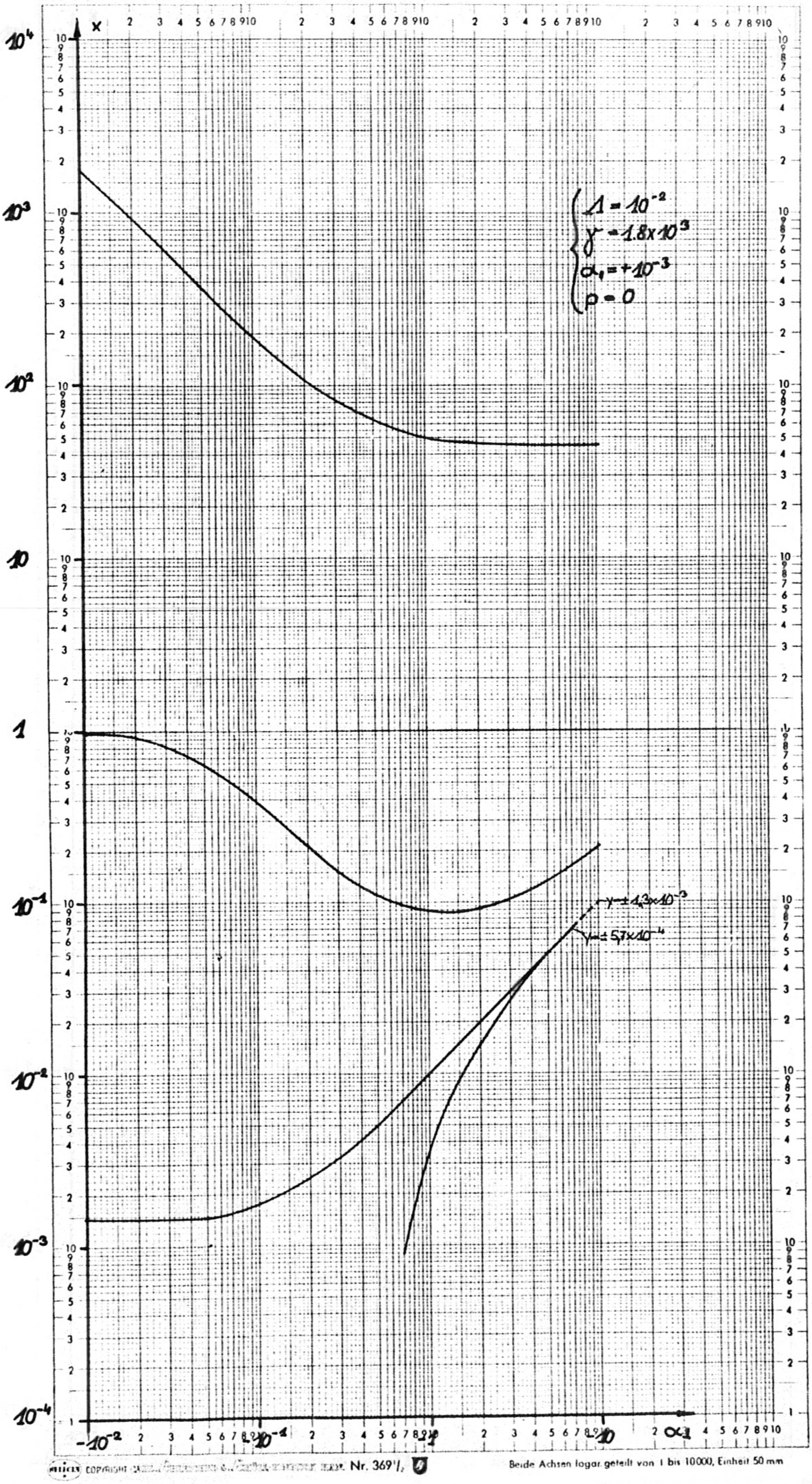
x
Λ = 10^-2
γ = 1.8×10^3
α_I = +10^-4
p = 0
y = ±1.4×10^-3
y = ±2.2×10^-4
α_I
Nr. 369 L
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 9



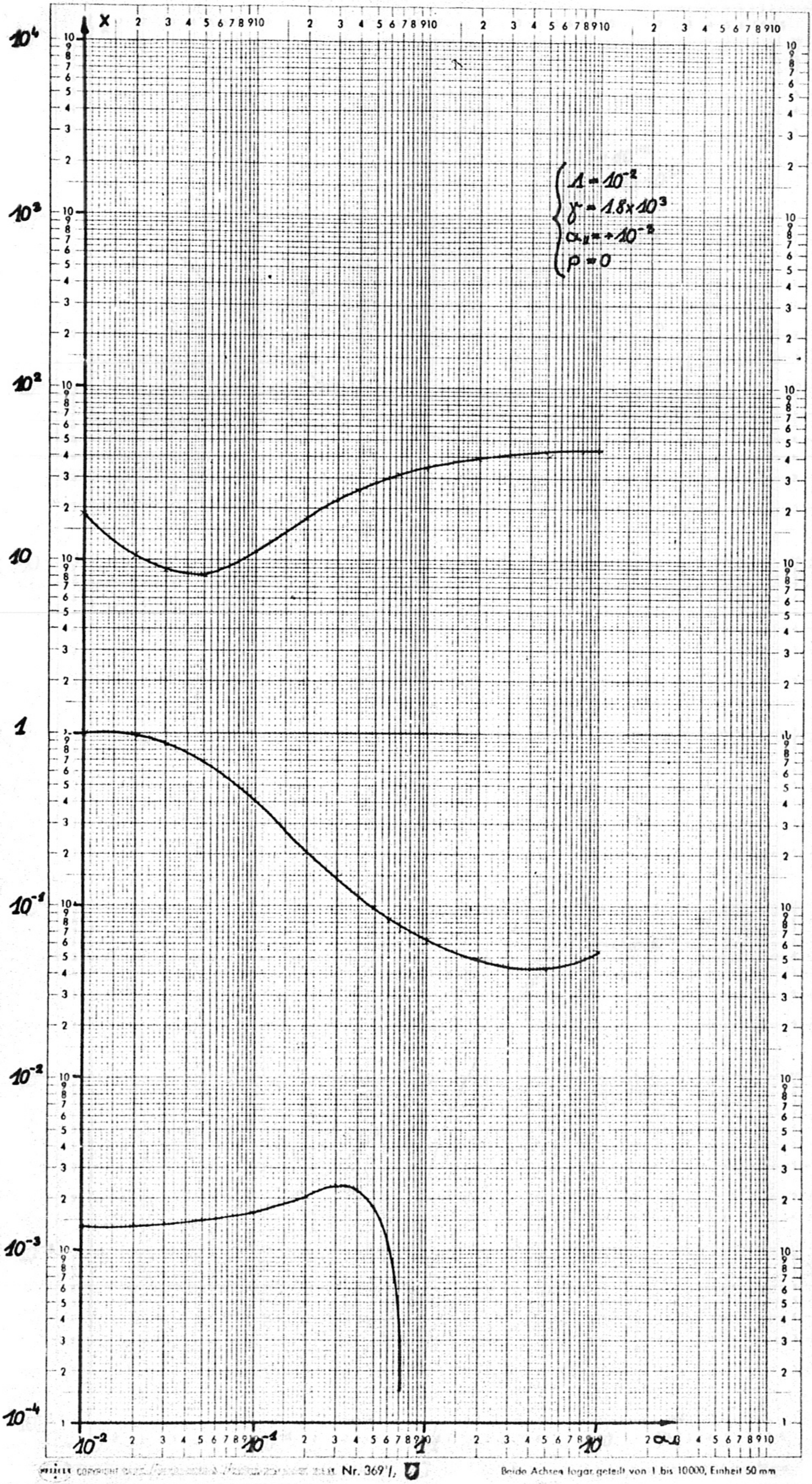
x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Delta = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_s = +10^{-4}$
$p = 0$
I
10^{-2}
10^{-1}
1
10
$\alpha_\perp$
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 10



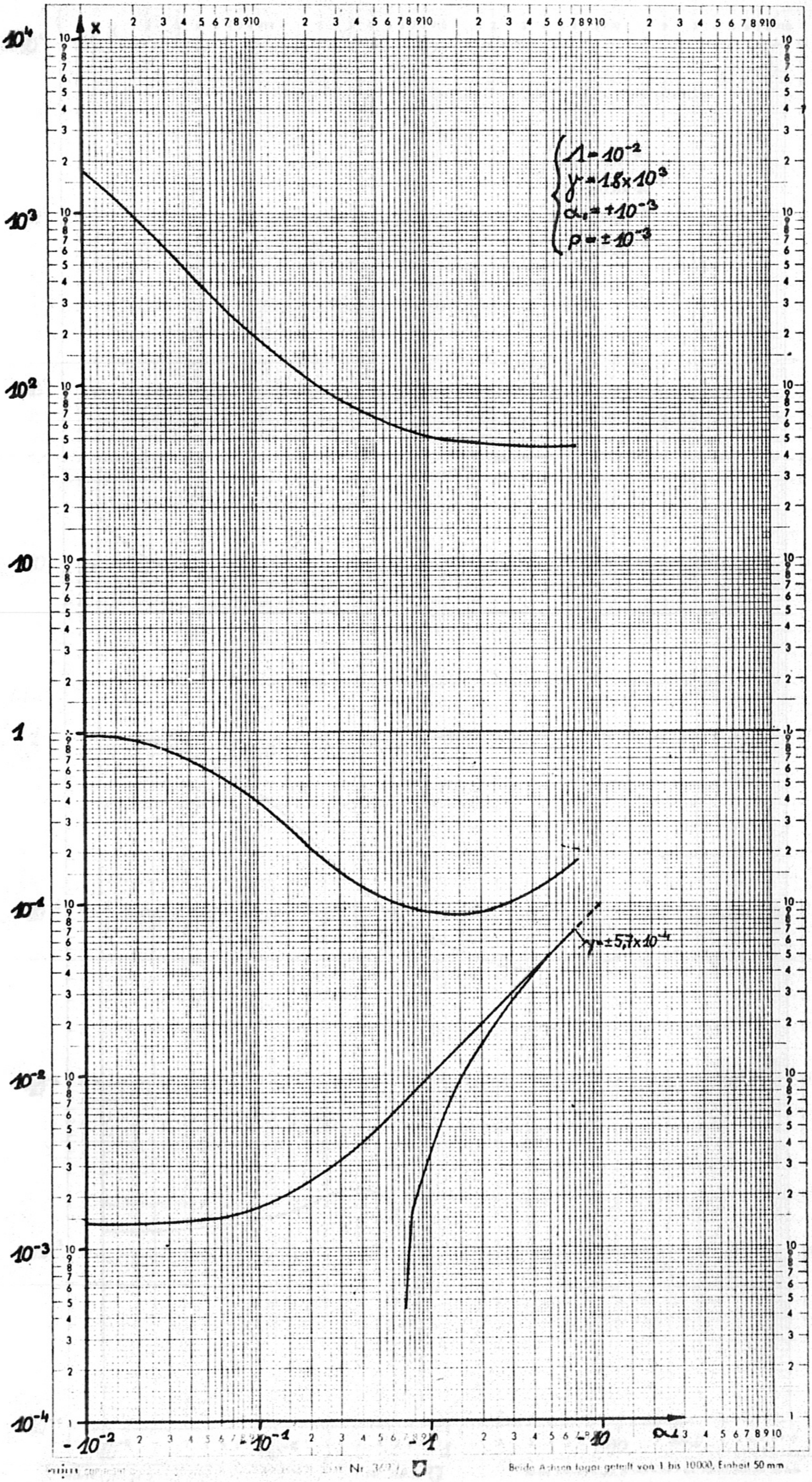
x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_1 = +10^{-3}$
$p = 0$
$\gamma = \pm 1.3 \times 10^{-3}$
$\gamma = \pm 5.7 \times 10^{-4}$
10^{-2}
10^{-1}
1
10
α
Nr. 369 1/2
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 11



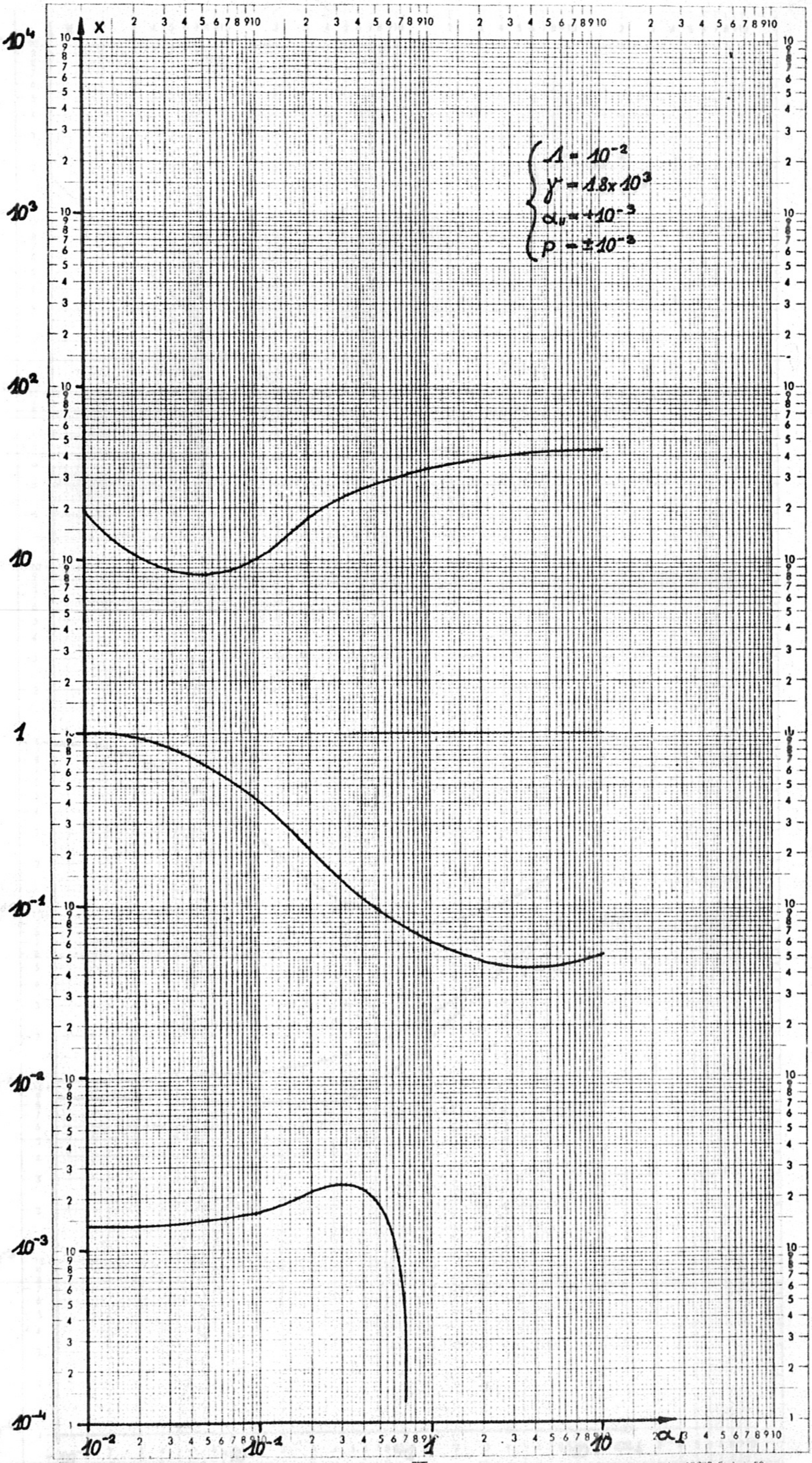
X
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_y = +10^{-5}$
$p = 0$
10^{-2}
10^{-1}
1
10
α_y
Nr. 369 1/
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 12



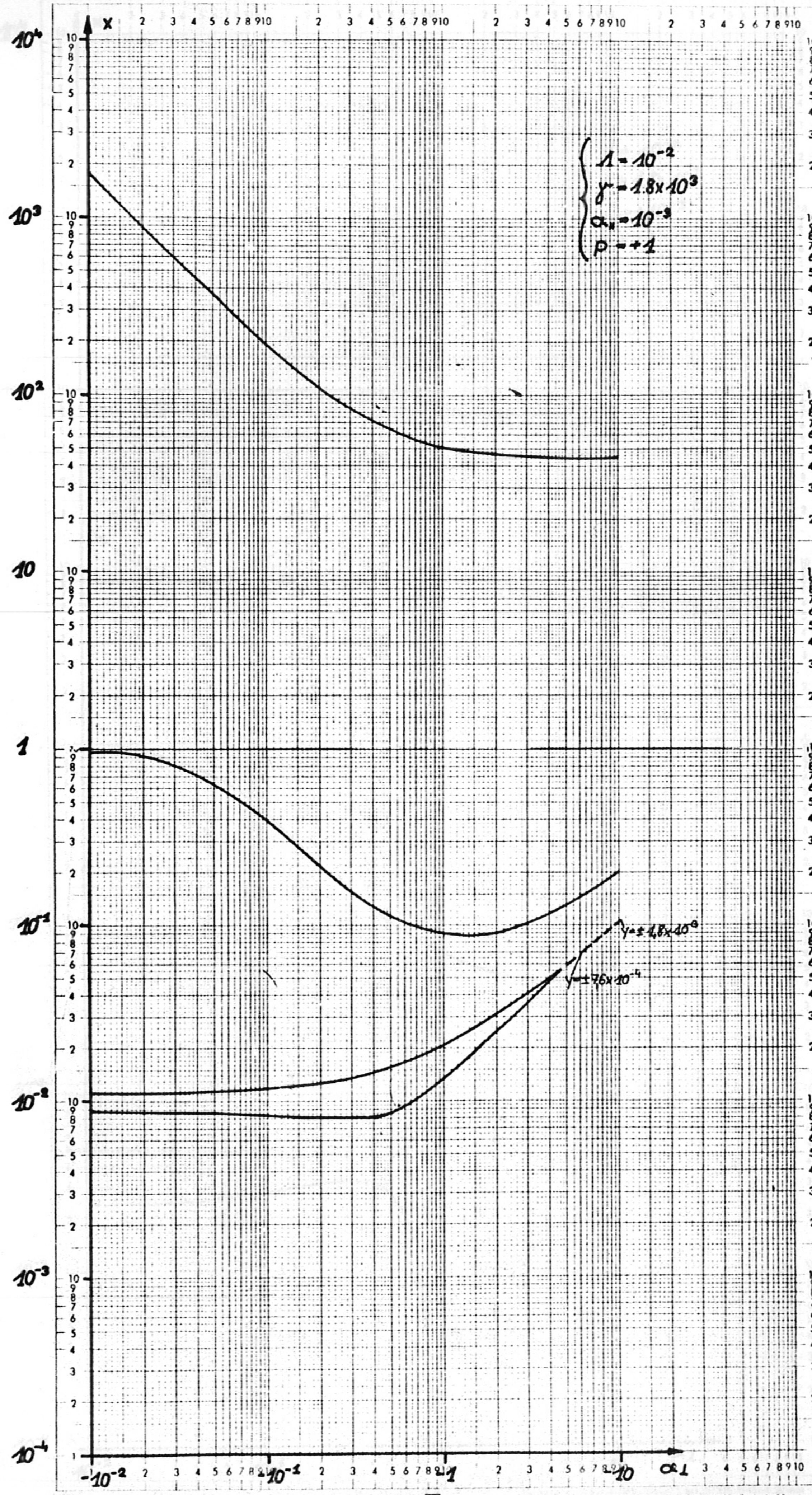
x
Λ = 10^-2
γ = 1,8 × 10^3
α₁ = ±10^-3
p = ±10^-3
γ = ±5,7×10^-4
10^4
10^3
10^2
10
1
10^-1
10^-2
10^-3
10^-4
- 10^-2
- 10^-1
- 1
- 10
α
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 13



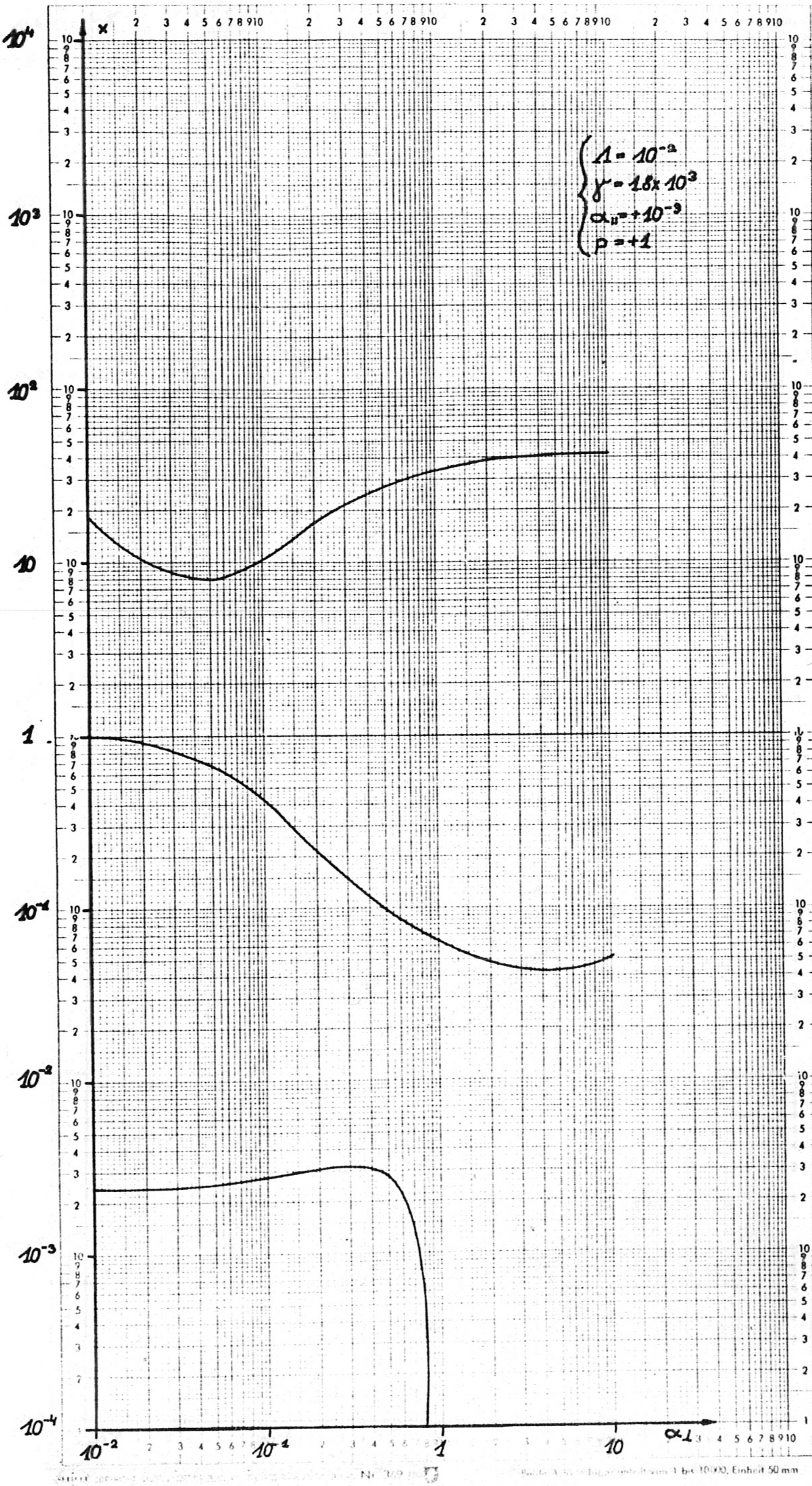
x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_\nu = +10^{-3}$
$P = \pm 10^{-3}$
10^{-2}
10^{-1}
1
10
α_1
Nr. 369 ½
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 14



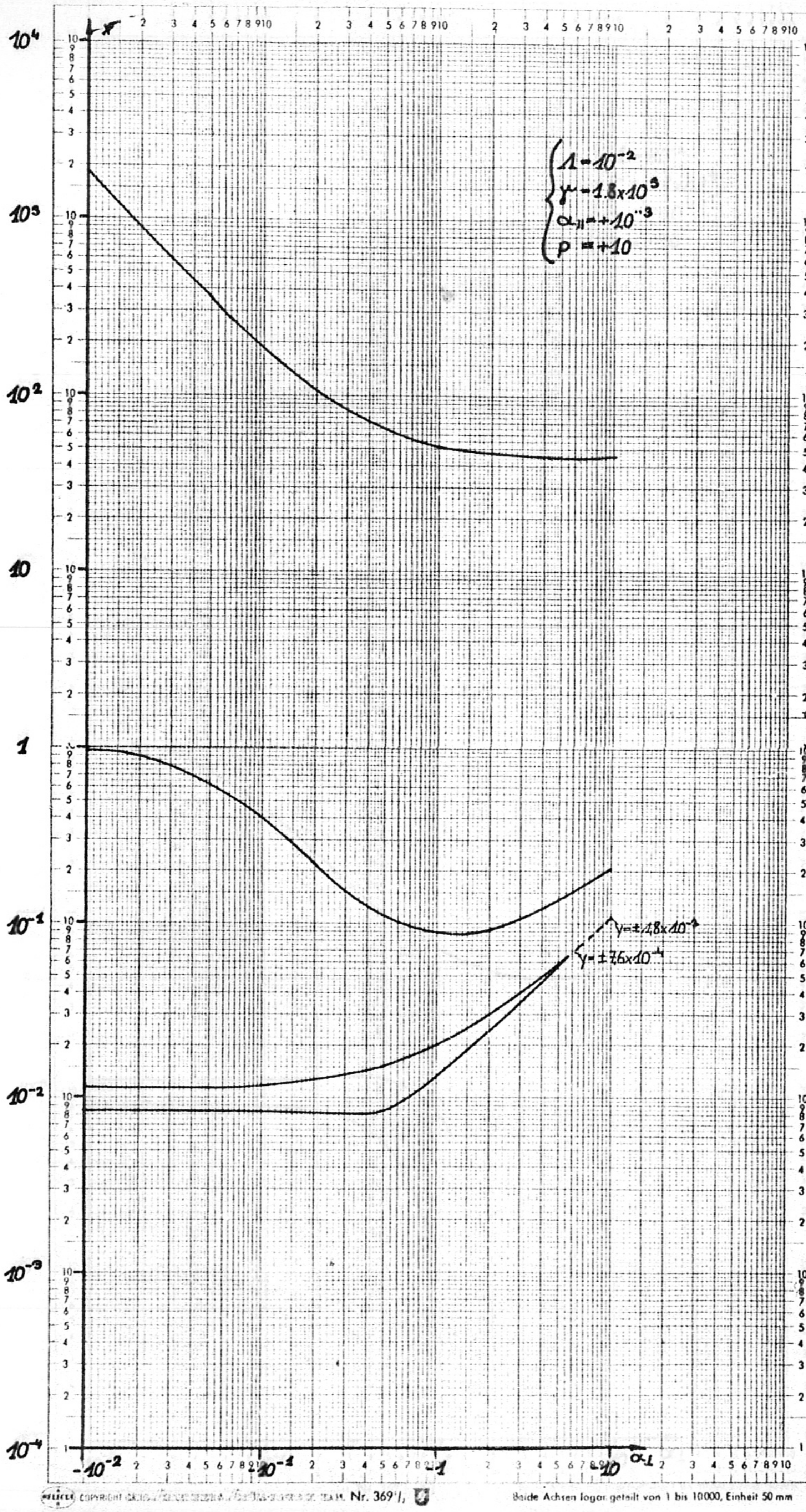
x
α
10⁴
10³
10²
10
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻²
10⁻¹
1
10
Λ = 10⁻²
γ = 1.8x10³
αₓ = 10⁻³
p = +1
y=±1,8x10
y=±1,76x10⁻⁴

Fig. 15



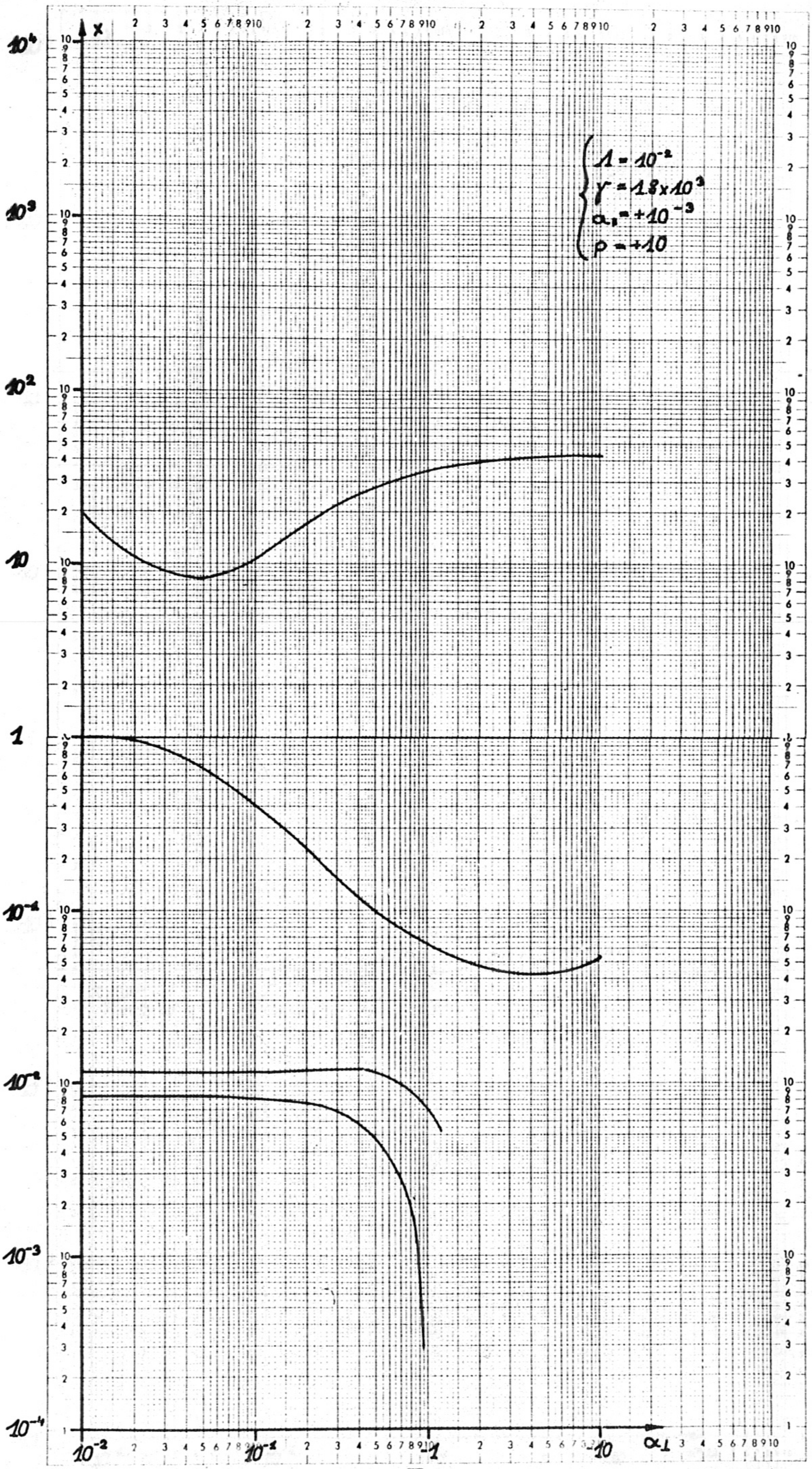
x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_{\parallel} = +10^{-3}$
$p = +1$
αl
10^{-2}
10^{-1}
1
10

Fig. 16



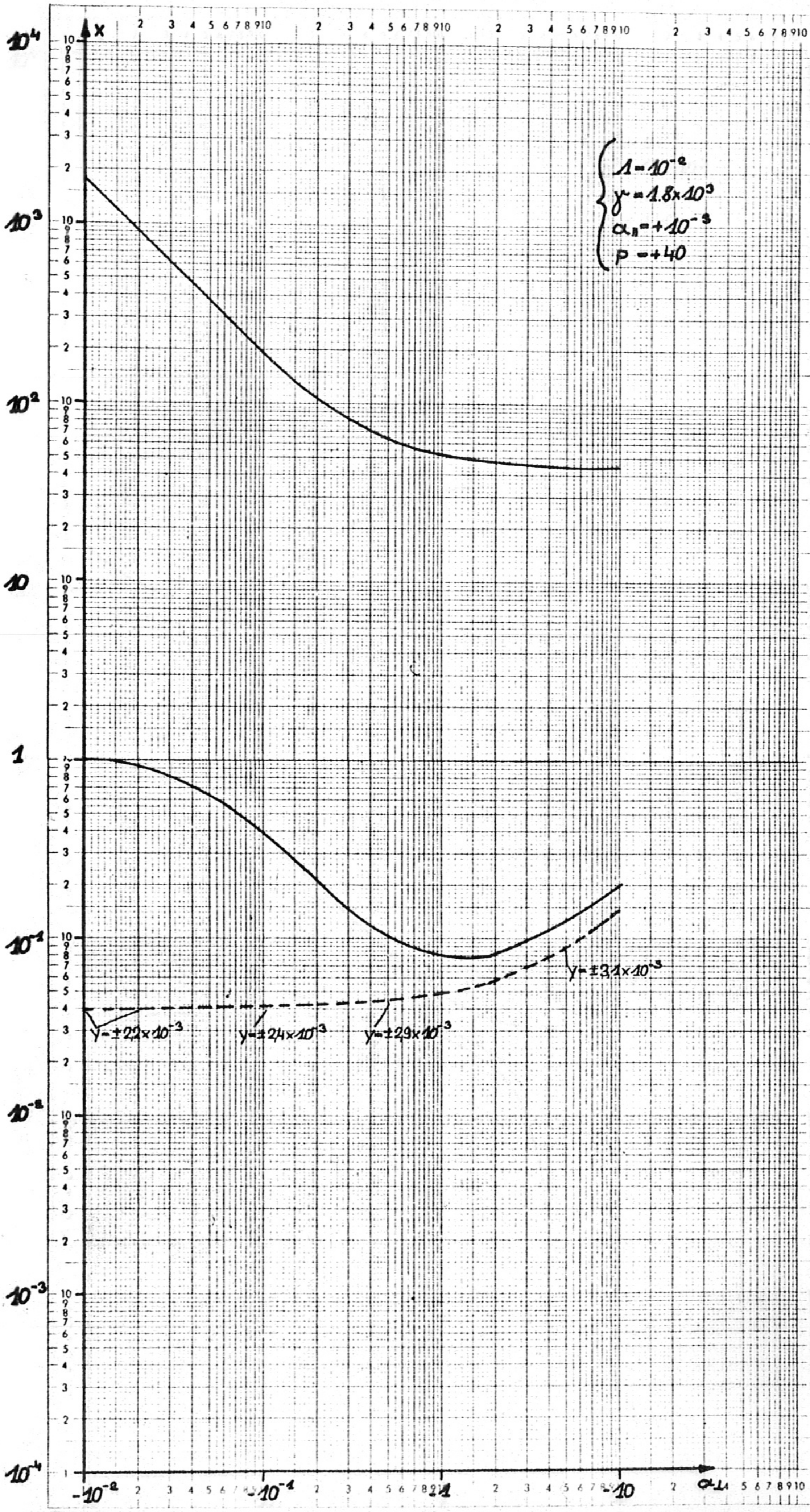
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^{3}$
$\alpha_{//} = +10^{-3}$
$p = +10$
$y = \pm 1.8 \times 10^{-4}$
$y = \pm 7.6 \times 10^{-4}$
-10^{-2}
-10^{-1}
-1
-10
$\alpha_\perp$
Nr. 369 ½
Beide Achsen logar geteilt von 1 bis 10000, Einheit 50 mm

Fig. 17



x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
1
10
$\alpha_\perp$
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_\parallel = +10^{-3}$
$\rho = +10$

Fig. 18



x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_\parallel = +10^{-3}$
$p = +40$
$\gamma = \pm 3.1 \times 10^{-3}$
$\gamma = \pm 2.2 \times 10^{-3}$
$\gamma = \pm 2.4 \times 10^{-3}$
$\gamma = \pm 2.9 \times 10^{-3}$
-10^{-2}
-10^{-1}
-1
-10
$\alpha_\perp$
Nr. 369 1/2
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 19

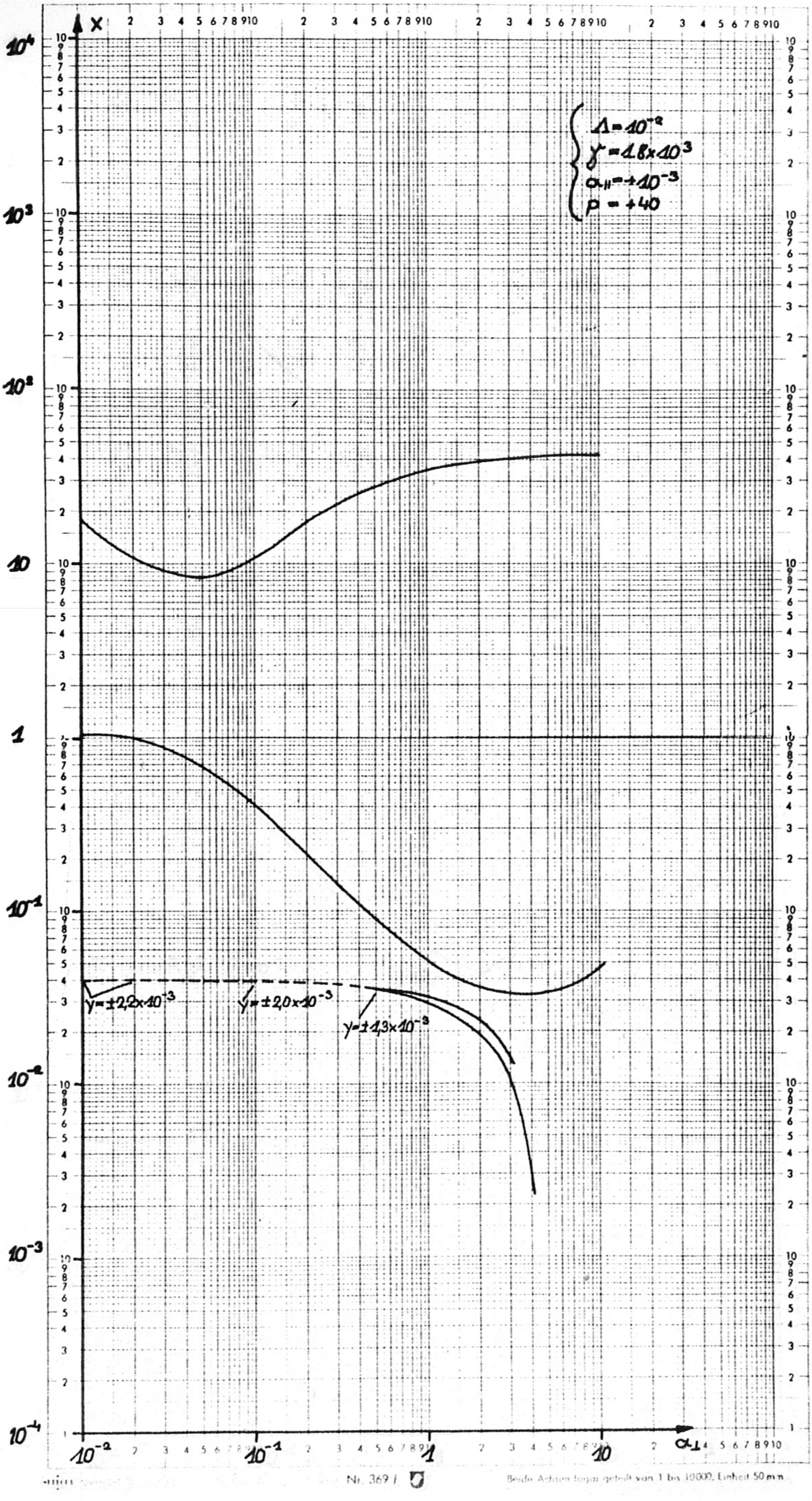

X
Λ = 10⁻²
γ = 1,8×10³
α∥ = +10⁻³
p = +40
y = ±2,2×10⁻³
y = ±2,0×10⁻³
y = ±1,3×10⁻³
α⊥
Nr. 369 I
Beide Achsen logar. geteilt von 1 bis 10000, Einheit 50 mm

Fig. 20

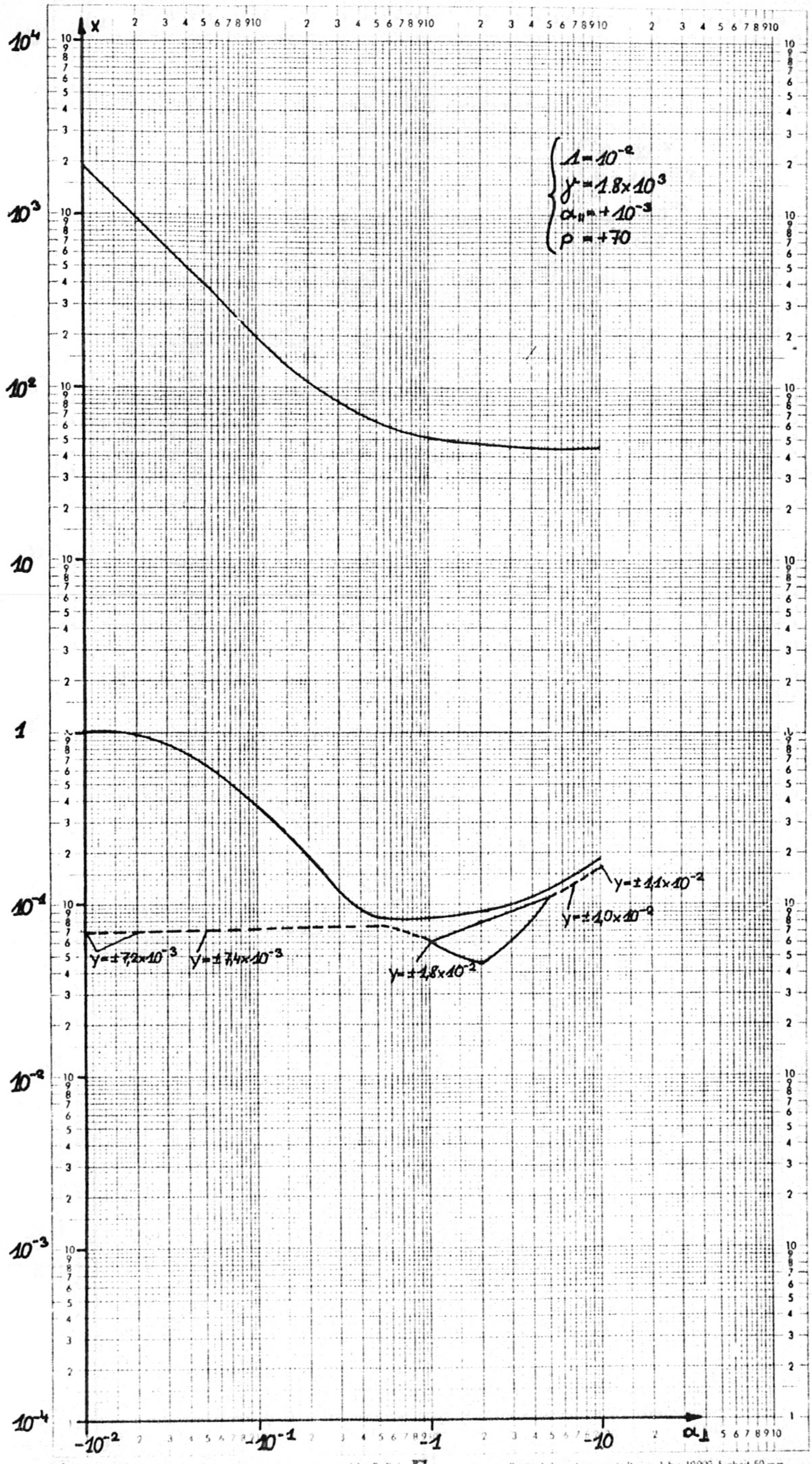

x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$1 = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_{\parallel} = +10^{-3}$
$p = +70$
$y = \pm 7,2 \times 10^{-3}$
$y = \pm 7,4 \times 10^{-3}$
$y = \pm 1,8 \times 10^{-2}$
$y = \pm 1,0 \times 10^{-2}$
$y = \pm 1,1 \times 10^{-2}$
-10^{-2}
-10^{-1}
-1
-10
$\alpha_{\perp}$

Fig. 21

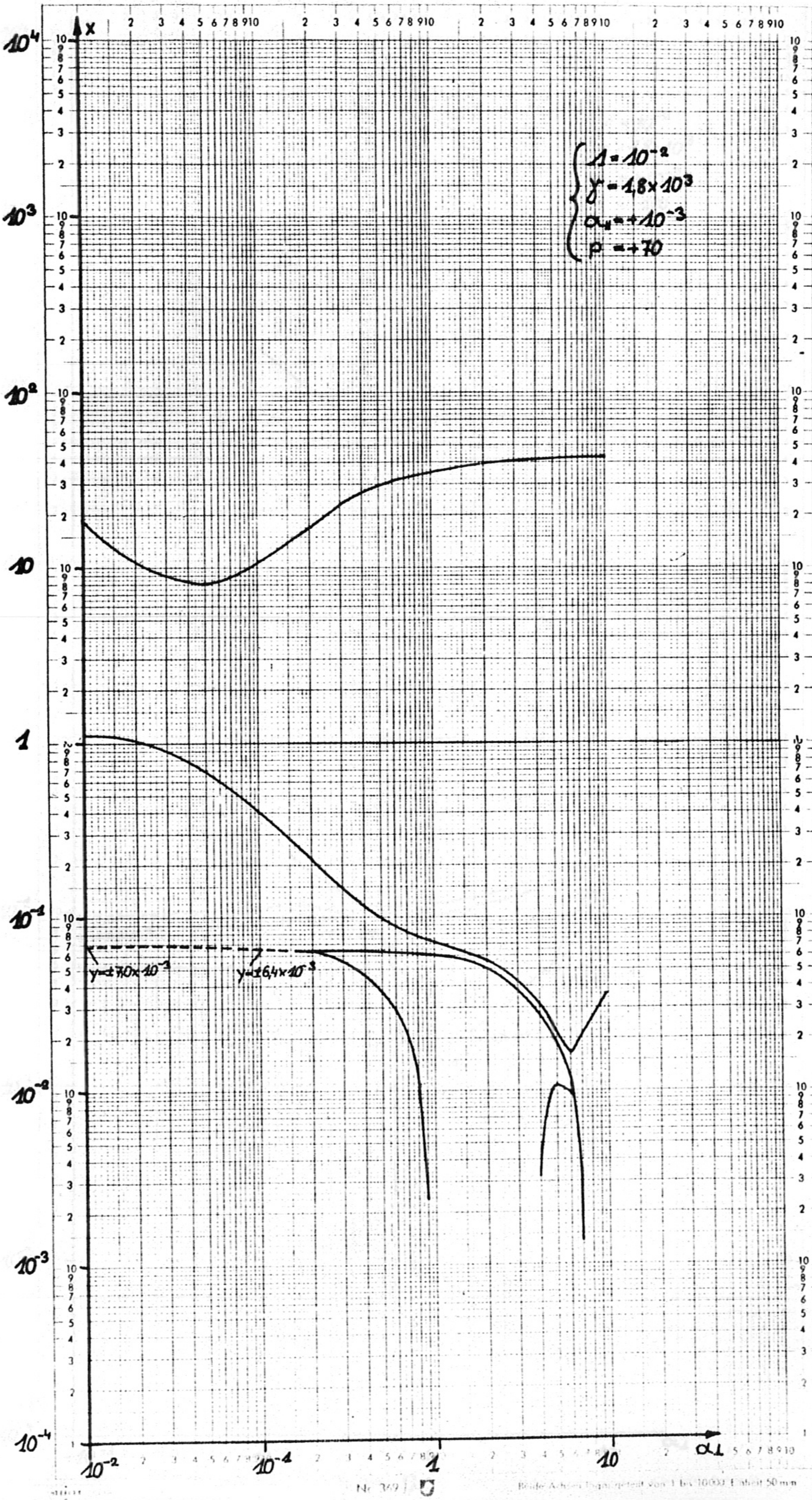
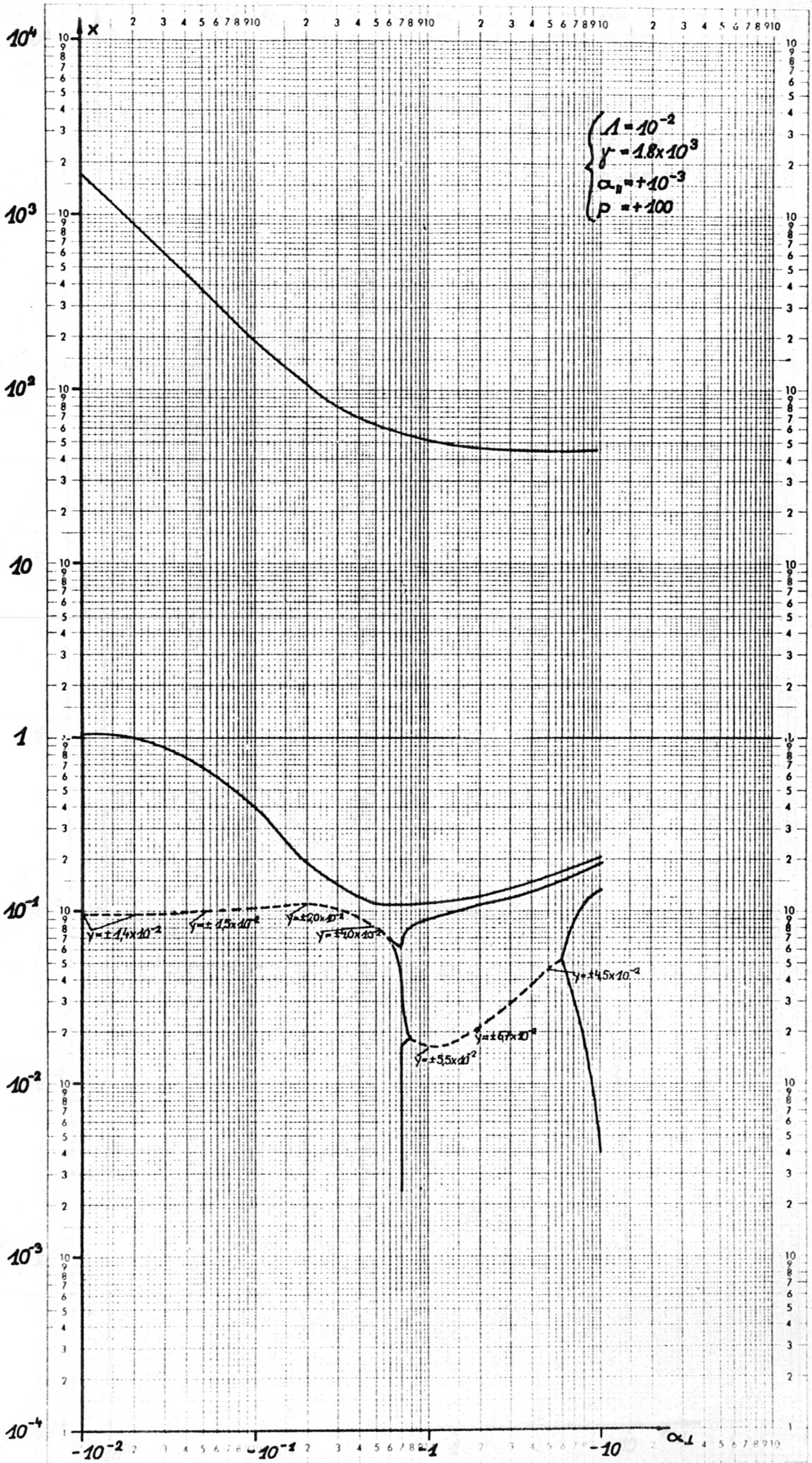
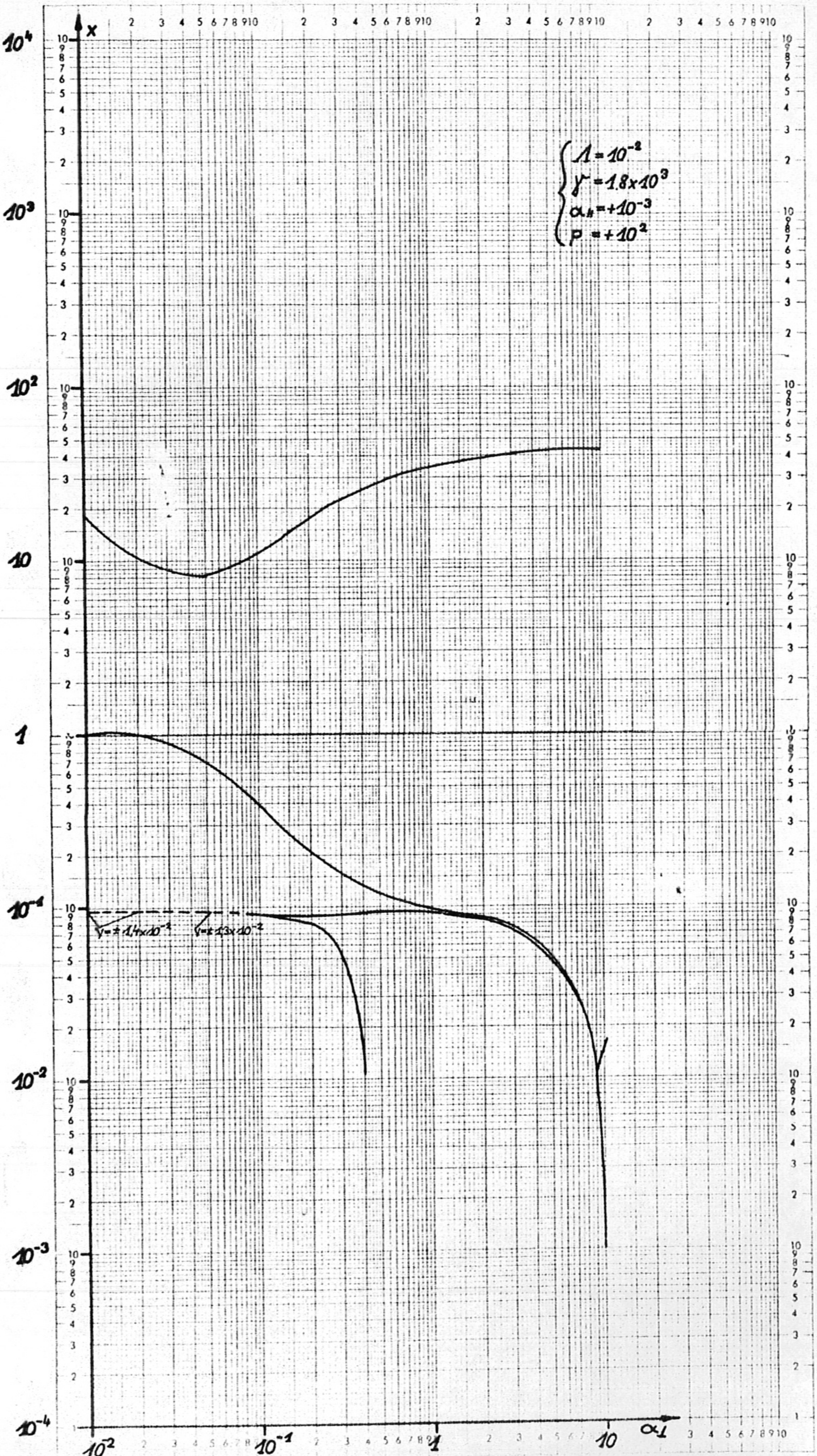

x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\Lambda = 10^{-2}$
$\gamma = 1{,}8 \times 10^3$
$\alpha_{\parallel} = +10^{-3}$
$p = +70$
$y = 170 \times 10^{-3}$
$y = 16{,}4 \times 10^{-3}$
10^{-2}
10^{-1}
1
10
$\alpha_{\perp}$

Fig. 22



$\Lambda = 10^{-2}$
$\gamma = 1.8 \times 10^3$
$\alpha_{\parallel} = +10^{-3}$
$p = +100$
x
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$y = \pm 1{,}4 \times 10^{-2}$
$y = \pm 1{,}5 \times 10^{-2}$
$y = \pm 2{,}0 \times 10^{-2}$
$y = \pm 4{,}0 \times 10^{-2}$
$y = \pm 5{,}5 \times 10^{-2}$
$y = \pm 4{,}5 \times 10^{-2}$
-10^{-2}
-10^{-1}
-1
-10
$\alpha_{\perp}$

Fig. 23


x
Λ = 10^-2
γ = 1.8x10^3
α* = +10^-3
p = +10^2
10^4
10^3
10^2
10
1
10^-1
10^-2
10^-3
10^-4
10^-2
10^-1
1
10
α1