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The Correlation Function in a Plasma.
Part III.

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Summary

We study a one species homogeneous plasma outside
equilibrium. We discuss the normalisation of the correla-
tion function which leads to a condition on the Fourier
transform in space of the density correlation $G(\vec{k})$ for
$\vec{k} \to 0$. We compute $G(\vec{k})$ by two different methods and
connect the result with the fluctuation spectrum of the
electric field. Integrals involved in the general case
are evaluated for small k. This result introduces
a limitation on the behaviour of the velocity distribution
for large velocities.

It is shown that fluctuation spectra may be obtained also
from test particle calculations. The test particle problem
helps furthermore to understand the limitation mentioned
above.

Finally we use $G(\vec{k})$ to compute the interaction part
of the equation of state.

1. Introduction.

The system considered here is a homogeneous infinite electron plasma in a continous neutralizing positive background. Collective effects leading to Debye screening are taken into account.

We have seen in Part I , section 5, that the correlation function in such a plasma relaxes to an asymptotic function in the fastest time scale τ_0 , i.e. in a time roughly of the order ω_p^{-1} , the inverse of the plasma frequency, while the distribution function $F \equiv F_1$ varies in the much slower time scale τ_1 and reaches thermodynamic equilibrium in a time of the order $n z_d^3 \omega_p^{-1}$, where $n z_d^3$, the number of particles in a Debye sphere, is much larger than unity.[1,2,3]

It is therefore possible and in agreement with the ideas of Bogoliubov[4] to define a meta-equilibrium plasma in which all S- body functions F_S $(S \geq 2)$ are functionals of the one body distribution function F which itself is in lowest order of the small parameter $(n z_d^3)^{-1}$ constant in time.

It is well known that the validity of these concepts is restricted to stable initial distributions that only allow for damped plasma oscillations on basis of the linearized Vlasov equation. (See Part I section 5 and Appendix B).

Our purpose here is to discuss the normalisation condition for the pair correlation function and its relation to complete screening (section 2), to give a complete solution for the correlation function by two different methods (sections 3 and 4), to connect the results with fluctuations of the electric field (section 5) and the test particle problem (section 6), and to show that the theory gives only correct results in view of complete screening and physically acceptable fluctuation spectra, if certain conditions on the distribution function are imposed. (Section 7)

Finally we use our results for the calculation of the
interaction part in the equation of state obtained
from the virial theorem derived in a new way exhibiting
more clearly its validity outside thermodynamic equi-
librium (section 8).

2. Normalisation of the pair correlation.

It has been observed by A.Schlüter[*] that the limit
hierarchy of eq. (I. 2.5) is exact (without the limit
$N, V \rightarrow \infty$) for renormalized s -body functions F_s^*
given by

$$F_s^* = \frac{N!}{(N-s)! \, N^s} F_s = \frac{N! \, V^{-s}}{(N-s)! \, N^s} \int D(1,--,N) \, d\xi_{s+1} --- d\xi_N \quad (2.1)$$

the notation being as in Part I, section 2.
We clearly have

$$F_1^* = F_1 \qquad F_2^* = \left(1 - \frac{1}{N}\right) F_2 \qquad (2.2)$$

and therefore the normalisations

$$\frac{1}{V} \int F_1^*(1) \, d\xi_1 = 1$$

$$\qquad (2.3)$$

$$\frac{1}{V} \int F_2^*(1,2) \, d\xi_2 = \left(1 - \frac{1}{N}\right) F_1^*(1)$$

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[*] Private communication. See also Reference 5.

Defining the correlation by

$$F_2^*(1,2) = F_1^*(1) F_1^*(2) + g^*(1,2)$$

(2.4)

we see from eq. (2.3)

$$\int g^*(1,2) d\xi_2 = -\frac{1}{n} F_1^*(1) \; , \quad n = \frac{N}{V}$$

(2.5)

For $F_1^* \equiv F_1 \equiv F$ and g^* we have the usual equations (I. 2.14) and (I. 2.15) where the integrations have to be carried out over the entire velocity space and over the volume V in configuration space.

Omitting zero order correlations we expand the dimensionless correlation $\overset{o}{g}{}^*(1,2)$ as

$$\overset{o}{g}{}^*(1,2) = \varepsilon \, \overset{o}{g}{}^{*(1)}(1,2) + \varepsilon^2 \overset{o}{g}{}^{*(2)}(1,2) + \cdots ,$$

(2.6)

$$\varepsilon = \left(n \, z_d^3 \right)^{-1}$$

In the dimensionless notation we measure distances, velocities and times in units of z_d (Debye length), V_T (thermal velocity) and $z_d V_T^{-1} = \omega_p^{-1}$ respectively.

The normalisation (2.5) reads then

$$\int \overset{o}{g}{}^*(1,2) d\overset{o}{\xi}_2 = -\varepsilon \overset{o}{F}(1)$$

(2.7)

It is clear from eq. (2.6) that this implies

$$\int \overset{o}{g}^{*(1)}(1,2)\,d\overset{o}{\xi}_2 = -\overset{o}{F}(1)$$

$$\int \overset{o}{g}^{*(2)}(1,2)\,d\overset{o}{\xi}_2 = 0\;,\quad \tau \geq 2 \tag{2.8}$$

The function $\overset{o}{g}^{*(1)}(1,2)$ obeys the following equation
($\overset{o}{\tau}_o$ is the fastest time scale and corresponds to $\omega_p t$, see Part I)

$$\left(\frac{\partial}{\partial \overset{o}{\tau}_o} + \overset{o}{\vec{V}}_1 \cdot \frac{\partial}{\partial \overset{o}{\vec{x}}_1} + \overset{o}{\vec{V}}_2 \cdot \frac{\partial}{\partial \overset{o}{\vec{x}}_2} \right) \overset{o}{g}^{*(1)}(1,2)$$

$$- \frac{1}{4\pi} \frac{\partial \overset{o}{\phi}_{12}}{\partial \overset{o}{\vec{x}}_1} \cdot \left(\frac{\partial}{\partial \overset{o}{\vec{V}}_1} - \frac{\partial}{\partial \overset{o}{\vec{V}}_2} \right) \overset{o}{F}(1)\,\overset{o}{F}(2)$$

$$- \frac{1}{4\pi} \frac{\partial \overset{o}{F}(1)}{\partial \overset{o}{\vec{V}}_1} \cdot \int \frac{\partial \overset{o}{\phi}_{13}}{\partial \overset{o}{\vec{x}}_1} \overset{o}{g}^{*(1)}(2,3)\,d\overset{o}{\xi}_3 \tag{2.9}$$

$$- \frac{1}{4\pi} \frac{\partial \overset{o}{F}(2)}{\partial \overset{o}{\vec{V}}_2} \cdot \int \frac{\partial \overset{o}{\phi}_{23}}{\partial \overset{o}{\vec{x}}_2} \overset{o}{g}^{*(1)}(1,3)\,d\overset{o}{\xi}_3 = 0$$

The integrations are over the entire $\overset{o}{\vec{V}}_3$ -space and over
the reduced volume $V r_d^{-3}$ in dimensionless configuration
space.
Writing

$$\overset{o}{g}^{*(1)} = \overset{o}{g}^{*(1,0)} + \lambda\,\overset{o}{g}^{*(1,1)} + \cdots$$

where[5]

$$\lambda = r_d^{-3}\,V = (\varepsilon N)^{-1} \ll 1$$

we get from eq. (2.8) in the limit $\lambda \to 0$ the same equation for $\overset{o}{g}*(1,0)$ with integrations over configuration space extended to infinity. This new equation is the usual one which we want to treat. The question arises which normalisation condition exists for $\overset{o}{g}*(1,0)$.

It is clear that $\overset{o}{g}*(1,0)$ as a function of the distance $\overset{o}{\vec{x_1}} - \overset{o}{\vec{x_2}} = \overset{o}{\vec{x}_{12}}$ should have a finite range of the order unity (or z_d for $g*(1,0)$). Otherwise the expansion in ε has no sense. Let us assume furthermore

$$\overset{o}{g}*(1,1) = \gamma_o\left(\overset{o}{\vec{V_1}}, \overset{o}{\vec{V_2}}, \overset{o}{z_o}\right) + \gamma_1\left(\overset{o}{\vec{x}_{12}}, \overset{o}{\vec{V_1}}, \overset{o}{\vec{V_2}}, \overset{o}{z_o}\right)$$

where γ_o is independent of $\overset{o}{\vec{x}_{12}}$ and $\gamma_1\left(\overset{o}{\vec{x}_{12}}\right)$ has a finite range of the order unity. Then we clearly have

$$\lim_{\lambda \to 0} \int \overset{o}{g}*(1)(1,2)d\overset{o}{\xi_2} = \int \overset{o}{g}*(1,0)(1,2)d\overset{o}{\xi_2} + \int \gamma_o(1,2)d^3\overset{o}{V_2}$$

where the integration in the right-hand side extends to infinity for both velocity and space variables.

Therefore the function $\overset{o}{g}*(1,0)$ does not obey the normalisation condition in the first line of eq. (2.8) in general. This is only the case if $\gamma_o = 0$, i.e. if the correlation has no part (of order λ or N^{-1}) with a non-zero average over the entire configuration space.

The case $\gamma_o = 0$ corresponds to complete screening of the electric field of every particle. This may be seen from eqs. (2.2) and (2.4) which imply

$$F_2(1,2) = \frac{N}{N-1}\left\{F(1)F(2) + g*(1,2)\right\} \qquad (2.10)$$

If particle *1* is completely shielded, i.e. exactly
compensated by the lack of other electrons (compared
with the average density) in its neighbourhood, then
one has far from particle 1 one electron more than
corresponds with the average density. Therefore the
conditional probability density for finding particle 2
somewhere in phase space far from particle 1 is equal
to the distribution function for a modified system
with the reduced volume $V - V/N$.

Therefore

$$x_{12} \to \infty, \quad F_2(1,2) \to \frac{N}{N-1} F(1) F(2) \qquad (2.11)$$

or, by eq.(2.10)

$$\vec{x}_{12} \to \infty, \quad g^*(1,2) \to 0$$

We have left again the dimensionless notation.

Eq.(2.11) is the product law for probabilities with a
correction connected with the right counting of the
particles. The relation between complete screening and
normalisation of $g^*(1,2)$ is further clarified in
section 6 where we calculate the field of a test particle.

We expect complete screening for the asymptotic $g^{*(1)}(\tau_0 \to \infty)$
Therefore we expect $g^*(1,0)$ to satisfy the normalisation
(2.5) in the limit $\tau_0 \to \infty$ but not necessarily for
finite τ_0. Introducing the Fourier-Laplace transform

$$\hat{g}(\vec{k}, \vec{v}_1, \vec{v}_2, p) = \frac{1}{8\pi^3} \int_0^\infty d\tau_0 \int d^3 x_{12} \, g^{*(1,0)}(\vec{x}_{12}, \qquad (2.12)$$

$$\vec{v}_1, \vec{v}_2, \tau_0) \exp\left(-i\vec{k} \cdot \vec{x}_{12} - p\tau_0\right)$$

we find again eq. (I. 5.29),

The asymptotic correlation

$$\hat{g}_A \left(\vec{k}, \vec{v_1}, \vec{v_2} \right) = \lim_{p \to 0+} p\, \hat{g} \left(\vec{k}, \vec{v_1}, \vec{v_2}, p \right)$$

is the solution of the integral equation (I.5.33) or

$$\hat{g}_A \left(\vec{k}, \vec{v_1}, \vec{v_2} \right) = \lim_{\delta \to 0+} \frac{u_p^2}{u_1 - u_2 - i\delta} \left[\frac{1}{8\pi^3 n} \left\{ F_{(2)} \frac{\partial F_{(1)}}{\partial u_1} \right. \right.$$

$$\left. - F_{(1)} \frac{\partial F_{(2)}}{\partial u_2} \right\} + \frac{\partial F_{(1)}}{\partial u_1} h_A^* \left(\vec{k}, \vec{v_2} \right) \qquad (2.13)$$

$$\left. - \frac{\partial F_{(2)}}{\partial u_2} h_A \left(\vec{k}, \vec{v_1} \right) \right]$$

where

$$u_p^2 = \frac{\omega_p^2}{k^2} \qquad u = \frac{\vec{k} \cdot \vec{v}}{k} \qquad h_A \left(\vec{k}, \vec{v_1} \right) = \int \hat{g}_A \left(\vec{k}, \vec{v_1}, \vec{v_2} \right) d^3 v_2 \qquad (2.14)$$

The normalisation condition corresponding to eq.(2.5) is

$$h_A \left(\vec{k} = 0, \vec{v_1} \right) = -\frac{1}{8\pi^3 n} F_{(1)} \qquad (2.15)$$

Indeed one finds from eq. (2.13) that the left-hand side is only bounded for $k \to 0$ if the expression between square brackets in the right-hand side vanishes, which condition leads to eq. (2.15)

3. Calculation of the pair correlation -
Integral equation method.

We start from eq. (2.13) and drop the subscript A and the argument $\vec{k}$ which occurs only as a parameter.

Following Lenard[6] we obtain

$$h(\vec{v})\frac{\partial \breve{F}(u)}{\partial u} - H(u)\frac{\partial F(\vec{v})}{\partial u} = -\frac{1}{8\pi^3 n}\Big\{$$

$$\frac{\partial \breve{F}(u)}{\partial u} F(\vec{v}) - \breve{F}(u)\frac{\partial F(\vec{v})}{\partial u}\Big\} 2\pi i u_p^2 \cdot$$

$$\cdot \frac{\partial \breve{F}^-(u)}{\partial u}\Big\{1 + 2\pi i u_p^2 \frac{\partial \breve{F}^-(u)}{\partial u}\Big\}^{-1} \tag{3.1}$$

$$H(u) = 2\pi i u_p^2 \Big[\frac{1}{8\pi^3 n}\Big\{\breve{F}^-(u)\frac{\partial \breve{F}(u)}{\partial u}$$

$$- \breve{F}(u)\frac{\partial \breve{F}^-(u)}{\partial u}\Big\} + \frac{\partial \breve{F}(u)}{\partial u} H^-(u) \tag{3.2}$$

$$- \frac{\partial \breve{F}^-(u)}{\partial u} H(u)\Big]$$

where u, u_p and h are defined in eq. (2.14) and $\breve{F}(u)$ and $H(u)$ by

$$\breve{F}(u) = \int F(\vec{v})\,\delta(\vec{x}\cdot\vec{v} - u)\,d^3v$$

$$H(u) = \int h(\vec{v})\,\delta(\vec{x}\cdot\vec{v} - u)\,d^3v \tag{3.3}$$

$$\vec{x} = \frac{\vec{k}}{k}$$

Moreover we have introduced negative frequency parts of functions in eqs. (3.1),(3.2). For real u the positive and negative frequency parts of a function $\psi(u)$ satisfy the relations [7,8)]

$$\psi^+(u) + \psi^-(u) = \psi(u)$$

$$\psi^+(u) - \psi^-(u) = \frac{1}{\pi i} P \int_{-\infty}^{+\infty} \frac{\psi(u')}{u'-u} du' \qquad (3.4)$$

It is easy to see that $\psi^+(u)$ and $\psi^-(u)$ as defined by eq. (3.4) can be continued analytically in the complex u -plane as

$$\psi^+(u) = \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{\psi(u')}{u'-u} du'$$

$$\psi^-(u) = -\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{\psi(u')}{u'-u} du' \qquad (3.5)$$

and that $\psi^+(u)$ and $\psi^-(u)$ are analytic without singularities in upper- and lower half plane respectively. The symbols $\oint$ and $\oint$ denote integration below and above the pole $u' = u$ respectively.

We note that integration by parts in eq.(3.5) leads to

$$\frac{\partial \psi^{+(-)}(u)}{\partial u} = \left\{ \frac{\partial \psi(u)}{\partial u} \right\}^{+(-)}$$

Lenard[6)] proved that $H(u)$ is a real function. Because only the imaginary part of $h(\vec{v})$ plays a role in the kinetic equation (I.5.35), Lenard only needed eq.(3.1) and not the solution of eq.(3.2). If one is interested in the correlation itself however, it is important to

to solve eq. (3.2). In order to do this we use the
first line of eq. (3.4) and write eq.(3.2) in the form
(see also Ref.9)

$$\frac{H^+(u)}{Z^+(u)} + \frac{H^-(u)}{Z^-(u)} = \frac{2\pi i\, u_p^2}{8\pi^3 n\, Z^+(u) Z^-(u)} \cdot$$

$$\cdot \left\{ \widetilde{F}^-(u) \frac{\partial \widetilde{F}(u)}{\partial u} - \widetilde{F}(u) \frac{\partial \widetilde{F}^-(u)}{\partial u} \right\} \tag{3.6}$$

with

$$Z^\pm(u) = 1 \mp 2\pi i\, u_p^2 \frac{\partial \widetilde{F}^\pm(u)}{\partial u} \tag{3.7}$$

The terms of the left-hand side of eq.(3.6) are positive-
and negative frequency functions provided that $Z^+(u)$
and $Z^-(u)$ have no zeros in upper- and lower half u-plane
respectively.

We note that $Z^-(u)$ is identical to the dielectric constant
$\mathcal{E}(\vec{k}, u)$ of eq.(I.5.37) and $Z^+(u)$ to its complex
conjugate $\mathcal{E}^*(\vec{k}, u)$. It follows therefore from the dis-
cussion in Part I, section 5, and from

$$Z^-(u) = \left\{ Z^+(u^*) \right\}^*$$

that our proviso is fulfilled if $F(\vec{k}, u)$ obeys the
Penrose criterium[10], eq. (I.5.40), for every direction
of $\vec{k}$. The solution of eq.(3.2) is obtained directly
by taking the positive- and negative frequency parts of
the right-hand side of eq. (3.6)

$$H(u) = Z^+(u)\Big[\text{r.h.s. } (3.6) \Big]^+ + Z^-(u)\Big[\text{r.h.s. } (3.6) \Big]^-$$

Performing some rearrangements described in the Appendix
we arrive at the result

$$H(u) = -\frac{2\pi i \, u_p^2}{8\pi^3 n} \frac{\partial \breve{F}(u)}{\partial u} \left[\left\{ \frac{\breve{F}(u) \, \partial \breve{F}^-(u)/\partial u}{\partial \breve{F}(u)/\partial u \, Z^-(u)} \right\}^+ \right.$$

$$\left. - \left\{ \frac{\breve{F}(u) \, \partial \breve{F}^+(u)/\partial u}{\partial \breve{F}(u)/\partial u \, Z^+(u)} \right\}^- \right] \quad (3.8)$$

or

$$8\pi^3 n \, H(u) = -\breve{F}(u) + \frac{\partial \breve{F}(u)}{\partial u} \left[\left\{ \frac{\breve{F}(u)}{\partial \breve{F}(u)/\partial u \, Z^-(u)} \right\}^+ \right.$$

$$\left. + \left\{ \frac{\breve{F}(u)}{\partial \breve{F}(u)/\partial u \, Z^+(u)} \right\}^- \right] \quad (3.9)$$

Special attention is paid in this paper to the purely
configurational correlation function

$$G(\vec{R}) = \int_{-\infty}^{+\infty} H(\vec{R}, u) \, du \quad (3.10)$$

It may be obtained from eq.(3.8) or (3.9) by an inter-
change of the order of integration, i.e. the u-integration
is carried out before the integrations connected with the
+ and - signs. The result may be written in the alternative
forms

$$G(\vec{R}) = -\frac{2\pi i u_p^2}{8\pi^3 n} \int_{-\infty}^{+\infty} \frac{\breve{F}(u)}{\partial \breve{F}(u)/\partial u} \cdot \quad (3.11)$$

$$\cdot \left\{ \frac{\left(\partial \breve{F}^-(u)/\partial u \right)^2}{Z^-(u)} - \frac{\left(\partial \breve{F}^+(u)/\partial u \right)^2}{Z^+(u)} \right\} du$$

and

$$8\pi^3 n \, \breve{G}(\vec{K}) = -1 + \int_{-\infty}^{+\infty} \frac{\breve{F}(u)}{\partial \breve{F}(u)/\partial u} \cdot$$

$$\cdot \left\{ \frac{\partial \breve{F}^-(u)/\partial u}{Z^-(u)} + \frac{\partial \breve{F}^+(u)/\partial u}{Z^+(u)} \right\} du \qquad (3.12)$$

The results of eqs.(3.11) and (3.12) have been obtained also by Wolff[11]) by a different method.

Equations (3.8) up to (3.12) are very appropriate if one wants to calculate the integrals in special cases by means of contour integration.

Eg.(3.12) may, however, be transformed into a much simpler form

$$8\pi^3 n \, \breve{G}(\vec{K}) = -1 + \int_{-\infty}^{+\infty} \frac{\breve{F}(u)}{\lambda^2(u) + \mu^2(u)} \, du \qquad (3.13)$$

with

$$\lambda(u) = \text{Re} \, Z^-(u) = 1 - u_p^2 \, P \int_{-\infty}^{+\infty} \frac{\partial \breve{F}(u')/\partial u'}{u' - u} \, du'$$

$$\mu(u) = \text{Im} \, Z^-(u) = \pi u_p^2 \frac{\partial \breve{F}(u)}{\partial u} \qquad (3.14)$$

It is difficult to carry out the integration in eq.(3.13) for all K. But a simple result may be obtained for small K. On first sight the integral is proportional to u_p^{-4} (or K^4) if $\vec{K} \rightarrow 0$. However, the resonance behaviour of the integrand can give a larger contribution.

For large u we have $\lambda(u) \simeq 1 - u_p^2/u^2$ and $\mu(u)$ very small. The integrand in eq. (3.13) has therefore sharp peaks at $u = \pm u_p$ if $u_p \gg V_T$ (V_T : thermal velocity). This leads to

$$u_p \to \infty, \quad 8\pi^3 n \, \tilde{G}(\vec{k}) \simeq -1 +$$

$$+ \frac{1}{2 u_p} \left\{ \left| \frac{\hat{F}(u)}{\partial \hat{F}(u)/\partial u} \right|_{u=u_p} + \left| \frac{\check{F}(u)}{\partial \check{F}(u)/\partial u} \right|_{u=-u_p} \right\} \quad (3.15)$$

This result is correct as long as the second term in the right-hand side is larger than u_p^{-4} for large u_p

We remark that, if $H(u)$ is known from eq.(3.8) or (3.9), $h(\vec{r})$ follows from eq. (3.1) and $\hat{g}_A(\vec{k}, \vec{v_1}, \vec{v_2})$ from eq. (2.13). The function $\tilde{G}(\vec{k})$ is not sufficient for determining $\hat{g}_A(\vec{k}, \vec{v_1}, \vec{v_2})$.

4. Calculation of the pair correlation-Frequency spectrum method.

Following Rostoker[12] we introduce an asymptotic correlation in phase space and time $W(\vec{x_1}, \vec{v_1}, t \,|\, \vec{x_2}, \vec{v_2}, t')$ It is the correlation part of the probability density for finding particle 1 in a phase space cell centered around $(\vec{x_1}, \vec{v_1})$ at time t and particle 2 near $(\vec{x_2}, \vec{v_2})$ at a later time t'. In a homogenous plasma we clearly have

$$W(\vec{x_1}, \vec{v_1}, t \,|\, \vec{x_2}, \vec{v_2}, t') = W(\vec{x_{12}}, \vec{v_1}, \vec{v_2}, \tau)$$

with

$$\vec{x_{12}} = \vec{x_1} - \vec{x_2} \qquad \tau = t' - t$$

and

$$W(\vec{x_{12}}, \vec{v_1}, \vec{v_2}, \tau = 0) = g^{*(1)}(1,2) \qquad (4.1)$$

$g^{*(1)}(1,2)$ being the first order asymptotic correlation of the static type discussed in the preceding sections. $W(\vec{x_{12}}, \vec{V_1}, \vec{V_2}, \tau)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} - \vec{V_2} \cdot \frac{\partial}{\partial \vec{x_{12}}}\right) W(\vec{x_{12}}, \vec{V_1}, \vec{V_2}, \tau) + \frac{e^2}{m} \frac{\partial F(\vec{V_2})}{\partial \vec{V_2}} \cdot$$

$$\cdot \frac{\partial}{\partial \vec{x_{12}}} \left\{ \frac{F(\vec{V_1})}{|\vec{x_{12}} + \vec{V_1}\tau|} + n \int \frac{W(\vec{x_{12}} + \vec{x'}, \vec{V_1}, \vec{V_3}, \tau)}{x'} d^3x' d^3v_3 \right\} = 0 \tag{4.2}$$

and its Fourier-Laplace transform defined by an equation of the type (2.12) may be obtained from

$$\hat{\widehat{W}}(\vec{k}, \vec{V_1}, \vec{V_2}, p) = \frac{1}{p - i\vec{k} \cdot \vec{V_2}} \left[\hat{g}_A(\vec{k}, \vec{V_1}, \vec{V_2}) \right.$$

$$\left. - \frac{i\omega_p^2}{k^2} \vec{k} \cdot \frac{\partial F(\vec{V_2})}{\partial \vec{V_2}} \left\{ \frac{F(\vec{V_1})}{8\pi^3 n (p - i\vec{k} \cdot \vec{V_1})} + \int \hat{\widehat{W}}(\vec{k}, \vec{V_1}, \vec{V_3}, p) d^3v_3 \right\} \right] \tag{4.3}$$

We define the dynamic density correlation by

$$G_p(\vec{k}) = \int \hat{\widehat{W}}(\vec{k}, \vec{V_1}, \vec{V_2}, p) d^3v_1 d^3v_2 \tag{4.4}$$

and find from a straightforward calculation

$$G_{p=ikz+\delta}(\vec{k}) = \frac{2\pi}{k Z^-(z)} \left\{ H^-(z) \right.$$

$$\left. - \frac{2\pi i u_p^2}{8\pi^3 n} \breve{F}^-(z) \frac{\partial \breve{F}^-(z)}{\partial z} \right\} \tag{4.5}$$

and for $G_{p=ikz-\delta}(\vec{k})$ just the complex conjugate of eq. (4.5) (z is real).

The inverse Laplace transform $G(\vec{k}, \tau)$ may be written as

$$G(\vec{k}, \tau) = \frac{k}{2\pi} \int G_z(\vec{k}) \exp(ikz\tau) dz \qquad (4.6)$$

where

$$G_z(\vec{k}) = G_z^+(\vec{k}) + G_z^-(\vec{k}) = G_{p=ikz-\delta}(\vec{k}) + G_{p=ikz+\delta}(\vec{k})$$

For $\tau > 0$ only $G_z^-(\vec{k})$ gives a contribution in eq.(4.6), for $\tau < 0$ only $G_z^+(\vec{k})$ and for $\tau = 0$ both $G_z^+(\vec{k})$ and $G_z^-(\vec{k})$.

Using the relation between $H^-(u)$ and $H(u)$ given in eq.(3.2) one derives easily

$$G_z(\vec{k}) = \frac{2\pi}{k} \frac{2\pi i u_p^2}{8\pi^3 \hbar} \frac{\breve{F}(z)}{\partial \breve{F}(z)/\partial z} \cdot$$
$$\cdot \left[\frac{(\partial \breve{F}^+(z)/\partial z)^2}{Z^+(z)} - \frac{(\partial \breve{F}^-(z)/\partial z)^2}{Z^-(z)} \right] \qquad (4.7)$$

It is now obvious that from eq.(4.6) with $\tau = 0$ the $G(\vec{k})$ is obtained as given by eq.(3.11)

It is interesting that we succeeded in deriving $G(\vec{k})$ without actually solving the integral equation (3.2) by considering dynamic correlation functions which contain more information about the system.

5. Relations between fluctuations and the correlation function.

The precise microscopic density of an actual system may be written as

$$n(\vec{x}, t) = \sum_{i=1}^{N} \delta\{\vec{x} - \vec{x}_i(t)\} \qquad (5.1)$$

For the ensemble average $\langle n(\vec{x}, t) \rangle$ we find of course

$$\langle n(\vec{x}, t) \rangle = n \equiv N/V$$

We now consider averages of the type

$$\langle n(\vec{x}, t) n(\vec{x}', t) \rangle = \int \sum_{i=1}^{N} \sum_{j=1}^{N} \delta(\vec{x}_i - \vec{x}) \cdot$$

$$\cdot \delta(\vec{x}_j - \vec{x}') D(1, \cdots, N) d\xi_1 \cdots d\xi_N$$

and find using the symmetry of D with respect to interchange of particles

$$\langle n(\vec{x}) n(\vec{x}') \rangle = N/V \, \delta(\vec{x} - \vec{x}') \int \delta(\vec{x} - \vec{x}_1) F(1) d\xi_1$$

$$+ N(N-1)/V^2 \int \delta(\vec{x}_1 - \vec{x}) \delta(\vec{x}_2 - \vec{x}') F_2(1,2) d\xi_1 d\xi_2$$

$$= n \delta(\vec{x} - \vec{x}') + n^2 \int \delta(\vec{x}_1 - \vec{x}) \cdot$$

$$\delta(\vec{x}_2 - \vec{x}') F_2^*(1,2) d\xi_1 d\xi_2$$

where $F_2^*(1,2)$ is defined by eq. (2.1) or (2.2). We note that the functions F_S^* and not F_S have direct physical meaning in calculating ensemble averages.

Defining the Fourier transform

$$P(\vec{R}) = \frac{1}{8\pi^3} \int \left\{ \langle n(\vec{x}) n(\vec{x} + \vec{X}) \rangle - n^2 \right\} \cdot$$
$$\cdot \exp\left(-i\,\vec{R}\cdot\vec{X}\right) d^3X \qquad (5.2)$$

and using eqs. (2.4) and (3.10) we find

$$P(\vec{R}) = \frac{n}{8\pi^3} \left\{ 1 + 8\pi^3 n \, G(\vec{R}) \right\} \qquad (5.3)$$

We may also define the Fourier transform of the actual density by

$$n(\vec{x}) - n = \int \rho(\vec{k}) \exp\left(i\,\vec{k}\cdot\vec{x}\right) d^3k \qquad (5.4)$$

Substituting this in eq. (5.2) we obtain

$$P(\vec{R}) = \int \langle \rho(\vec{k}_1) \rho(\vec{k}) \rangle \exp\left\{ i(\vec{k}_1 + \vec{R}) \cdot \vec{x} \right\} d^3k_1$$

or, because $P(\vec{R})$ is independent of $\vec{x}$, integration over configuration space gives

$$P(\vec{R}) = \frac{8\pi^3}{V} \langle \rho(\vec{R}) \rho(-\vec{R}) \rangle$$

The reality of $n(\vec{x}) - n$ in eq. (5.4) implies $\rho(-\vec{R}) = \rho^*(\vec{R})$
and therefore

$$P(\vec{R}) = \frac{8\pi^3}{V} \langle |\rho(\vec{R})|^2 \rangle \qquad (5.5)$$

Following quite similar lines we find for the
fluctuations of the electric field

$$\langle E_\alpha(\vec{x}) E_\beta(\vec{x}') \rangle = \frac{2 e^2 n}{\pi} \int \frac{k_\alpha k_\beta}{k^4} \cdot$$

$$\cdot \left\{ 1 + 8\pi^3 n \, G(\vec{k}) \right\} \exp\left\{ i \vec{k} \cdot (\vec{x} - \vec{x}') \right\} d^3k \tag{5.6}$$

We get therefore for the mean electrostatic energy density

$$\mathcal{E} = \frac{1}{8\pi} \langle E^2(\vec{x}) \rangle = \int W(\vec{k}) d^3k \tag{5.7}$$

with

$$W(\vec{k}) = \frac{e^2 n}{4\pi^2} \frac{1 + 8\pi^3 n \, G(\vec{k})}{k^2} = \frac{2\pi e^2 P(\vec{k})}{k^2} \tag{5.8}$$

We now want to express $W(\vec{k})$ in terms of the Fourier
transform of the electric field in order to derive
eq. (5.8) more directly.

The total energy is

$$\mathcal{E} V = \frac{1}{8\pi} \int \langle E^2(\vec{x}) \rangle d^3x$$

$$= \frac{1}{8\pi} \int \langle \hat{\vec{E}}(\vec{k_1}) \hat{\vec{E}}(\vec{k_2}) \rangle \exp\left\{ i(\vec{k_1} + \vec{k_2}) \cdot \vec{x} \right\} d^3k_1 \, d^3k_2 \, d^3x$$

$$= \pi^2 \int \langle \hat{\vec{E}}(\vec{k}) \hat{\vec{E}}(-\vec{k}) \rangle d^3k$$

$$= \pi^2 \int \langle |\hat{E}(\vec{k})|^2 \rangle d^3k$$

Therefore

$$W(\vec{K}) = \frac{\pi^2}{V} < |\hat{E}(\vec{K})|^2 >$$

(5.9)

The Poisson equation now leads to eq.(5.8) as follows

$$W(\vec{K}) = \frac{\pi^2}{V}(4\pi e)^2 \frac{1}{k^2} < |\rho(\vec{K})|^2 > = \frac{2\pi e^2 P(\vec{K})}{k^2}$$

Substituting eq. (3.13) into eq. (5.8) we have

$$W(\vec{K}) = \frac{m u_p^2}{16 \pi^3} \int \frac{\breve{F}(u)}{\lambda^2(u) + \mu^2(u)} du$$

(5.10)

where $\lambda(u)$ and $\mu(u)$ are given in eq. (3.14).
For small K we find from eq. (3.15)

$$W(\vec{K}) \backsim \frac{m u_p}{32 \pi^3} \left\{ \left| \frac{\breve{F}(u)}{\partial \breve{F}(u)/\partial u} \right|_{u=u_p} \right.$$
$$\left. + \left| \frac{\breve{F}(u)}{\partial \breve{F}(u)/\partial u} \right|_{u=-u_p} \right\}$$

(5.11)

In equilibrium we recover the familiar result

$$W(\vec{K}) = \frac{1}{8\pi^3} \frac{\theta}{2} \frac{k_d^2}{k^2 + k_d^2}$$
$$k_d^2 = z_d^{-2} = \frac{4\pi n e^2}{\theta}$$

(5.12)

6. Test particle problem.

We consider a test particle performing a straight
trajectory $\vec{X} = \vec{X_0} + \vec{V_0}t$.

The field particles have an equlibrium distribution
$F(\vec{v})$ with the normalisation $\int F(\vec{v})d^3v = 1$.
The test particle causes a perturbation of the distribu-
tion of the field particles and an electric potential
in the plasma.

The distribution function of the field particles takes
the form

$$f(\vec{x}, \vec{v}, \vec{x_0}, \vec{v_0}, t) = F(\vec{v}) + f_1(\vec{x}-\vec{x_0}-\vec{v_0}t, \vec{v}, \vec{v_0}, t)$$

and the potential may be written as

$$\phi(\vec{x}-\vec{x_0}-\vec{v_0}t, \vec{v_0}, t)$$

It is simpler to describe the phenomena in the reference
system of the test particle. In lowest order of $\varepsilon = (n\lambda_d^3)^{-1}$
we may use the linearized Vlasov equation

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial\vec{x}} + \vec{v_0}\cdot\frac{\partial}{\partial\vec{X}}\right)f_1(\vec{x}-\vec{X}, \vec{v}, \vec{v_0}, t) =$$

$$= -\frac{e}{m}\frac{\partial F(\vec{v})}{\partial\vec{v}}\cdot\frac{\partial\phi(\vec{x}-\vec{X}, \vec{v_0}, t)}{\partial\vec{x}} \quad (6.1)$$

and the Poisson equation

$$\frac{\partial}{\partial\vec{x}}\cdot\frac{\partial\phi(\vec{x}-\vec{X}, \vec{v_0}, t)}{\partial\vec{x}} = 4\pi e\left[\delta(\vec{x}-\vec{X})\right. \quad (6.2)$$

$$\left. + n\int f_1(\vec{x}-\vec{X}, \vec{v}, \vec{v_0}, t)d^3v\right]$$

It is already clear from eq. (6.2) that if ϕ is completely shielded the integral in configuration space over the right-hand side should vanish and therefore

$$\int f_1(\vec{x}, \vec{v}, \vec{v_0}, t)\, d^3v\, d^3x = -\frac{1}{n} \tag{6.3}$$

This is similar to eq. (2.5) if one replaces $g^*(1,2)$ by $F(\vec{v}) f_1(\vec{x}, \vec{v}, \vec{v_0}, t)$.

Indeed, performing a straightforward analysis along the lines of Rosenbluth and Rostoker[13] we find in the limit $t \to \infty$ for the Fourier transforms in space

$$\hat{\phi}(\vec{k}, u_0) = -\frac{e}{2\pi^2 k^2}\, \frac{1}{Z^-(\vec{k}, u_0)} \tag{6.4}$$

$$\int \hat{f_1}(\vec{k}, \vec{v}, u_0)\, d^3v = -\frac{1}{8\pi^3 n}\, \frac{2\pi i\, u_p^2\, \partial F(\vec{x}, u_0)/\partial u_0}{Z^-(\vec{k}, u_0)} \tag{6.5}$$

Compare for the notation with eqs. (2.14, (3.3), (3.4), (3.5) and (3.7). It is seen immediately that

$$\int \hat{f_1}(\vec{k}=0, \vec{v}, u_0)\, d^3v = -\frac{1}{8\pi^3 n}$$

in agreement with eq. (6.3).

It should be remarked that, although both f_1 and the correlation function describe screening and satisfy the same normalisation corresponding to complete screening, there exists no simple relation between the two in

general. It is, however, possible to derive the fluctuation spectra $P(\vec{k})$ and $W(\vec{k})$ and therefore also the integrated correlation function $G(\vec{k})$ from eq. (6.4) and (6.5).

From eq. (5.9) and a natural interpretation of averages in the test particle problem we get

$$W(\vec{k}) = \pi^2 n \, k^2 \int_{-\infty}^{+\infty} \hat{\phi}(\vec{k}, u_0) \hat{\phi}^*(\vec{k}, u_0) \, \breve{F}(\vec{x}, u_0) \, d u_0 \qquad (6.6)$$

$n \vec{V} \breve{F}(u_0) d u_0$ being the number of test particles in the range $d u_0$, if one stops to distinguish test particles from field particles.

Substitution of eq.(6.4) in eq.(6.6) leads directly to eq.(5.10). The function $\rho(\vec{k})$ defined in eq.(5.4) may be compared with

$$\rho'(\vec{k}, u_0) = \frac{1}{8\pi^3} + n \int \hat{f_1}(\vec{k}, \vec{v}, u_0) \, d^3 v \qquad (6.7)$$

where the first term of the right-hand side is the Fourier transform of a delta function and represents the contribution to the density of the test particle itself.

From eq.(6.5) we see that

$$\rho'(\vec{k}, u_0) = \frac{1}{8\pi^3 Z^-(\vec{k}, u_0)} \qquad (6.8)$$

From eq. (5.5) and an averaging process as in eq.(6.6) follows

$$P(\vec{k}) = 8\pi^3 n \int_{-\infty}^{+\infty} e'(\vec{k}, u_0) e'^*(\vec{k}, u_0) \breve{F}(\vec{x}, u_0) du_0$$

(6.9)

Substitution of eq.(6.8) in eq.(6.9) shows that the relation (5.8) exists between this $P(\vec{k})$ and the $W(\vec{k})$ of eq.(5.10). This finishes the task of connecting the results of test particle calculations with the exact theory of the foregoing sections. It is clear that this connection exists as far as spatial correlations are concerned but not with respect to correlations in velocity space. The complete correlation function $g(1,2)$ cannot be found from test particle calculations.

We shall derive now some special results for small k which allow for an interesting physical interpretation.

The energy per unit time emitted by a test particle due to the drag exerted by its asymmetric cloud is $e\vec{v_0} \cdot \dfrac{\partial \phi(\vec{x}-\vec{X}, \vec{v_0})}{\partial \vec{X}}$

The emission per unit time, unit volume and per Fourier mode is therefore, according to eq.(6.4), given by

$$W_e(\vec{k}) = \frac{e^2 n}{2\pi^2} \int_{-\infty}^{+\infty} \frac{i k u_0 \breve{F}(\vec{x}, u_0)}{k^2 Z^-(\vec{k}, u_0)} du_0$$

(6.10)

For small k the integrand has sharp peaks at $u = \pm u_p$ and the integral is easily seen to yield

$$W_e(\vec{k}) \backsim -\frac{k m u_p^4}{16\pi^2} \left\{ \breve{F}(\vec{x}, u_p) + \breve{F}(\vec{x}, -u_p) \right\}$$

(6.11)

It is clear that the right-hand side corresponds
to plasma waves emitted by fast particles with
velocities in the neighbourhood of $\pm u_p$, the
phase velocities of the waves. The energy density
of the waves consists of two parts, namely the energy
density of the electrostatic field $E^2/8\pi$ and the
kinetic energy density of the ordered motion connected
with the electric current density. This energy density
follows simply from

$$\frac{1}{2} n m \left(\frac{\dot{j}}{en}\right)^2 = \frac{2\pi \dot{j}^2}{\omega_p^2} \backsim \frac{E^2}{8\pi} \tag{6.12}$$

because

$$\vec{\dot{j}} = -\frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t} \backsim -\frac{\omega_p}{4\pi} \vec{E}$$

Per unit volume and Fourier mode we therefore have
from eq.(5.11) an energy density of the waves given
by

$$2 W(\vec{k}) \backsim \frac{m u_p}{16\pi^3} \left\{ \left| \frac{\tilde{F}(u)}{\partial \tilde{F}(u)/\partial u} \right|_{u=u_p} + \left| \frac{\tilde{F}(u)}{\partial \tilde{F}(u)/\partial u} \right|_{u=-u_p} \right\} \tag{6.13}$$

In thermodynamic equilibrium this is the Rayleigh
Jeans spectrum[*]

$$2 W(\vec{k}) \backsim \frac{\theta}{8\pi^3} \tag{6.14}$$

Applying now Kirchhoff's law separately to the waves

---

[*] Usually the Rayleigh Jeans spectrum contains an extra
factor 2 in the right-hand side. This is correct for
electromagnetic waves which have twice as many degrees
of freedom than electrostatic waves because of the
magnetic field.

in positive and negative direction we find from
eqs.(6.11) and (6.13) the absorption coefficient
$A(\vec{k})$ as

$$A_\pm(\vec{k}) = \pi k u_p^3 \left| \frac{\partial \breve{F}(\vec{x},u)}{\partial u} \right|_{u=\pm u_p} = 2\, \gamma_{L\pm}(\vec{k}) \tag{6.15}$$

where γ_L is the Landau damping coefficient.
This is exactly the absorption coefficient we should
have expected.

In Ref. 12 and 13 Kirchhoff's law was already stated
in the form (for thermal equilibrium)

$$W_e(\vec{k}) = \frac{\theta}{8\pi^3}\, 2\,\gamma_L(\vec{k}) \tag{6.16}$$

but no explanation was given there why the right-hand
side contains $2\,W(\vec{k})$ instead of $W(\vec{k})$.

The physical interpretation of our small k-results is
quite clear. Fast particles emit plasma waves which are
absorbed again by the Landau damping mechanism. In the
meta-equilibrium investigated here these processes ba-
lance each other and the energy density of the waves
may be considered as following from Kirchhoff's law.

7. Discussion of the results.

It should be pointed out that the expressions $H(\vec{k},u)$
and $G(\vec{k})$ do not necessarily converge uniformly to a
limit if $\vec{k} \to 0$. It is true that if one takes first
the limit $\vec{k} \to 0$ before evaluating the integrals in
in eqs.(3.8) up to (3.13) one finds

$$H(u) \to -\breve{F}(u)\,, \quad 8\pi^3 n\, G(\vec{k}) \to -1$$

This, however, is only correct if the integrands in eqs.(3.9) and (3.12) approach zero faster than $|u|^{-1}$ for large $|u|$.

We examine in fact the three following points for $\vec{k} \to 0$

a) uniform convergence of $H(\vec{k}, u)$

b) uniform convergence of $\Sigma(\vec{k}) \to -(8\pi^3 n)^{-1}$ which corresponds to the complete screening of a particle.

c) finite limit for $W(\vec{k})$

From eqs.(3.9),(3.12) and (5.10) it may be seen that in all these cases we have to impose a condition on expressions of the form $u^n F(u) \left(\frac{\partial F}{\partial u}\right)^{-1}$ when $|u| \to \infty$.

These conditions lead to an asymptotic behaviour of the form

$$F(u) \to P_m(u) \exp\left(-\alpha |u|^\beta\right)$$

where $P_m(u)$ is an arbitrary polynomial and α a positive number.

The values for n and β are indicated below for the three cases:

a) $\quad n = 0 \qquad \beta > 1$

b) $\quad n = -1 \qquad \beta > 0 \qquad\qquad$ (7.1)

c) $\quad n = 1 \qquad \beta \geq 2$

We consider as example the often used class of distribution functions[11]

$$F(\vec{v}) = \frac{C_z}{(v^2 + a^2)^z} \;,\quad \tilde{F}(u) = \frac{A_z}{(u^2 + a^2)^{z-1}} \;,\quad A_z = \frac{\pi C_z}{z-1} \quad (7.2)$$

which violates all conditions (7.1) and leads to both an incomplete screening and a divergence of the long wavelength energy density components. We find namely in this case

$$G(\vec{k}) = -\frac{\lambda}{8\pi^3 n} \frac{k_o^2}{k^2 + k_o^2}$$

$$W(\vec{k}) = \frac{ne^2}{4\pi^2 k^2} \frac{k^2 + (1-\lambda)k_o^2}{k^2 + k_o^2} \qquad (7.3)$$

$$k_o^2 = \frac{(2z-3)\omega_p^2}{a^2} \qquad \lambda = \frac{z - 3/2}{z - 1}$$

It is interesting to note that the violation of the normalisation condition $G(\vec{k}=0) = -(8\pi^3 n)^{-1}$ (complete screening) is removed if one uses the non-uniform convergence of $G(\vec{k})$ in order to write

$$G(\vec{k}) = -\frac{1}{8\pi^3 n} \left\{ \frac{\lambda k_o^2}{k^2 + k_o^2} + (1-\lambda)\delta_{k,0} \right\} \qquad (7.4)$$

where $\delta_{k,0}$ is a Kronecker delta corresponding to V^{-1} in configuration space. This has clearly no sense because the basic equations were valid in the limit $V \to \infty$ but it shows that the results do not violate the neutrality condition for the system as a whole. For finite V the $\vec{k}$-spectrum is discrete and one finds also from eq.(7.4) that $W(\vec{k})$ is proportional to $V^{2/3}$ for small (discrete) K .

In order to get a physical insight in our difficulties we should realize that the usual treatment is a first order theory in a small parameter ε . It is a good approximation only for $\varepsilon < \varepsilon_0$, say, where ε_0 may still be a function of K. We suggest that at least for some velocity distributions $\varepsilon_0(k) \to 0$, if $k \to 0$.

This situation is well known in another related
problem namely the damping of plasma oscillations.
In zero order of $\mathcal{E}$ one finds the Landau damping
which is exponentially small for small K[8,14] In
first order of $\mathcal{E}$ Ichikawa[15] finds a damping
proportional to K^2. Therefore the validity of the
zero order theory is clearly restricted to a range
of $\mathcal{E}$ which decreases with decreasing K.

We notice that our difficulties with small K-components
are associated with large velocity properties of the
distribution function. For example one may visualize
the meta-equilibrium spectrum as resulting from the
competition between the emission of electrostatic
waves by fast particles and their absorption by the
Landau damping mechanism as described in section 6.
The breakdown of the expansion in $\mathcal{E}$ is connected
with the fact that for small K other mechanisms (like
the collisional damping of Ichikawa)take over Landau
damping. To each new absorption effect corresponds a
new emission effect. At thermal equilibrium the result-
ing spectrum is always the Rayleigh Jeans one no matter
which processes are involved but this no more true out-
side thermodynamic equlibrium.

Of course, as soon as we cut off the velocity distribution
all difficulties disappear for small enough $\mathcal{E}$, the
critical $\mathcal{E}_o$ decreasing for increasing cut-off velocity.

8. Equation of state.

An interesting application of the correlation
function may be found in the interaction part of the
equation of state as given by the virial theorem.
Usually the virial theorem is derived by means of a

time averaging process on the equations of motion
of the particles. It is a bit unsatisfactory that
the interaction part of the equation of state is
to be calculated with the aid of the correlation
function usually obtained as an ensemble average.
Although one should expect time- and ensemble aver-
ages to be equal, this has never been proven and
it is therefore more elegant to derive the virial
theorem purely by means of ensemble averages. This
is done in this section. The procedure shows more-
over more clearly under which conditions the result-
ing equation is valid. It is shown for instance that
the virial theorem is also valid in our meta-equi-
librium situation.

We start from the Liouville equation

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{N}\left\{\vec{v_i}\cdot\frac{\partial}{\partial\vec{x_i}} - \frac{1}{m}\sum_{j\neq i,\,j=1}^{N}\frac{\partial\phi_{ij}}{\partial\vec{x_i}}\cdot\frac{\partial}{\partial\vec{v_i}} + \right.\right.$$

$$\left.\left. + \frac{1}{m}\vec{K_i}\cdot\frac{\partial}{\partial\vec{v_i}}\right\}\right] D\left(1,---,N\right) = 0 \qquad (8.1)$$

where ϕ_{ij} is the interaction potential between
particles i and j and $\vec{K_i}$ the external force on
particle i.

We perform the following manipulations:

a) multiplication of eq.(8.1) with $\sum_{k=1}^{N}\vec{x_k}\cdot\vec{v_k}$
    and integration over the entire Γ-space.

b) Reduction of the sums by means of the symmetry
    of D in its arguments.

c) Demonstration that some terms vanish, because
    they can be written as integrals over the
    divergence of a vectorfield vanishing at infinity
    or at the surface of the system.

The result is in the notation of section 2

$$
n \frac{\partial}{\partial t} \int \vec{x_1} \cdot \vec{v_1} \, F(1) d\xi_1 + n \int \vec{x_1} \cdot \vec{v_1} \, \vec{v_1} \cdot \frac{\partial F(1)}{\partial \vec{x_1}} d\xi_1
$$

$$
- \frac{n^2}{m} \int \vec{x_1} \cdot \vec{v_1} \frac{\partial \phi_{12}}{\partial \vec{x_1}} \cdot \frac{\partial F_2^*(1,2)}{\partial \vec{v_1}} d\xi_1 d\xi_2 + \tag{8.2}
$$

$$
+ \frac{n}{m} \int \vec{K}(\vec{x_1}) \cdot \frac{\partial F(1)}{\partial \vec{v_1}} \vec{x_1} \cdot \vec{v_1} \, d\xi_1 = 0
$$

The first term vanishes in the approximation where $F(1)$ is stationary, i.e. in our meta-equilibrium. It vanishes also for distribution functions isotropic in velocity - or configuration space even if they were non-stationary.

We introduce

$$
\vec{K}(\vec{x}) = - \frac{\partial \phi_e(\vec{x})}{\partial \vec{x}} + \vec{K}_w(\vec{x}) \tag{8.3}
$$

where $\phi_e(\vec{x})$ is some external potential and $\vec{K}_w(\vec{x})$ the wall force on a particle, which is assumed to be zero everywhere in the system except in a small layer around the system where it increases very fast to infinity. This implies that $F(1) = 0$ if particle 1 is at the surface of the system and $F_2(1,2) = 0$ if particle 1 or 2 is at the surface.

The second term in eq. (8.2) may be written as (in index notation and without the subscript 1)

$$
n \int x_\alpha v_\alpha v_\beta \frac{\partial F}{\partial x_\beta} d\xi = n \int x_\alpha \frac{\partial \langle v_\alpha v_\beta \rangle}{\partial x_\beta} d^3x =
$$

$$
= n \int \left\{ \frac{\partial}{\partial x_\beta} \left(x_\alpha \langle v_\alpha v_\beta \rangle \right) - \langle v^2 \rangle \right\} d^3x = -n \int \langle v^2 \rangle d^3x \tag{8.4}
$$

because $\langle v_\alpha v_\beta \rangle = \int v_\alpha v_\beta F d^3v$ vanishes at the surface. The third term in eq. (8.2) becomes for central interaction forces

$$\frac{n^2}{m}\int \vec{x_1}\cdot\vec{v_1}\,\frac{\partial\phi_{12}}{\partial\vec{x_1}}\cdot\frac{\partial F_2^*}{\partial\vec{v_1}}\,d\xi_1 d\xi_2 =$$

$$= -\frac{n^2}{m}\int \vec{x_1}\cdot\frac{\partial\phi_{12}}{\partial\vec{x_1}}\,F_2\,d\xi_1 d\xi_2 = \qquad (8.5)$$

$$= -\frac{n^2}{2m^2}\int z\,\frac{\partial\phi_{12}(z)}{\partial z}\,\mu(\vec{z},\vec{x_1})\,d^3x_1\,d^3x_2$$

where

$$\vec{z} = \vec{x_1} - \vec{x_2} \qquad \mu(\vec{z},\vec{x_1}) = \int F_2(1,2)\,d^3v_1\,d^3v_2 \qquad (8.6)$$

The last term in the left-hand side of eq.(8.2) may be transformed as

$$\frac{n}{m}\int \vec{K}(\vec{x})\cdot\frac{\partial F(\xi)}{\partial\vec{v}}\,\vec{x}\cdot\vec{v}\,d\xi =$$

$$\qquad (8.7)$$

$$= -\frac{n}{m}\int \vec{x}\cdot\vec{K}(\vec{x})F(\xi)\,d\xi = -\frac{1}{m}\int n(\vec{x})\vec{x}\cdot\vec{K}(\vec{x})d^3x$$

where $n(\vec{x}) = n\int F(\xi)d^3v$ is the local density. We substitute eq.(8.3) in the right-hand side of eq.(8.7) and notice that $n(\vec{x})\vec{K}_w(\vec{x})$ is the force per unit volume exerted by the walls on the system. It exists only in the small boundary layer and may be identified with $-\vec{n}p\,dS$ where p is the hydrostatic pressure of the system exerted on the walls, dS a surface element and $\vec{n}$ the unit vector normal to it. If p is position independent we now have

$$\int n(\vec{x})\vec{x}\cdot\vec{K}_w(\vec{x})d^3x = -p\int\vec{x}\cdot\vec{n}\,dS = -3pV \qquad (8.8)$$

If p does depend on position eq.(8.8) defines an average hydrostatic pressure.

From eqs. (8.2) upt to (8.8) we derive the quite general equation of state

$$p = \frac{1}{3} nm \frac{1}{V} \int \langle v^2 \rangle d^3x$$
$$- \frac{n^2}{6V} \int z \frac{\partial \phi_{12}(z)}{\partial z} \mu \left(\vec{z}, \vec{x}_1 \right) d^3x_1 d^3x_2$$
$$- \frac{1}{3V} \int n(\vec{x}) \vec{x} \cdot \frac{\partial \phi_e(\vec{x})}{\partial \vec{x}} d^3x \tag{8.9}$$

In the case of a homogeneous electron gas and a neutralizing continuous background we write

$$\mu(\vec{z}) = 1 + \Gamma(\vec{z}) \tag{8.10}$$

where $\Gamma(\vec{z})$ is the configurational correlation function and we identify $\phi_e(\vec{x})$ with the potential of the background

$$\phi_e(\vec{x}) = -n \int \phi_{12}(z) d^3x_2 \tag{8.11}$$

Then the third term of the right-hand side of eq.(8.9) cancels against that part of the second term which comes from the constant in eq.(8.10). Extending the volume integrals to infinity and using the smallness of the range of $\Gamma(\vec{z})$ we get

$$p = \frac{1}{3} nm \langle v^2 \rangle - \frac{n^2}{6} \int z \frac{\partial \phi_{12}(z)}{\partial z} \Gamma(\vec{z}) d^3z \tag{8.12}$$

Substituting the Coulomb potential and Fourier transforms we may write

$$P = \frac{1}{3} n m \langle v^2 \rangle + \frac{ne^2}{12\pi^2} \int \frac{8\pi^3 n \, G(\vec{k})}{k^2} d^3k$$

(8.13)

where $G(\vec{k})$ is given in e.g. eq.(3.13).
For isotropic distributions $G(\vec{k})$ depends only on the magnitude k and eq.(8.13) may be written as

$$P = \frac{1}{3} n m \langle v^2 \rangle \left(1 - \frac{\alpha}{24\pi n \, z_d^3} \right)$$

(8.14)

In thermal equilibrium $\alpha = 1$ corresponding to the usual result of the Debye Hückel theory and for the class of distribution functions (7.2) we find

$$\alpha = 3 \left(3 \, \frac{2z-3}{2z-5} \right)^{1/2} \frac{z-3/2}{z-1} \quad , \quad z = 3, 4, ---$$

if the Debye length is defined by

$$z_d^2 = \frac{\langle v^2 \rangle}{\omega_p^2} = \frac{3a^2}{(2z-5)\omega_p^2}$$

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Appendix. Solution of the Integral Equation (3.6).

From eq. (3.6) we have

$$8\pi^3 n\, H^+ = 2\pi i u_p^2 Z^+ \left[\frac{\breve{F}^- \partial\breve{F}/\partial u - \breve{F}\,\partial\breve{F}^-/\partial u}{Z^+ Z^-}\right]^+$$

and a similar equation for H^-. We have omitted the argument u for the sake of brevity.
We express $\partial\breve{F}/\partial u$ and $\partial\breve{F}^-/\partial u$ in terms of $Z^{\pm}$
by means of eq. (3.7) and obtain

$$8\pi^3 n\, H^+ = Z^+ \left[\frac{\breve{F} - \breve{F}^- Z^+ - \breve{F}^+ Z^-}{Z^+ Z^-}\right]^+$$

$$= -\breve{F}^+ + \left(\frac{\breve{F}}{Z^+ Z^-}\right)^+ Z^+$$

Adding the corresponding equation for H^- and using eq. (3.7) in the form

$$Z^- = Z^+ - 2\pi i u_p^2 \frac{\partial\breve{F}}{\partial u}$$

we see that

$$8\pi^3 n\, H = -\breve{F} + \frac{\breve{F}}{Z^-} - 2\pi i u_p^2 \frac{\partial\breve{F}}{\partial u}\left(\frac{\breve{F}}{Z^+ Z^-}\right)^-$$

or, using again eq.(3.7) in order to split up the denominator $Z^+ Z^-$,

$$8\pi^3 n\, H = -\breve{F} + \frac{\breve{F}}{Z^-} - \frac{\partial\breve{F}}{\partial u}\left\{\frac{\breve{F}}{\partial\breve{F}/\partial u}\left(\frac{1}{Z^+} - \frac{1}{Z^-}\right)\right\}^-$$

By means of the idendity

$$\frac{\breve{F}}{Z^-} \equiv \frac{\partial \breve{F}}{\partial u} \left\{ \left(\frac{\widetilde{F}}{\partial \breve{F}/\partial u \, Z^-} \right)^+ + \left(\frac{\widetilde{F}}{\partial \breve{F}/\partial u \, Z^-} \right)^- \right\}$$

we now arrive directly at eq. (3.9).

The same idendity again with unity instead of Z^-
leads with the aid of eq. (3.7) from eq. (3.9) to
eq. (3.8).