

I N S T I T U T F Ü R P L A S M A P H Y S I K
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Kinetic Equations for Plasmas.
Part I.

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Summary

A short review is given of the methods of Bogoliubov and of Rosenbluth and Rostoker for the derivation of kinetic equations. The procedure of Sandri based on the concept of multiple time scales is then followed for weakly coupled and long range homogeneous systems. The advantages of this procedure are exposed and the Landau equation is rederived in a more compact form. Also the Bogoliubov-Lenard-Balescu equation is obtained again.

New results are found for the relaxation into the kinetic regime and the conditions for kinetic behaviour.

1. Introduction.

We consider a one species many body system obeying the laws of classical mechanics. Such a system is described in all details by the Liouville equation or the B.B.G.K.Y.-hierarchy derived from it. We want to derive from this starting point a kinetic equation for the one particle distribution describing an irreversible behaviour. This irreversibility is introduced by means of multiple time scales as used by Sandri¹⁾, McCune²⁾ and Frieman³⁾. Their method is closely connected with the synchronisation hypothesis of Bogoliubov⁴⁾ and gives a more systematic presentation of his ideas. The method makes clear, as pointed out by McCune⁵⁾, that irreversibility comes in mainly by considering the asymptotic results of fast processes as initial values for slow processes and that assumptions about a statistical independency (molecular chaos) of the particles do not seem to be essential. Indeed it is shown that the usual assumptions may be weakened.

In this paper the interaction forces are assumed to be central and we restrict ourselves to the cases of weak coupling and long range forces. The precise meaning of these cases is given in section 2 which contains also the derivation of the hierarchy and a discussion about the application of the theory to plasmas. Section 3 gives a short review of some previous methods leading to the kinetic equation, in section 4 we explain the concept of multiple time scales, and in section 5 we treat the simple initial value problem¹⁾ for the weak coupling and long range cases. In section 6 we consider the relaxation of the system into the kinetic regime and in section 7 we discuss the conditions which should be fulfilled for the theory in sections 5 and 6 to be valid on basis of the complete initial value problem. In section 8 finally we collect our main results.

2. Foundations.

A classical system of N particles in a volume V obeys the Liouville equation

$$\frac{\partial D}{\partial t} + \sum_{i=1}^N \vec{v}_i \cdot \frac{\partial D}{\partial \vec{x}_i} - \frac{1}{m} \sum_{i \neq j=1}^N \frac{\partial \phi_{ij}}{\partial \vec{x}_i} \cdot \frac{\partial D}{\partial \vec{v}_i} + \frac{1}{m} \sum_{i=1}^N \vec{k}_i \cdot \frac{\partial D}{\partial \vec{v}_i} = 0 \quad (2.1)$$

where

D	density of systems in Γ -space
\vec{x}_i	position vector of particle i
\vec{v}_i	velocity of particle i
m	mass of a particle
ϕ_{ij}	interaction potential of particles i and j
\vec{k}_i	external force on particle i

Reduced distribution functions F_s are defined by

$$F_s(1, \dots, s, t) = V^s \int D(1, \dots, N, t) d\xi_{s+1} \dots d\xi_N \quad (2.2)$$

where i is an abbreviation for (\vec{x}_i, \vec{v}_i) and $d\xi_i$ for $d^3x_i d^3v_i$. It is clear that

$$V^{-s} F_s(1, \dots, s, t) d\xi_1 \dots d\xi_s$$

represents the probability of finding particle 1 in the element $d\xi_1$, particle 2 in $d\xi_2$, and so on up to particle s , no matter where particles $s+1, \dots, N$ are in phase space.

Integration of (2.1) over all coordinates and velocities except those of the first s particles gives

$$\begin{aligned} & \frac{\partial F_s}{\partial t} + \sum_{i=1}^s \vec{v}_i \cdot \frac{\partial F_s}{\partial \vec{x}_i} + V^s \sum_{i=s+1}^N \int \frac{\partial}{\partial \vec{x}_i} \cdot (\vec{v}_i D) d\xi_{s+1} \dots d\xi_N \\ & - \frac{1}{m} \sum_{i \neq j=1}^s \frac{\partial \phi_{ij}}{\partial \vec{x}_i} \cdot \frac{\partial F_s}{\partial \vec{v}_i} - V^s \sum_{i=1}^s \sum_{j=s+1}^N \frac{1}{m} \int \frac{\partial \phi_{ij}}{\partial \vec{x}_i} \cdot \frac{\partial D}{\partial \vec{v}_i} d\xi_{s+1} \dots d\xi_N \\ & - V^s \sum_{i=s+1}^N \sum_{j=1}^s \frac{1}{m} \int \frac{\partial}{\partial \vec{v}_i} \cdot \left(\frac{\partial \phi_{ij}}{\partial \vec{x}_i} D \right) d\xi_{s+1} \dots d\xi_N + \frac{1}{m} \sum_{i=1}^s \vec{k}_i \cdot \frac{\partial F_s}{\partial \vec{v}_i} = 0 \end{aligned} \quad (2.3)$$

The last but one term in the left-hand side of eq.(2.3) obviously vanishes if D decreases quickly enough for increasing velocity of any particle.

Using the symmetry of D with respect to interchange of particles the sums in the third and fifth terms in the left-hand side of eq.(2.3) are easily simplified.

The third term can be written as

$$\frac{N-s}{V} \int \frac{\partial}{\partial \vec{x}_{s+1}} \cdot (\vec{V}_{s+1} F_{s+1}) d\vec{x}_{s+1} = \frac{N-s}{V} \int_{\Omega} \int_S \vec{n} \cdot \vec{V}_{s+1} F_{s+1} d^2 x_{s+1} d^3 V_{s+1} \quad (2.4)$$

Ω is the velocity space, S the surface of the system and \vec{n} the unit vector perpendicular to S .

If we restrict our attention to a part of the system sufficiently far from the walls we may note that in the integrand in the right-hand side of eq.(2.4) particle $s+1$ is far from the others and therefore statistically independent of the others, i.e.

$$F_{s+1}(1, \dots, s+1) = F_s(1, \dots, s) F_1(s+1)$$

Then we get from eq.(2.4)

$$\frac{N-s}{N} F_s \int_S \vec{n} \cdot \vec{u}(\vec{x}) n(\vec{x}) d^2 x = \frac{N-s}{N} F_s \frac{\partial N}{\partial t} = 0$$

where $n(\vec{x})$ and $\vec{u}(\vec{x})$ are the local particle density and average velocity in the hydrodynamic sense respectively. The contribution of eq.(2.4) vanishes, because already in the Liouville equation it is assumed that one deals with a fixed number of particles. The boundary conditions should be such that the total number of particles remains constant.

The fifth term in the left-hand side of eq.(2.3) becomes

$$\frac{1}{m} \frac{N-s}{V} \sum_{l=1}^s \int \frac{\partial \phi_{l,s+1}}{\partial \vec{x}_l} \cdot \frac{\partial F_{s+1}}{\partial \vec{V}_l} d\vec{x}_{s+1}$$

For $S \ll N$ we now obtain the usual B.B.G.K.Y.-hierarchy

$$\begin{aligned} \frac{\partial F_S}{\partial t} + \sum_{i=1}^S \vec{v}_i \cdot \frac{\partial F_S}{\partial \vec{x}_i} + \frac{1}{m} \sum_{i=1}^S \vec{k}_i \cdot \frac{\partial F_S}{\partial \vec{v}_i} - \frac{1}{m} \sum_{i \neq j=1}^S \frac{\partial \phi_{ij}}{\partial \vec{x}_i} \cdot \frac{\partial F_S}{\partial \vec{v}_i} = \\ = \frac{n}{m} \sum_{i=1}^S \int \frac{\partial \phi_{i, S+1}}{\partial \vec{x}_i} \cdot \frac{\partial F_{S+1}}{\partial \vec{v}_i} d\vec{v}_{S+1} \end{aligned} \quad (2.5)$$

where $n = N/V$ is the particle density.

The system of eqs.(2.5) is open. The problem is to close it. This can be achieved sometimes by means of an expansion in some small parameter.

All quantities in eq.(2.5) may be made dimensionless by means of ϕ_0 , an average value of the interaction potential, r_0 , the range of the potential and v_T , the thermal velocity. In that case two dimensionless parameters appear in eq.(2.5). The last term of the left-hand side representing the mutual interaction of the S particles gets the factor ϕ_0/mv_T^2 and the right-hand side the factor $nr_0^3 \phi_0/mv_T^2$.

According to the order of magnitude of these factors we can distinguish as Frieman⁶⁾ three interesting cases ($\epsilon \ll 1$)

A) Dilute gas with short range forces

$$\phi_0/mv_T^2 \ll 1 \quad nr_0^3 \ll \epsilon \quad (2.6)$$

B) Weak coupling

$$\phi_0/mv_T^2 \ll \epsilon \quad nr_0^3 \ll 1 \quad (2.7)$$

C) Long range forces

$$\phi_0/mv_T^2 \ll \epsilon \quad nr_0^3 \ll 1/\epsilon \quad (2.8)$$

Case A) leads to the Boltzmann equation, B) to the Landau⁷⁾ equation and C) to the Bogoliubov⁴⁾, Lenard⁸⁾ Balescu⁹⁾ equation.

The case of a plasma^{*}) offers some difficulties, because the quantities ϕ_0 and r_0 mentioned above do not really exist there.

In a plasma we have three characteristic lengths

a) distance of closest approach or Landau cut-off

$$r_L = e^2 / m v_T^2 \quad (2.9)$$

($-e$: electron charge)

b) mean interparticle distance $n^{-1/3}$

c) Debye length

$$r_d = \left(\frac{m v_T^2}{4\pi n e^2} \right)^{1/2} \quad (2.10)$$

These three lengths are not independent but define only one dimensionless parameter which can be expressed as

$$\lambda = (n r_d^3)^{-1} \quad (2.11)$$

i.e. the inverse of the number of particles in a Debye sphere. We assume everywhere

$$r_L \ll n^{-1/3} \ll r_d, \quad \lambda \ll 1 \quad (2.12)$$

In this case we have many particles interacting with each other simultaneously. It seems natural to consider such a plasma as being case C).

Indeed taking $r_0 = r_d$ and $\phi_0 = e^2 / r_d$ we find the conditions (2.8) with $\epsilon = \lambda$.

* In this paper we treat only a homogeneous electron plasma in a continuous neutralizing background of positive charge.

We should note however that an expansion in λ cannot give correct results for small interaction distances. This expansion has only sense if in the dimensionless form of eq.(2.5) all quantities except the parameter ε do not deviate too much from unity. This condition is clearly not satisfied in the term

$$\sum_{j \neq i}^s \frac{\partial \phi_{ij}}{\partial \vec{x}_i} \cdot \frac{\partial F_s}{\partial \vec{v}_i}$$

for small distances $|\vec{x}_i - \vec{x}_j|$. Therefore we expect the need for a cut-off at short distance.

A natural cut-off is z_L , because it corresponds to the distance of closest approach of two particles with the relative velocity $2 v_T$.

The procedure C) gives a correct description of weak interactions and the collective effect (screening for large distances) but neglects close collisions.

If we treat the plasma as a system of type A) we describe correctly the close collisions but not the collective effect, because such a treatment amounts to a restriction to binary interactions. In this case the right-hand side of eq.(2.5) diverges because of the lack of screening, and we have to introduce an artificial upper cut-off at z_D .

Procedure B) suffers from both difficulties and has to be provided with two cut-offs at z_L and z_D . Nevertheless procedure B) gives under some conditions an approximately correct result corresponding to that of A) if one neglects close collisions there and to that of C) if one neglects interactions over distances larger than z_D there. This behaviour of procedure B) indicates indeed that it is not too bad to extend procedure A) up to z_D or procedure C) down to z_L .

We summarize the three procedures in the following table.

	z_0	ϕ_0	ε	range of interactions for which the expansion is valid	range of interactions for which the theory gives correct results	resulting kinetic equations
A	z_L	$e^2 z_L^{-1}$	λ^2	$0 < z < z_1$	$0 < z < z_d$	Boltzmann equation
B	$n^{-1/3}$	$e^2 n^{1/3}$	$\lambda^{2/3}$	$z_1 < z < z_2$	$z_L < z < z_d$	Landau equation
C	z_d	$e^2 z_d^{-1}$	λ	$z_2 < z$	$z_L < z$	Bogoliubov-Lenard-Balescu equation

where z_1 and z_2 are distances obeying the inequalities

$$z_L < z_1 < n^{-1/3} \quad n^{-1/3} < z_2 < z_d$$

In order to make the following discussion easier we define s-body correlation functions g_s by a cluster expansion

$$F_s(1, \dots, s) = \prod_{1 \leq i \leq s} F_1(i) + \sum_{1 \leq i < j \leq s} g_2(i, j) \prod_{\substack{1 \leq k \leq s \\ k \neq i, j}} F_1(k) + \sum_{1 \leq i < j < k \leq s} g_3(i, j, k) \prod_{\substack{1 \leq l \leq s \\ l \neq i, j, k}} F_1(l) + \dots + g_s(1, \dots, s) \quad (2.13)$$

and write down explicitly the first two hierarchy equations for the distribution function $F \equiv F_1$ and the correlation function $g \equiv g_2$ following from (2.5) for a homogeneous electron plasma without external forces.

$$\frac{\partial F(1)}{\partial t} = \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} g(1, 2) d\vec{x}_2 \quad (2.14)$$

$$\begin{aligned}
 & \left\{ \frac{\partial}{\partial t} + (\vec{v}_1 - \vec{v}_2) \cdot \frac{\partial}{\partial \vec{x}_1} \right\} g(1,2) - \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_1} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) \left\{ F(1)F(2) + \right. \\
 & \left. + g(1,2) \right\} - \frac{n}{m} \frac{\partial F(1)}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{13}}{\partial \vec{x}_1} g(2,3) d\xi_3 \\
 & - \frac{n}{m} \frac{\partial F(2)}{\partial \vec{v}_2} \cdot \int \frac{\partial \phi_{23}}{\partial \vec{x}_2} g(1,3) d\xi_3 = \\
 & = \frac{n}{m} \int \left(\frac{\partial \phi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} + \frac{\partial \phi_{23}}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{v}_2} \right) g_3(1,2,3) d\xi_3
 \end{aligned} \tag{2.15}$$

One can estimate the order of magnitude of the contributions of different ranges of interaction distances to the right-hand side of eq.(2.14).

Treating the range $0 < z < z_L$ with procedure A) one sees easily that

$$O\left[\left(\frac{\partial F}{\partial t}\right)_A\right] = z_L^2 v_T n O[F] = \omega_p \lambda O[F] \tag{2.16}$$

Using C) for $z > z_d$ one finds the same order of magnitude, while B) applied to the range $z_L < z < z_d$ gives

$$O\left[\left(\frac{\partial F}{\partial t}\right)_B\right] = v_T n^{1/3} \lambda^{1/3} \ln \frac{1}{\lambda} O[F] = \omega_p \lambda \ln \frac{1}{\lambda} O[F] \tag{2.17}$$

The logarithmic factor in (2.17) comes from the integration in the collision integral over the absolute value of the interaction distance between the cut-offs z_L and z_d .

In deriving (2.16) and (2.17) one should take

$$O\left[\frac{g(1,2)}{F(1)F(2)}\right] = 1 \quad \text{in case A)}$$

$$O\left[\frac{g(1,2)}{F(1)F(2)}\right] = \varepsilon \quad \text{in cases B) and C)}$$

because the correlations are assumed to be due to the interactions and therefore

$$O\left[\frac{g^{(1,2)}}{F^{(1)}F^{(2)}}\right] = \phi_0/mv_T^2$$

From the results (2.16) and (2.17) it is seen immediately that both close collisions and interactions over distances larger than the Debye length may be neglected, if

$$\ln \frac{1}{\lambda} \gg 1 \quad (2.18)$$

We shall restrict ourselves to this logarithmic accuracy throughout this paper.

Although our considerations seem to indicate that procedure B), which is the easiest one, gives sufficient information for our purposes, procedure C) is often highly preferable, because

- a) it avoids one artificial cut-off
- b) it gives insight in the shielding mechanism
- c) it imposes certain restrictions on the validity of the theory.

3. Previous methods.

The review of previously published methods in this section is far from complete. The work of Prigogine and Balescu^{10,11,9)} based on a direct systematic solution of the Fourier transformed Liouville equation for instance is not discussed here.

We give some comment on the theories of Bogoliubov⁴⁾ and Rosenbluth and Rostoker¹²⁾.

For the sake of simplicity we illustrate these theories for the weak coupling case.

We write the hierarchy in the abbreviated form

$$\frac{\partial F_S}{\partial t} + K_S F_S + \varepsilon \Omega_S F_S = \varepsilon L_{S+1} F_{S+1} \quad (3.1)$$

where ε is the small parameter of weak coupling, if eq. (3.1) is written in dimensionless form. If not, one may nevertheless expand in powers of ε and put eventually $\varepsilon = 1$. The operators K_S , Ω_S and L_{S+1} are defined by

$$\begin{aligned} K_S &= \sum_{i=1}^S \vec{v}_i \cdot \frac{\partial}{\partial \vec{x}_i} & \Omega_S &= -\frac{1}{m} \sum_{i \neq j=1}^S \frac{\partial \phi_{ij}}{\partial \vec{x}_i} \cdot \frac{\partial}{\partial \vec{v}_i} \\ L_{S+1} &= \frac{n}{m} \sum_{i=1}^S \int d\vec{x}_{S+1} \frac{\partial \phi_{i,S+1}}{\partial \vec{x}_i} \cdot \frac{\partial}{\partial \vec{v}_i} \end{aligned} \quad (3.2)$$

We treat a homogeneous system without external forces. The kinetic equation we are looking for should have the form

$$\frac{\partial F(1)}{\partial t} = A(1; F) \quad (3.3)$$

where $A(1; F)$ is an expression completely defined for every t by $F(t)$.

Clearly such an equation does not follow from the general solution of the hierarchy, which depends on all initial conditions $F_S(t=0)$.

a) Bogoliubov's theory.

Bogoliubov now assumes that for all initial conditions admissible from a physical point of view all functions $F_S (S > 1)$ relax quickly to functionals of F . (Synchronization hypothesis). This relaxation occurs in a time of the order of a collision duration τ_0/v_T . The distribution itself is approximately constant in this initial stage.

F changes in a much longer time scale, the kinetic stage, of the order $\tau_0/v_T \varepsilon^2$ in the weak coupling case. In this time thermodynamic equilibrium is reached, or, in the inhomogeneous case, local thermal equilibrium.

Following Bogoliubov we now expand $A(1; F)$ of eq.(3.3) and all F_S for $S \geq 2$

$$\frac{\partial F}{\partial t} = A_0(1; F) + \varepsilon A_1(1; F) + \varepsilon^2 A_2(1; F) + \dots \quad (3.4)$$

$$F_S(1, \dots, S; F) = F_S^{(0)}(1, \dots, S; F) + \varepsilon F_S^{(1)}(1, \dots, S; F) + \dots$$

where the form of $F_S^{(2)}(1, \dots, S; F)$ for every t is uniquely defined by $F(t)$.

The crucial point is now the expansion of the $\frac{\partial}{\partial t}$ -operator in eq.(3.1) for $S \geq 2$

$$\frac{\partial}{\partial t} = D_0 + \varepsilon D_1 + \dots \quad (3.5)$$

where D_2 denotes a differentiation with respect to t through the dependence on F with the replacement of $\partial F / \partial t$ by A_2 .

The equations (3.1) have to be solved by these expansions and together with a boundary condition corresponding to vanishing correlations in the infinite past, i.e.

$$\lim_{\tau \rightarrow \infty} F_S(\vec{x}_{ij} - \vec{v}_{ij} \tau, \vec{v}_1, \dots, \vec{v}_S) = \prod_{i=1}^S F(\vec{v}_i) \quad (3.6)$$

$$\vec{x}_{ij} = \vec{x}_i - \vec{x}_j, \quad \vec{v}_{ij} = \vec{v}_i - \vec{v}_j, \quad 2 \leq j \leq S$$

For the expansion coefficients $F_S^{(2)}$ we have

$$\lim_{\tau \rightarrow \infty} F_S^{(0)}(\vec{x}_{ij} - \vec{v}_{ij} \tau, \vec{v}_1, \dots, \vec{v}_S) = \prod_{i=1}^S F(\vec{v}_i) \quad (3.7)$$

$$\lim_{\tau \rightarrow \infty} F_S^{(2)}(\vec{x}_{ij} - \vec{v}_{ij} \tau, \vec{v}_1, \dots, \vec{v}_S) = 0, \quad 2 \geq 1$$

From eq.(3.1) with $S=1$ we obtain

$$\begin{aligned} A_0 &= 0 \\ A_1 &= L_2 F_2^{(0)}(1, 2; F) \\ A_2 &= L_2 F_2^{(1)}(1, 2; F) \end{aligned} \quad (3.8)$$

From the definition of D_2 follows $\Delta_0 = 0$ because $A_0 = 0$ and we find from eqs.(3.1), (3.4) and (3.5) for $s \geq 2$

$$K_S F_S^{(0)} = 0 \quad (3.9)$$

$$D_1 F_S^{(0)} + K_S F_S^{(1)} + \Omega_S F_S^{(0)} = L_{S+1} F_{S+1}^{(0)} \quad (3.10)$$

Eq.(3.9) can be written as

$$\sum_{j=2}^S \vec{v}_{ij} \cdot \frac{\partial F_S^{(0)}(\vec{x}_{1k}, \vec{v}_1, \dots, \vec{v}_S; F)}{\partial \vec{x}_{ij}} = 0 \quad 2 \leq k \leq S$$

or substituting $\vec{x}_{ij} - \vec{v}_{ij} \tau$ for \vec{x}_{ij} as arguments of $F_S^{(0)}$

$$\frac{\partial F_S^{(0)}(\vec{x}_{ij} - \vec{v}_{ij} \tau, \vec{v}_1, \dots, \vec{v}_S; F)}{\partial \tau} = 0 \quad 2 \leq j' \leq S$$

We see therefore immediately that the only solution satisfying eq.(3.7) is

$$F_S^{(0)} = \prod_{i=1}^S F(\vec{v}_i) \quad (3.11)$$

These solutions imply $L_S F_S^{(0)} = 0$ and therefore also $A_1 = 0$ and $D_1 = 0$. Eq.(3.10) becomes simply with the aid of (3.11)

$$K_S F_S^{(1)} = -\Omega_S \prod_{i=1}^S F(\vec{v}_i)$$

Using the same method as the one leading to eq.(3.11) we obtain as the only solution satisfying eq.(3.7)

$$F_S^{(1)}(\vec{x}_{ij} - \vec{v}_{ij} \tau, \vec{v}_1, \dots, \vec{v}_S; F) = \int_{-\infty}^{\infty} \Omega_S(\vec{x}_{ij} - \vec{v}_{ij} \tau') d\tau' \prod_{i=1}^S F(\vec{v}_i)$$

and therefore

$$F_S^{(1)}(\vec{x}_{ij}, \vec{v}_1, \dots, \vec{v}_S; F) = \theta_S \prod_{i=1}^S F(\vec{v}_i) \quad (3.12)$$

with

$$\theta_s = \int_0^\infty \Omega_s (\vec{x}_{ij} - \vec{v}_{ij} \tau') d\tau' \quad (3.13)$$

From the last line of eqs.(3.8) we now get as the lowest order of the kinetic equation (3.3)

$$\frac{\partial F(\vec{v}_1)}{\partial t} = \varepsilon^2 L_2 \theta_2 F(\vec{v}_1) F(\vec{v}_2) \quad (3.14)$$

The Bogoliubov theory can be applied also to inhomogeneous systems. In the short range case this results in a generalized Boltzmann equation showing an interference between streaming and collision term. In the case of a homogeneous plasma Bogoliubov arrived at a closed system of two equations for F and the correlation g . Lenard⁸⁾ succeeded in deriving an explicit kinetic equation from this system.

The main point in Bogoliubov's theory is the synchronisation hypothesis. This hypothesis is not proven and the theory fails in two respects:

- 1) it gives no information about the fast processes leading to the synchronisation
 - 2) it offers no basis for a discussion on the question which class of initial conditions leads to kinetic behaviour.
- The situation with respect to these points is improved by the multiple time scale method discussed in the following sections. This method is at the same time easier.

b. Theory of Rosenbluth and Rostoker.

Rosenbluth and Rostoker expand all functions F_s including $F_i \equiv F$ in powers of ε in a straight forward way. These expansions are only valid in a short time of the order of magnitude τ_0/ν_T , the characteristic time by means of which time is made dimensionless in eq.(3.1).

In zero order we have

$$\frac{\partial F^{(0)}}{\partial t} = 0 \quad (3.15)$$

$$\frac{\partial F_S^{(0)}}{\partial t} + K_S F_S^{(0)} = 0, \quad S \geq 2 \quad (3.16)$$

The natural solution of eq.(3.16) is because of the absence of interactions in this equation

$$F_S^{(0)} = \prod_{i=1}^S F^{(0)}(\vec{v}_i) \quad (3.17)$$

In first order one then obtains because $L_S \prod_{i=1}^S F^{(0)}(\vec{v}_i) = 0$

$$\frac{\partial F^{(1)}}{\partial t} = 0 \quad (3.18)$$

$$\frac{\partial F_S^{(1)}}{\partial t} + K_S F_S^{(1)} + \Omega_S F_S^{(0)} = 0, \quad S \geq 2 \quad (3.19)$$

Eq.(3.19) is satisfied by

$$F_S^{(1)} = \sum_{1 \leq i < j \leq S} F_2^{(1)}(i,j) \prod_{\substack{k=1 \\ k \neq i,j}}^S F^{(0)}(k) \quad (3.20)$$

for all S if

$$\frac{\partial F_2^{(1)}}{\partial t} + K_2 F_2^{(1)} + \Omega_2 F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_2) = 0 \quad (3.21)$$

and this equation follows also from (3.19) together with (3.17) for $S = 2$.

The solution of (3.21) is

$$F_2^{(1)} = F_2^{(1)}(\vec{x}_{12} - \vec{v}_{12}t, \vec{v}_1, \vec{v}_2, t=0) - \int_0^t \Omega_2(\vec{x}_{12} - \vec{v}_{12}\tau) d\tau F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_2) \quad (3.22)$$

where $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$, $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ and the first term in the right-hand side involves the initial condition for $F_2^{(1)}$. For finite \vec{x}_{12} and non vanishing \vec{v}_{12} $F_2^{(1)}$ approaches the limit $F_{2A}^{(1)}$ given by (see eq.(3.13))

$$F_{2A}^{(1)} = \theta_2 F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_2) \quad (3.23)$$

if the initial $F_2^{(1)}$ vanishes for large \vec{x}_{12} .

In the second order we have

$$\frac{\partial F^{(2)}}{\partial t} = L_2 F_2^{(1)} \quad (3.24)$$

Substituting the asymptotic result (3.23) in (3.24) we obtain a kinetic equation similar to (3.14) but with different orders of F in the left and right-hand sides. This equation is clearly only valid for small times. It describes a linear variation of $F^{(2)}$ in time because $F^{(0)}$ is constant.

The method of Rosenbluth and Rostoker gives the possibility to investigate the fast synchronisation process because one retains the time derivative in eq.(3.19). On the other hand it fails to give a kinetic equation valid for long times. Furthermore there is no clear justification for substituting the asymptotic $F_{2A}^{(1)}$ of eq.(3.23) into eq.(3.24).

These disadvantages are absent in the multiple time scale method described in the following sections.

It should be remarked that the theory of Rosenbluth and Rostoker was originally applied to the plasma case with the collective effect. This theory was extended to include magnetic interaction by Harris and Simon^{13,14,15)}. The weak coupling case, as described here, was first treated with this method by Frieman⁶⁾.

4. Multiple time scales.

As already observed in the discussion of Bogoliubov's theory there exist in a many body system several time scales with very different orders of magnitude.

In an ordinary not too dense gas for example we have

- a) $t_0 = \tau_0 / v_T$, the duration of a collision. In this time scale all F_s ($s \geq 2$) relax to functionals of F_1 according to Bogoliubov.
- b) $t_1 = \varepsilon^{-1} t_0$, $\varepsilon = n \tau_0^3$, the average time between two successive collisions. In this time scale the distribution function relaxes to a local Maxwellian. This process is described by the Boltzmann equation.
- c) $t_H = L / v_T$, the hydrodynamic time (L : characteristic length of inhomogeneities). It is the characteristic time for hydrodynamic phenomena as sound waves and the relaxation to a stationary flow pattern.

These phenomena are described by the Euler equations of hydrodynamics.

- d) $t_D = \delta^{-1} t_1$, $\delta = t_1 / t_H$, the diffusion time. In this time scale transport phenomena take place and the system relaxes into thermodynamic equilibrium. This relaxation is described by the Navier Stokes equation.

It is clear that the above separation of time scales is only possible if $\varepsilon, \delta \ll 1$. Then indeed we have

$$t_0 \ll t_1 \ll t_H \ll t_D$$

The Chapman Enskog theory¹⁶⁾ consists of an expansion of the Boltzmann equation in powers of δ . It assumes a fast relaxation (in the t_H time scale) of the distribution function to a functional of density, mean flow velocity and temperature.

Rather in the same way the Bogoliubov theory expands the hierarchy equations in powers of ε and assumes a relaxation in the t_0 time scale of the functions F_s to functionals of the distribution function.

It should be noted that in the weak coupling case we have $t_1 = \varepsilon^{-2} t_0$, $\varepsilon = \phi_0 / m v_T^2$ and in the long range case $t_0 = r_d / v_T = \omega_p^{-1}$ and $t_1 = \varepsilon^{-1} t_0$, $\varepsilon = (n r_d^3)^{-1}$

In these cases the definition of t_1 as "the average time between two successive collisions" has no sense and should be replaced by "the average time for a particle trajectory to undergo a $\pi/2$ deflection."

We remark also that the treatment of an inhomogeneous system with long range forces is extremely hard because already small inhomogeneities can introduce phenomena of the fastest time scale, t_0 , namely the plasma oscillations. (See e.g. Ref. 17,18,19,20)

We now return to homogeneous systems and exploit the existence of different time scales in a systematic way following Sandri¹⁾. We give a new and very simple justification for Sandri's method.

We write the hierarchy equations in the abbreviated form

$$\frac{\partial F_s}{\partial t} + \alpha_s F_s = L_{s+1} F_{s+1} \quad (4.1)$$

where α_s and L_{s+1} are operators the form of which is not interesting at the moment. It is sufficient to note that they depend linearly on some small parameter ε . Now we introduce "extended" functions \bar{F}_s which are functions of an infinite number of time coordinates $\tau_0, \tau_1, \tau_2, \dots$ and obey the "extended" hierarchy

$$\frac{\partial \bar{F}_s}{\partial \tau_0} + \varepsilon \frac{\partial \bar{F}_s}{\partial \tau_1} + \varepsilon^2 \frac{\partial \bar{F}_s}{\partial \tau_2} + \dots + \alpha_s \bar{F}_s = L_{s+1} \bar{F}_{s+1} \quad (4.2)$$

Solutions of eqs.(4.2) may be expressed as

$$\bar{F}_s = \bar{F}_s(\tau_0, \tau_1 - \varepsilon \tau_0, \tau_2 - \varepsilon^2 \tau_0, \dots, 1, \dots, s)$$

if the dependence on the first argument τ_0 and the \vec{x}_i, \vec{v}_i ($1 \leq i \leq s$) is obtained from

$$\frac{\partial \bar{F}_S}{\partial \tau_0} + \alpha_S \bar{F}_S = L_{S+1} \bar{F}_{S+1}$$

It is clear therefore that to every solution

$$\bar{F}_S(\tau_0, \tau_1, \tau_2, \dots, 1, \dots, S) \quad \text{of eqs. (4.2) a}$$

$$\text{solution } \bar{F}_S(t, 1, \dots, S) = \bar{F}_S(\tau_0 = t, \tau_1 = \varepsilon t, \tau_2 = \varepsilon^2 t, \dots, 1, \dots, S)$$

of eqs. (4.1) corresponds.

The advantage of eqs. (4.2) lies in the separation of different time scales, τ_0 describes the fastest phenomena (in a dimensionless notation τ_0 would correspond to t/t_0), τ_1 slower ones, and so on. The solution of eqs. (4.2) is, of course, not unique. We substitute an expansion of the form

$$\bar{F}_S = \bar{F}_S^{(0)} + \varepsilon \bar{F}_S^{(1)} + \varepsilon^2 \bar{F}_S^{(2)} + \dots$$

into eqs. (4.2) and require the coefficients of every power of ε to vanish.

The additional requirement that $\bar{F}_S^{(0)}, \bar{F}_S^{(1)}, \dots$ are bounded for all values of τ_0, τ_1, \dots assures that the expansion is valid for long times.

This "removal of secularities" is the essential point of the theory. It makes the solution of eqs. (4.2) unique.

It should be remarked that this method usually breaks down in orders higher than the one in which the kinetic equation is obtained^{1,21)}. The method fails therefore up to now, as all older theories, to give finite corrections to the kinetic equation.

In the following sections we treat the extended hierarchy (4.2). We do not distinguish typographically between the functions F_S and \bar{F}_S and we do not write down in general all the variables τ_0, τ_1, \dots as arguments of the functions \bar{F}_S .

In sections 5 and 6 we treat the simple initial value problem. This means that all correlations are assumed to be zero at $\tau_0 = 0$. The initial condition for the distribution is incorporated in its zero order.

In section 7 we discuss the complete initial value problem in order to see under which conditions the influence of the initial correlations vanishes quickly enough to make the theory of the simple initial value problem valid.

5. Derivation of kinetic equations. Simple initial value problems.

We apply the extension method of section 4 to eqs.(2.14) and (2.15).

A) Weak coupling.

In zero order we have

$$\frac{\partial F^{(0)}}{\partial \tau_0} = 0 \quad (5.1)$$

$$\left\{ \frac{\partial}{\partial \tau_0} + (\vec{v}_1 - \vec{v}_2) \cdot \frac{\partial}{\partial \vec{x}_1} \right\} g^{(0)}(1,2) = 0 \quad (5.2)$$

From $g^{(0)}(\tau_0=0) = 0$ and eq.(5.2) follows $g^{(0)}(1,2) = 0$ for all τ_0 .

More generally it follows from eqs. (3.1) that

$$F_s^{(0)}(1, \dots, s) = \prod_{i=1}^s F^{(0)}(i) \quad (5.3)$$

In first order one has from eqs. (2.14), (2.15) and $g^{(0)}(1,2) = 0$

$$\frac{\partial F^{(0)}}{\partial \tau_1} + \frac{\partial F^{(1)}}{\partial \tau_0} = 0 \quad (5.4)$$

and

$$\left\{ \frac{\partial}{\partial \tau_0} + (\vec{v}_1 - \vec{v}_2) \cdot \frac{\partial}{\partial \vec{x}_1} \right\} g^{(1)}(1,2) =$$

$$= \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_1} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) F^{(0)}(1) F^{(0)}(2) \quad (5.5)$$

The simple initial conditions are

$$g^{(1)}(\tau_0=0)=0 \quad F^{(1)}(\tau_0=0)=0$$

Therefore eq.(5.4) gives because of eq.(5.1) immediately

$$F^{(1)} = -\tau_0 \frac{\partial F^{(0)}}{\partial \tau_1}$$

This means that $F^{(1)}$ grows linearly in the fastest time scale. Now we apply for the first time the principle of removal of secular terms and find

$$F^{(1)} = \frac{\partial F^{(0)}}{\partial \tau_1} = 0 \quad (5.6)$$

Eq. (5.5) can be solved by integration along its characteristics. The result is $(\vec{x}_{12} = \vec{x}_1 - \vec{x}_2, \vec{v}_{12} = \vec{v}_1 - \vec{v}_2)$

$$g^{(1)}(1,2) = \frac{1}{m} \frac{\partial}{\partial \vec{x}_{12}} \cdot \int_0^{\tau_0} \phi_{12} (|\vec{x}_{12} - \vec{v}_{12} s|) ds \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) F^{(0)}(1) F^{(0)}(2) \quad (5.7)$$

For large τ_0 $g(1,2)$ approaches an asymptotic limit which is obviously given by eq.(5.7) if one extends the integral from zero to infinity. The Fourier transform in space of the correlation, which is often useful, is defined

by

$$\hat{g}(\vec{k}, \vec{v}_1, \vec{v}_2, \tau_0) = \frac{1}{8\pi^3} \int g^{(0)}(1,2) \exp(-i \vec{k} \cdot \vec{x}_{12}) d^3 x_{12} \quad (5.8)$$

This function may be obtained from eq.(5.7) or as the solution of the Fourier transform of eq.(5.5).

$$\hat{g}(\vec{k}, \vec{v}_1, \vec{v}_2, \tau_0) = \hat{\phi}(k) \frac{1 - \exp(-i \vec{k} \cdot \vec{v}_{12} \tau_0)}{m \vec{k} \cdot \vec{v}_{12}} \quad (5.9)$$

$$\vec{k} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) F^{(0)}(1) F^{(0)}(2)$$

For large τ_0 \hat{g} approaches a limit \hat{g}_A in the sense of generalized functions. This limit is given by

$$\hat{g}_A(\vec{k}, \vec{v}_1, \vec{v}_2) = \frac{2\pi i}{m} \hat{\phi}(k) \delta^-(\vec{k} \cdot \vec{v}_{12}) \vec{k} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) F^{(0)}(1) F^{(0)}(2) \quad (5.10)$$

$\hat{\phi}(k)$ is the Fourier transform of the interaction potential and is defined by a formula similar to eq. (5.8). The generalized function $\delta^-(x)$ is the negative frequency part of the Dirac delta function $\delta(x)$ and can be written as

$$\delta^-(x) = \frac{1}{2} \delta(x) + \frac{1}{2\pi i} P \frac{1}{x} \quad (5.11)$$

where the symbol P means "principle part of".

The general solution of the hierarchy equations (3.1) in first order is

$$F_s^{(1)} = \sum_{1 \leq i < j \leq s} g^{(1)}(i, j) \prod_{\substack{k=1 \\ k \neq i, j}}^s F^{(0)}(k) \quad (5.12)$$

exactly as in eq. (3.20).

Of the second order theory we only treat the equation for the distribution function because we are looking for the kinetic equation and we are not interested in the complete second order solution of the hierarchy. We have

$$\frac{\partial F^{(2)}}{\partial \tau_0} + \frac{\partial F^{(1)}}{\partial \tau_1} + \frac{\partial F^{(0)}}{\partial \tau_2} = \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} g^{(1)}(1, 2) d\vec{x}_2 \quad (5.13)$$

The second term of the left-hand side drops out because of eq. (5.6). Eq. (5.13) can therefore be integrated over τ_0 with the aid of eqs. (5.1) and (5.7). For large τ_0 the result is

$$F^{(2)} = -\tau_0 \left[\frac{\partial F^{(0)}}{\partial \tau_2} - \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} g_A^{(1)}(1, 2) d\vec{x}_2 \right] \quad (5.14)$$

if the expression

$$\vec{R} = \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} g^{(1)}(1, 2) d\vec{x}_2$$

approaches its asymptotic value \vec{R}_A faster than as τ_0^{-1}

i.e. if

$$\lim_{z_0 \rightarrow \infty} z_0^{1+\delta} (\vec{R} - \vec{R}_A) = 0, \delta > 0$$

This condition is proven to hold in the next section. We now apply again the principle of removal of secularities because $F^{(2)}$ should be finite (and small) for all z_0 . This condition gives the kinetic equation

$$\frac{\partial F^{(0)}}{\partial z_2} = \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} g_A^{(1)}(1,2) d\xi_2 \quad (5.15)$$

The right-hand side is the "collision integral". It can be expressed also in terms of the Fourier transforms of the potential and the correlation function (Parseval's theorem).

$$\frac{\partial F^{(0)}}{\partial z_2} = -8\pi^3 i \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \vec{k} \hat{\phi}(k) \hat{g}_A(\vec{k}, \vec{v}_1, \vec{v}_2) d^3k d^3v_2 \quad (5.16)$$

When substituting \hat{g}_A from eq. (5.10) we only need the odd part in \vec{k} i.e. the imaginary part of \hat{g}_A and therefore only the first term of the right-hand side of eq. (5.11).

$$\frac{\partial F^{(0)}}{\partial z_2} = \frac{8\pi^4 n}{m^2} \frac{\partial}{\partial v_\alpha} \int \hat{\phi}^2(k) \delta(\vec{k}, \vec{u}) k_\alpha k_\beta L_\beta(\vec{v}, \vec{u}) d^3k d^3u \quad (5.17)$$

where we have omitted the index i of \vec{v}_i , the Greek subscripts denote cartesian components of vectors and obey the summation convention, $\vec{u} \equiv \vec{v}_u$ is chosen as integration variable and

$$\vec{G} = F^{(0)}(\vec{v} - \vec{u}) \frac{\partial F^{(0)}(\vec{v})}{\partial \vec{v}} - F^{(0)}(\vec{v}) \frac{\partial F^{(0)}(\vec{v} - \vec{u})}{\partial \vec{v}} \quad (5.18)$$

The tensor

$$S_{\alpha\beta} = \int \hat{\phi}^2(k) \delta(\vec{k}, \vec{u}) k_\alpha k_\beta d^3k \quad (5.19)$$

is symmetrical and depends only on the vector \vec{u} . It is therefore clear that it must have the form

$$S_{\alpha\beta} = A \frac{u_\alpha u_\beta}{u^2} + B \delta_{\alpha\beta} \quad (5.20)$$

($\delta_{\alpha\beta}$: Kronecker delta)

Taking the scalar product of eq. (5.20) and $u_\alpha u_\beta$ and $\delta_{\alpha\beta}$ respectively we find

$$A = \frac{1}{2} S_{\alpha\beta} \left(3 \frac{u_\alpha u_\beta}{u^2} - \delta_{\alpha\beta} \right), \quad B = \frac{1}{2} S_{\alpha\beta} \left(\delta_{\alpha\beta} - \frac{u_\alpha u_\beta}{u^2} \right) \quad (5.21)$$

and therefore substituting $S_{\alpha\beta}$ from eq. (5.19) and integrating out the direction of \vec{k}

$$B = -A = \frac{\pi}{u} \int_0^\infty k^3 \hat{\phi}^2(k) dk \quad (5.22)$$

Collecting eqs. (5.22) and (5.20) the kinetic equation (5.17) takes the simple form

$$\frac{\partial F^{(0)}}{\partial \tau_2} = C \frac{\partial}{\partial k_\alpha} \int \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3} E_\beta(\vec{v}, \vec{u}) d^3 u \quad (5.23)$$

where C is a constant given by

$$C = \frac{8\pi^5 n}{m^2} \int_0^\infty k^3 \hat{\phi}^2(k) dk \quad (5.24)$$

Eq. (5.23) is indeed the simplest representation of the Landau equation.

In Bogoliubov's work the constant C is given, if one writes his representation of the Landau equation in velocity variables instead of momentum variables, as

$$C = \frac{n\pi}{2m^2} \int_0^\infty z^3 f^2(z) dz \quad (5.25)$$

$$f(z) = \int_{-\infty}^{+\infty} \frac{\phi'(\sqrt{z^2 + x^2})}{\sqrt{z^2 + x^2}} dx$$

where ϕ' is the derivative of the potential with respect to its argument.

It is proven in appendix A that eqs. (5.24) and (5.25) are indeed identical.

In the case of a Coulomb potential the integral in eq. (5.24) diverges for small and large k .

Taking a modified Coulomb potential

$$\phi(z) = \frac{e^2}{z} \exp(-k_d z) \{1 - \exp(-k_L z)\}, \quad k_d = z_d^{-1}, \quad k_L = z_L^{-1}$$

we have using $k_L \gg k_d$

$$\hat{\phi}(k) = \frac{e^2}{2\pi^2} \frac{k_L^2}{(k^2 + k_L^2)(k^2 + k_d^2)}$$

and from eq. (5.24)

$$C = \frac{2\pi e^4 n}{m^2} \ln \frac{k_L}{k_d} \quad (5.26)$$

B) Long range.

We treat here a homogeneous electron plasma in a continuous neutralizing background of positive charge.

The long range treatment has already been referred to in section 2. It offers the advantage of giving a correct description of the screening effect.

We start from eqs. (2.14) and (2.15) and find again in zero order the results of eqs. (5.1) and (5.3).

In first order we have

$$\frac{\partial F^{(0)}}{\partial z_1} + \frac{\partial F^{(1)}}{\partial z_0} = \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} g^{(1)}(1,2) d\xi_2 \quad (5.27)$$

$$\left\{ \frac{\partial}{\partial \tau_0} + (\vec{v}_1 - \vec{v}_2) \cdot \frac{\partial}{\partial \vec{x}_1} \right\} g^{(1)}(1,2) - \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_1} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) F^{(0)}(1) F^{(0)}(2) \\ - \frac{n}{m} \frac{\partial F^{(0)}(1)}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{13}}{\partial \vec{x}_1} g^{(1)}(2,3) d\xi_3 - \frac{n}{m} \frac{\partial F^{(0)}(2)}{\partial \vec{v}_2} \cdot \\ \cdot \int \frac{\partial \phi_{23}}{\partial \vec{x}_2} g^{(1)}(1,3) d\xi_3 = 0 \quad (5.28)$$

The Fourier-Laplace-transform of the first order correlation function defined by

$$\tilde{g}(\vec{k}, \vec{v}_1, \vec{v}_2, p) = \int_0^\infty d\tau_0 \frac{1}{8\pi^3} \int d^3x_{12} \exp(-p\tau_0 - i\vec{k} \cdot \vec{x}_{12}) g^{(1)}(1,2)$$

satisfies, if $g^{(1)}(\tau_0=0)=0$ the transformed equation

$$\tilde{g}(\vec{k}, \vec{v}_1, \vec{v}_2, p) = \frac{u_p^2}{u_1 - u_2 - i p/k} \left[\frac{1}{8\pi^3 n p} \left\{ F^{(0)}(2) \frac{\partial F^{(0)}(1)}{\partial u_1} \right. \right. \\ \left. \left. - F^{(0)}(1) \frac{\partial F^{(0)}(2)}{\partial u_2} \right\} + \frac{\partial F^{(0)}(1)}{\partial u_1} \tilde{h}^*(\vec{k}, \vec{v}_2, p) \right. \\ \left. - \frac{\partial F^{(0)}(2)}{\partial u_2} \tilde{h}(\vec{k}, \vec{v}_1, p) \right] \quad (5.29)$$

where we have introduced the Coulomb potential and its Fourier transform

$$\phi_{12} = \frac{e^2}{x_{12}} \quad \hat{\phi}(k) = \frac{e^2}{2\pi^2 k^2} \quad (5.30)$$

and furthermore

$$u = \frac{\vec{k} \cdot \vec{v}}{k} \quad (\text{velocity component parallel to } \vec{k})$$

$$u_p^2 = \frac{\omega_p^2}{k^2}, \quad \omega_p^2 = \frac{4\pi n e^2}{m} \quad (\text{plasma frequency}) \quad (5.31)$$

$$\tilde{h}(\vec{k}, \vec{v}, \rho) = \int \tilde{g}(\vec{k}, \vec{v}, \vec{v}', \rho) d^3v'$$

and \tilde{h}^* is the complex conjugate of \tilde{h} . Use has been made of the symmetry condition

$$\tilde{g}(-\vec{k}, \vec{v}, \vec{v}', \rho) = \tilde{g}^*(\vec{k}, \vec{v}, \vec{v}', \rho)$$

following from the fact that $g^{(1,2)}$ is real. The function $\hat{g}(\vec{k}, \vec{v}_1, \vec{v}_2, \tau_0)$ defined by eq.(5.8) approaches for large τ_0 the asymptotic limit $\hat{g}_A(\vec{k}, \vec{v}_1, \vec{v}_2)$ which can be expressed as

$$\hat{g}_A(\vec{k}, \vec{v}_1, \vec{v}_2) = \lim_{\rho \rightarrow 0} \rho \tilde{g}(\vec{k}, \vec{v}_1, \vec{v}_2, \rho) \quad (5.32)$$

if the limit in the right-hand side exists, i.e. if there exists a solution of an equation which is obtained by multiplication of eq. (5.29) with ρ and taking the limit $\rho \rightarrow 0$

This equation is

$$\hat{g}_A(\vec{k}, \vec{v}_1, \vec{v}_2) = 2\pi i u_p^2 \delta^-(u_1 - u_2) \left[\frac{1}{8\pi^3 n} \left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2} \right) F^{(0)}(1) F^{(0)}(2) + \frac{\partial F^{(0)}(1)}{\partial u_1} \hat{h}_A^*(\vec{k}, \vec{v}_1) - \frac{\partial F^{(0)}(2)}{\partial u_2} \hat{h}_A(\vec{k}, \vec{v}_2) \right] \quad (5.33)$$

where

$$\hat{h}_A(\vec{k}, \vec{v}) = \int \hat{g}_A(\vec{k}, \vec{v}, \vec{v}') d^3 v'$$

The conditions under which a solution of eq. (5.33) exists are discussed in Part III of this thesis. We shall see, however, that for the kinetic equation we only need the existence of the imaginary part of $\hat{h}_A(\vec{k}, \vec{v})$

We proceed therefore to the kinetic equation and integrate eq. (5.27) with respect to τ_0 . For large τ_0 we obtain

$$F^{(1)} = -\tau_0 \left[\frac{\partial F^{(0)}}{\partial \tau_1} - \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} g_A^{(1)}(1,2) d\xi_2 \right] \quad (5.34)$$

This is quite similar to eq. (5.14) and valid if a condition of the type mentioned there is fulfilled. Requiring that $F^{(1)}$ is small for all τ_0 we obtain the kinetic equation in a form analogous to eq. (5.15). In terms of Fourier transformed functions we can write this equation, using eqs. (5.30) (Coulomb potential), (5.34) and (5.31), as (omitting the subscript 1 of \vec{v}_1)

$$\frac{\partial F^{(0)}}{\partial \tau_1} = - \frac{\partial}{\partial \vec{v}} \cdot \int u_p^2 \vec{k} \text{Im} \hat{h}_A(\vec{k}, \vec{v}) d^3 k \quad (5.35)$$

Lenard⁸⁾ succeeded in deriving from the integral equation (5.33) a purely algebraic equation for $\text{Im } \hat{h}_A$. Substitution of this result in eq. (5.35) gives the explicit form of the kinetic equation. The result may be written as

$$\frac{\partial F^{(0)}}{\partial \tau_1} = \frac{\omega_p^4}{8\pi^2 n} \frac{\partial}{\partial k_\alpha} \int \frac{k_\alpha k_\beta \delta(\vec{k} \cdot \vec{u})}{k^4 |\epsilon(\vec{k}, \vec{k} \cdot \vec{v})|^2} G_\beta(\vec{v}, \vec{u}) d^3 k d^3 u \quad (5.36)$$

with $G_\beta(\vec{v}, \vec{u})$ given by eq. (5.18) and the dielectric constant $\epsilon(\vec{k}, u)$ by

$$\epsilon(\vec{k}, u) = 1 + 2\pi i u_p^2 \frac{\partial \tilde{F}^-(\vec{x}, u)}{\partial u} \quad (5.37)$$

and

$$\vec{x} = \frac{\vec{k}}{k} \quad \tilde{F}^-(\vec{x}, u) = \int_{-\infty}^{+\infty} \delta^-(u-u') \tilde{F}(\vec{x}, u') du'$$

$$\tilde{F}(\vec{x}, u) = \int F^{(0)}(\vec{v}) \delta(u - \vec{x} \cdot \vec{v}) d^3 v \quad (5.38)$$

We see that \tilde{F}^- is the negative frequency part of an one-dimensional distribution function \tilde{F} in which all velocity components are integrated out except the one parallel to \vec{k} .

The kinetic equation (5.36) is very similar to eq.(5.17). It is essentially eq. (5.17) if one interprets the $\hat{\phi}(k)$ in it as the screened Coulomb potential between two particles

$$\frac{e^2}{2\pi^2 k^2 |\epsilon(\vec{k}, \vec{k}, \vec{v})|^{-1}}$$

It is in general not possible, however, to carry out the integrations with respect to the direction of \vec{k} as we did with eq.(5.17), because the screened interaction potential depends not only on the magnitude of \vec{k} but also on its direction.

Lenard carried out the integration over the magnitude and showed that in logarithmic accuracy, i.e.

$\ln k/k_d \gg 1$, eq.(5.36) reduces to the Landau equation (5.17) with C given by eq. (5.26).

As it has been said already eq. (5.36) is only valid if $\text{Im } \hat{h}_A(\vec{k}, \vec{v})$ exists. This condition corresponds to the requirement that $\epsilon(\vec{k}, u)$ has no zeros for real u .

We investigate this condition in more detail. We have

$$|\epsilon(\vec{k}, u)|^2 = \left[1 - u_p^2 P \int_{-\infty}^{+\infty} \frac{\partial \tilde{F}(\vec{x}, u') / \partial u'}{u' - u} du' \right]^2 + \pi^2 u_p^4 \left\{ \frac{\partial \tilde{F}(\vec{x}, u)}{\partial u} \right\}^2 \quad (5.39)$$

This expression can only vanish for those real values of u for which both terms in the right-hand side vanish. The second term vanishes in all points with zero slope and

the first term is zero for some value of k if, and only if, the integral is positive.

Therefore eq. (5.36) is valid if, and only if, for every direction of \vec{x} the inequality

$$P \int_{-\infty}^{+\infty} \frac{\partial \tilde{F}(\vec{x}, u) / \partial u}{u - V_{\vec{x}}} du \leq 0 \quad (5.40)$$

is satisfied in all points $u = V_{\vec{x}}$ for which $\tilde{F}(\vec{x}, u)$ has zero slope.

The same condition restricted to all minima of $\tilde{F}(\vec{x}, u)$ is exactly the Penrose criterion²²⁾ for electrostatic stability of Vlasov plasmas. This criterion is necessary and sufficient. If eq. (5.40) is violated in a minimum $\tilde{E}^*(\vec{k}, u)^{+}$ has zeros for some values of k on the real u -axis and in the upper half of the u -plane corresponding to stationary and growing electrostatic oscillations respectively of a plasma obeying the linearized Vlasov equation with the one-dimensional zero order distribution $\tilde{F}(\vec{x}, u)$ and an arbitrary inhomogeneous initial perturbation. Penrose shows that a stable plasma, i.e. obeying eq. (5.40) in all minima, can only exhibit stationary oscillations if eq. (5.40) is violated in inclination points defined by

$$\frac{\partial \tilde{F}(\vec{x}, u)}{\partial u} = 0, \quad \frac{\partial^2 \tilde{F}(\vec{x}, u)}{\partial u^2} = 0$$

Only in that case $\tilde{E}^*(\vec{k}, u) = 0$ has solutions for real values of u but not for values of u with $\text{Im } u > 0$. We conclude therefore that eq. (5.40) is satisfied for all maxima, if it is satisfied for all minima. Indeed it follows from the paper of Penrose, that the integral of eq. (5.40) in a minimum is always smaller than in the surrounding maxima.

+) It is \tilde{E}^* which corresponds to the dielectric constant in the paper of Penrose.

Therefore we may restrict condition (5.40) to all points $u = V_{\vec{x}}$ for which $\tilde{F}(\vec{x}, u)$ has a minimum or inclination point. If $\tilde{F}(\vec{x}, u)$ vanishes in a certain range of values of u eq. (5.40) should be satisfied for all values $u = V_{\vec{x}}$ in that range.

It should be remarked that we have to require that $\tilde{F}(\vec{x}, u)$ is stable and allows for damped oscillations only at every value of the time coordinate τ . The stability of $\tilde{F}(\vec{x}, u)$ is not necessarily conserved by the Bogoliubov-Lenard-Balescu equation. If, however, $F(\vec{v})$ is isotropic, then it remains isotropic and $\tilde{F}(\vec{x}, u)$ is always single-humped and therefore stable. This matter is discussed in Appendix B.

6. Relaxation into the kinetic regime.

The extension method of sections 4 and 5 enables us also to investigate the transient behaviour of the distribution function, i.e. the way in which it approaches asymptotically a solution of the kinetic equation.

We discuss this in the simplest case, the weak coupling case. It may be seen from eqs. (5.13), (5.15) and 5.6) that the relaxation into the kinetic regime is described by

$$\frac{\partial F^{(2)}}{\partial \tau_0} = \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_1} \{g^{(1)}(1,2) - g_A^{(1)}(1,2)\} d\xi_2 \quad (6.1)$$

Using Fourier transforms as in eq. (5.16) we obtain with the aid of eq. (5.9)

$$\begin{aligned} \frac{\partial F^{(2)}}{\partial \tau_0} = & - \frac{8\pi^3 n}{m^2} \frac{\partial}{\partial v_\alpha} \int_{\tau_0}^{\infty} d\tau_0' \int d^3k d^3u \\ & \exp(-i \vec{k} \cdot \vec{u} \tau_0') \hat{\phi}^2(k) k_\alpha k_\beta \Gamma_\beta(\vec{v}, \vec{u}) \end{aligned} \quad (6.2)$$

$\vec{G}(\vec{v}, \vec{u})$ being given in eq.(5.18).

Writing $\vec{\xi} = \vec{u} \tau_0'$ we have

$$\int d^3 u \exp(-i \vec{k} \cdot \vec{u} \tau_0') G_\beta(\vec{v}, \vec{u}) = \\ = \frac{1}{\tau_0'^3} \int d^3 \xi \exp(-i \vec{k} \cdot \vec{\xi}) G_\beta(\vec{v}, \frac{\vec{\xi}}{\tau_0'})$$

The asymptotic behaviour may now be obtained by expanding $G_\beta(\vec{v}, \vec{\xi}/\tau_0')$ in a Taylor series in powers of $\vec{\xi}/\tau_0'$. From eq. (5.18) we see that

$$\vec{G}(\vec{v}, \vec{u}=0) = 0$$

Moreover we note that for symmetry reasons only the cosine part of the exponential contributes to the \vec{k} -integral in eq. (6.2). Consequently only even terms of the Taylor expansion of $\vec{G}(\vec{v}, \vec{u})$ contribute to the \vec{u} -integral, namely

$$\frac{1}{2!} \left\{ \frac{\partial^2 G_\beta(\vec{v}, \vec{u})}{\partial u_\lambda \partial u_\mu} \right\}_{\vec{u}=0} \frac{\xi_\lambda \xi_\mu}{\tau_0'^2} + \frac{1}{4!} \left\{ \frac{\partial^4 G_\beta(\vec{v}, \vec{u})}{\partial u_\lambda \partial u_\mu \partial u_\nu \partial u_\rho} \right\}_{\vec{u}=0} \frac{\xi_\lambda \xi_\mu \xi_\nu \xi_\rho}{\tau_0'^4} + \dots$$

The result is therefore

$$\frac{\partial F^{(2)}}{\partial \tau_0} \rightarrow - \frac{Q(\vec{v})}{\tau_0^4} + O\left(\frac{1}{\tau_0^6}\right) \quad (6.3)$$

with

$$Q(\vec{v}) = \frac{\pi^3 n}{m^2} \frac{\partial}{\partial v_\alpha} \left\{ \frac{\partial^2 G_\beta(\vec{v}, \vec{u})}{\partial u_\lambda \partial u_\mu} \right\}_{\vec{u}=0} \int d^3 k d^3 \xi \quad (6.4)$$

$$k_\alpha k_\beta \xi_\lambda \xi_\mu \hat{\phi}^2(k) \cos(\vec{k} \cdot \vec{\xi})$$

It is interesting that the relaxation is not exponential but follows a power law.*) Now it is well known that the asymptotic relaxation to thermodynamic equilibrium found from the linearized kinetic equation is exponential. It is therefore clear that in the approach to equilibrium after a sufficiently long time the deviation from equilibrium is mainly of non kinetic origin. However, $F^{(2)}$ is of order ε^2 , ε being the parameter of weak coupling, and the importance of the phenomenon depends therefore on the ratio ε^2/δ if δ is the order of magnitude of the ratio of the initial value for the linearized kinetic equation and the final Maxwell distribution.

The asymptotic series in inverse powers of τ_0^{-2} is only correct if the integrals of the type occurring in eq.(6.4) exist and if some derivatives of \vec{G} , at least the second derivatives, exist for all \vec{v} .

Let us assume that the last even derivatives of \vec{G} existing everywhere are of order $2p$. The Taylor series is now replaced by an exact expression namely the first $2p-1$ terms of it plus a rest term which can be written as

$$\frac{\xi_{\lambda_1} \xi_{\lambda_2} \dots \xi_{\lambda_{2p}}}{(2p)! \tau_0'^{2p}} T_{\beta, \lambda_1 \dots \lambda_{2p}} (\vec{v}, \theta \vec{f}/\tau_0')$$

where $0 < \theta < 1$ and

$$T_{\beta, \lambda_1 \dots \lambda_{2p}} (\vec{v}, \vec{u}) = \frac{\partial^{2p} G_{\beta} (\vec{v}, \vec{u})}{\partial u_{\lambda_1} \dots \partial u_{\lambda_{2p}}}$$

*) Our result differs from the τ_0^{-2} law found by Sandri ¹⁾.

Now we assume that $\frac{\partial}{\partial k_\alpha} T_{\beta, \lambda, \dots, \lambda_{2p}}$ is bounded,

$$\left| \frac{\partial}{\partial k_\alpha} T_{\beta, \lambda, \dots, \lambda_{2p}} \right| \leq M_{\alpha, \beta, \lambda, \dots, \lambda_{2p}}$$

then the rest term clearly gives a contribution to $\partial F^{(2)}/\partial \tau_0$ which has an absolute value less than

$$\frac{1}{\tau_0^{2p+2}} \frac{8\pi^3 n}{m^2} \frac{M_{\alpha, \beta, \lambda, \dots, \lambda_{2p}}}{(2p)!(2p+2)} |I_{\alpha, \beta, \lambda, \dots, \lambda_{2p}}| \quad (6.5)$$

where

$$I_{\alpha, \beta, \lambda, \dots, \lambda_{2p}} = \int d^3k d^3\xi k_\alpha k_\beta \xi_{\lambda_1} \dots \xi_{\lambda_{2p}} \hat{\phi}^2(k) c_n(\vec{k}, \vec{\xi}) \quad (6.6)$$

It is obvious that the expression (6.5) becomes smaller than all the foregoing terms resulting from the Taylor expansion, if τ_0 is large enough.

This proves that our expansion (6.3) is really asymptotic if all the integrals $I_{\lambda m}$, $1 \leq m \leq p$ exist. We study now in some detail these integrals.

In a way similar to the one leading from eq. (5.19) to eq. (5.22) we get

$$\begin{aligned} \int d^3k k_\alpha k_\beta \hat{\phi}^2(k) c_n(\vec{k}, \vec{\xi}) &= \\ &= \frac{4\pi}{\xi^3} \left\{ C_1(\xi) \xi^2 \delta_{\alpha\beta} + C_2(\xi) \xi_\alpha \xi_\beta \right\} \end{aligned} \quad (6.7)$$

where

$$C_1(\xi) = \int_0^{\infty} k^3 \hat{\phi}^2(k) \left[-\frac{\cos k\xi}{k\xi} + \frac{\sin k\xi}{k^2 \xi^2} \right] dk$$

$$C_2(\xi) = \int_0^{\infty} k^3 \hat{\phi}^2(k) \left[\sin k\xi + 3 \frac{\cos k\xi}{k\xi} - 3 \frac{\sin k\xi}{k^2 \xi^2} \right] dk \quad (6.8)$$

It may be seen immediately that $C_1(\xi)$ and $C_2(\xi)$ exist as finite integrals, if the integral in eq. (5.24) giving the constant before the collision term in the kinetic equation converges.

Now we assume that

$$L(k) \equiv k^3 \hat{\phi}^2(k)$$

can be continued as an analytic function in a strip of the complex plane around the real axis.

$$-k_1 \leq \text{Im}(k) \leq k_2$$

We may write eq. (6.8) in the form

$$C_1(\xi) = -\frac{i}{2\xi} \int_{-\infty}^{+\infty} L(k) \exp(ik\xi) \left(1 + \frac{i}{k\xi}\right) \frac{1}{k} dk$$

$$C_2(\xi) = -\frac{i}{2} \int_{-\infty}^{+\infty} L(k) \exp(ik\xi) \left(1 + \frac{3i}{k\xi} - \frac{3}{k^2 \xi^2}\right) dk$$

Shifting the integration path in the positive imaginary direction we obtain

$$C_1(\xi) = -\frac{i}{2\xi} \exp(-k_2 \xi) \int_{-\infty}^{+\infty} L(k+i'k_2) \exp(i'k\xi) \left\{ 1 + \frac{i'}{(k+i'k_2)\xi} \right\} \frac{dk}{k+i'k_2}$$

$$C_2(\xi) = -\frac{i}{2} \exp(-k_2 \xi) \int_{-\infty}^{+\infty} L(k+i'k_2) \exp(i'k\xi) \left\{ 1 + \frac{3i'}{(k+i'k_2)\xi} - \frac{3}{(k+i'k_2)^2 \xi^2} \right\} dk$$

If $L(k+i'k_2)$ is absolutely integrable ^{*)} it is easily seen now that

$$|C_1(\xi)| \leq \frac{M}{2k_2 \xi} \exp(-k_2 \xi) \left(1 + \frac{1}{k_2 \xi} \right)$$

$$|C_2(\xi)| \leq \frac{M}{2} \exp(-k_2 \xi) \left(1 + \frac{3}{k_2 \xi} + \frac{3}{k_2^2 \xi^2} \right) \quad (6.9)$$

with

$$M = \int_{-\infty}^{+\infty} |L(k+i'k_2)| dk$$

On the other hand we see substituting eq. (6.7) in eq. (6.6) that

$$|I_{\alpha, \beta, \lambda_1, \dots, \lambda_{2p}}| \leq (4\pi)^2 \int_0^\infty \xi^{2p+1} \{ |C_1(\xi)| + |C_2(\xi)| \} d\xi \quad (6.10)$$

*) Note that $L(k)$ has been assumed already as integrable in eq. (5.24).

and it is obvious from eq. (6.9) that the integral in the right-hand side of eq. (6.10) converges for all $\rho \geq 1$. This proves that all integrals occurring in the asymptotic expansion of $\partial F^{(2)}/\partial \tau_0$ exist.

It is interesting to look into a special example. We take a Maxwellian for $F^{(0)}$ and study the relaxation of the correlation Function from zero to its asymptotic value and the resulting behaviour of $\partial F^{(2)}/\partial \tau_0$. As interaction potential we choose a Debye shielded Coulomb potential.

$$F^{(0)}(\vec{v}) = \left(\frac{a}{\pi}\right)^{3/2} \exp(-a v^2)$$

$$\hat{\phi}(k) = \frac{e^2}{2\pi^2} \frac{1}{k^2 + k_D^2}, \quad \phi(z) = \frac{e^2}{z} \exp(-k_D z) \quad (6.11)$$

We then have

$$\vec{G} = -2a \vec{U} F^{(0)}(\vec{v}) F^{(0)}(\vec{v} - \vec{U})$$

From eq. (5.10) we see using eq. (5.11)

$$\hat{g}_A(\vec{k}, \vec{v}_1, \vec{v}_2) = -\frac{2a}{m} \hat{\phi}(k) F^{(0)}(\vec{v}_1) F^{(0)}(\vec{v}_2) \quad (6.12)$$

This is the well known equilibrium correlation function.

For the transient correlation function $\hat{g} - \hat{g}_A \equiv \hat{g}_T$ we obtain from eq. (5.9)

$$\hat{g}_T(\vec{k}, \vec{v}, \vec{U}, \tau_0) = \frac{2a}{m} \hat{\phi}(k) \exp(-i \vec{k} \cdot \vec{U} \tau_0) F^{(0)}(\vec{v}) F^{(0)}(\vec{v} - \vec{U}) \quad (6.13)$$

The correlations in configuration space corresponding to eq. (6.12) and (6.13) are

$$g_A(1,2) = -\frac{2a}{m} \phi(r) F^{(0)}(\vec{r}_1) F^{(0)}(\vec{r}_2)$$

$$g_T(1,2) = \frac{2a}{m} \phi(|\vec{r} - \vec{u}\tau_0|) F^{(0)}(\vec{r}_1) F^{(0)}(\vec{r}_2) \quad (6.14)$$

It is interesting to note that with the potential of eq. (6.11) the transient correlation $g_T(1,2)$ decreases exponentially but the resulting $\partial F^{(2)}/\partial \tau_0$ as τ_0^{-4} . Substituting eq. (6.13) in eq. (6.2) it is possible again to integrate out the directions of \vec{k} . We find

$$\frac{\partial F^{(2)}}{\partial \tau_0} = -\frac{512 \pi^5 n a^2}{m^2} F^{(0)}(\vec{r}) \left(\frac{a}{\pi}\right)^{3/2} \int_0^\pi d\theta \int_0^\infty d\kappa \kappa^2 \sin\theta \cdot$$

$$\cdot (2v \cos\theta - \kappa) \exp[-a(v^2 + \kappa^2 - 2v\kappa \cos\theta)] \cdot$$

$$\cdot \{C_1(\kappa\tau_0) + C_2(\kappa\tau_0)\}$$

$C_1(\xi)$ and $C_2(\xi)$ being given in eq. (6.8).

The θ -integration is elementary, the κ - and k -integrations for the potential of eq. (6.11) are possible with the aid of integral tables.^{23,24)} The result is

$$\frac{\partial F^{(2)}}{\partial \tau_0} = \frac{8\pi n a^2 e^4}{m^2} F^{(0)}(\vec{r}) \left[\left(v - \frac{3}{2av} - \frac{\kappa_d^2 \tau_0^2}{4a^2 v} \right) \cdot \right.$$

$$\cdot \exp\left(\frac{\kappa_d^2 \tau_0^2}{4a}\right) \left\{ \exp(\kappa_d \tau_0 v) \operatorname{Erfc}\left(\frac{\kappa_d \tau_0}{2\sqrt{a}} + v\sqrt{a}\right) \right. \quad (6.15)$$

$$\left. - \exp(-\kappa_d \tau_0 v) \operatorname{Erfc}\left(\frac{\kappa_d \tau_0}{2\sqrt{a}} - v\sqrt{a}\right) \right\} - \frac{1}{\sqrt{a}} \exp(-av^2) \Big]$$

where the error function is defined by

$$\operatorname{Erfc}(x) = \int_x^{\infty} \exp(-t^2) dt$$

The asymptotic expansion of the error function

$$\operatorname{Erfc}(x) = \exp(-x^2) \left(\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} - \dots \right)$$

leads directly to an asymptotic series in inverse powers of τ_0^2 . The resulting τ_0^{-4} -law may be written in the form

$$\frac{\partial F^{(2)}}{\partial \tau_0} \rightarrow - \frac{64 n a^5 e^4}{\pi^{1/2} m^2 k_B^4} \exp(-2a v^2) \frac{3 + 4a v^2}{\tau_0^4} \quad (6.16)$$

7. Complete initial value problem.

Up to now we only treated the simple initial value problem. That means that we chose the correlations and the distributions of higher than zeroth order to vanish initially. We now want to discuss the effect of non vanishing initial correlations. It is still possible to incorporate all initial conditions in the zero order functions, i.e.

$$F^{(2)}(t=0) = g_s^{(2)}(1, \dots, s, t=0) = 0, \quad s \geq 1$$

We restrict our attention again to the homogeneous weak coupling case.

In zero order we have again eqs. (5.1) and (5.2).

The solution of eq. (5.2) is now

$$g^{(0)}(1,2) = g^{(0)}(\vec{x}_{12} - \vec{v}_{12} \tau_0, \vec{v}_{12}, \vec{v}_1, \tau_0 = 0) \quad (7.1)$$

in obvious notation. The Fourier transform is

$$\hat{g}^{(0)}(\vec{k}, \vec{v}_{12}, \vec{v}_1, \tau_0) = \hat{g}^{(0)}(\vec{k}, \vec{v}_{12}, \vec{v}_1, \tau_0 = 0) \exp(-i \vec{k} \cdot \vec{v}_{12} \tau_0) \quad (7.2)$$

In first order we have for the distribution function

$$\frac{\partial F^{(1)}}{\partial \tau_0} + \frac{\partial F^{(1)}}{\partial \tau_1} = -8\pi^3 i \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \vec{k} \hat{\phi}(k) \hat{g}^{(0)}(\vec{k}, \vec{v}_{12}, \vec{v}_1, \tau_0) d^3 v_{12} d^3 k \quad (7.3)$$

Using eqs. (5.1) and 7.2) and integrating over τ_0 we obtain

$$F^{(1)} = -\tau_0 \frac{\partial F^{(1)}}{\partial \tau_1} - 8\pi^3 \frac{n}{m} \frac{\partial}{\partial \vec{v}} \cdot \int \frac{1 - \exp(-i \vec{k} \cdot \vec{u} \tau_0)}{\vec{k} \cdot \vec{u}} \vec{k} \hat{\phi}(k) \hat{g}^{(0)}(\vec{k}, \vec{u}, \vec{v}, \tau_0 = 0) d^3 u d^3 k \quad (7.4)$$

where we have omitted the subscript of \vec{v}_1 and have written $\vec{v}_{12} \equiv \vec{u}$. Because the τ_0 -dependence of the second term in the right-hand side of eq. (7.4) is different for different functions $\hat{g}^{(0)}(\tau_0 = 0)$ we have to require that both terms

in the right-hand side are bounded separately. This leads to eq. (5.6) and to the condition that the integral in eq. (7.4) has a finite limit if $z_0 \rightarrow \infty$.

We denote this integral as $\vec{I}(z_0, \vec{v})$, say, and write it in the form

$$\vec{I}(z_0, \vec{v}) = \int \frac{1 - \exp(-i k \eta)}{k \eta} \vec{k} \hat{\phi}(k) \Gamma(\vec{k}, \eta/z_0, \vec{v}) d^3 k d\eta \quad (7.5)$$

where

$$\Gamma(\vec{k}, \eta/z_0, \vec{v}) = \int \hat{g}^{(0)}(\vec{k}, \vec{u}, \vec{v}, z_0=0) \delta(\frac{\vec{k} \cdot \vec{u}}{k} - \eta/z_0) d^3 u$$

The function Γ exists certainly because $g^{(0)}(1,2)$ being directly related to a probability density through its definition eq. (2.13) must be integrable with respect to all its velocity arguments and Γ is nothing else than $\hat{g}^{(0)}$ integrated over the \vec{u} -components perpendicular to \vec{k} .

The existence of $\lim_{z_0 \rightarrow \infty} \vec{I}(z_0, \vec{v})$ clearly depends on the behaviour of $\Gamma(\vec{k}, \eta/z_0, \vec{v})$ for small η/z_0 . The limit exists if $\Gamma(\vec{k}, \eta/z_0, \vec{v})$ is bounded for small η/z_0 and if the \vec{k} -integral exists. We may write

$$|\vec{I}(z_0, \vec{v})| \leq \int \hat{\phi}(k) |\Gamma_0(\vec{k}, \vec{v})| d^3 k$$

with

$$\Gamma_0(\vec{k}, \vec{v}) = 2\pi \int_{-\infty}^{+\infty} \Gamma(\vec{k}, u, \vec{v}) \frac{du}{u}$$

There are no difficulties for small \vec{k} if $\phi(k)$ and $g^{(0)}(1,2, z_0=0)$ have a finite range in configuration space such that the expressions

$$\int \phi(z) d^3z = 8\pi^3 \hat{\phi}(k=0)$$

$$\int g^{(0)}(1,2, \tau_0=0) d^3x_{12} = 8\pi^3 \hat{g}^{(0)}(\vec{k}=0, \vec{v}_k, \vec{v}_l, \tau_0=0)$$

are finite for nearly all values of \vec{v}_{12} and \vec{v}_l . The \vec{k} -integral exists if $\hat{\phi}(k) \Gamma_0(\vec{k}, \vec{v})$ is bounded for all k , as is certainly the case if $\phi(z)$ and $g^{(0)}(1,2, \tau_0=0)$ are also absolutely integrable in configuration space, and if this product decreases sufficiently fast for large k .

The condition for small u on $\Gamma(\vec{k}, u, \vec{v})$ is related to the principle of "absence of parallel motions" of Sandri¹⁾, namely in the case where $\hat{g}^{(0)}(\vec{k}, \vec{u}, \vec{v}, \tau_0=0)$ depends only on \vec{u} through the absolute value u for small \vec{u} -components. In that case the condition may be written as

$$\lim_{u \rightarrow 0} u^{2-\delta} \hat{g}^{(0)}(\vec{k}, \vec{u}, \vec{v}, \tau_0=0) = 0, \quad \delta > 0 \quad (7.6)$$

We note furthermore that $\vec{I}(\tau_0, \vec{v})$ vanishes always according to eq. (7.5) if $\Gamma(\vec{k}, \tau/\tau_0, \vec{v})$ is an even function of \vec{k} .

We might be interested in the condition under which the relaxation of $\partial F^{(1)}/\partial \tau_0$ is faster than that of $\partial F^{(2)}/\partial \tau_0$ as calculated in the preceding section, i.e. faster than τ_0^{-4} . It follows from eq. (7.5) that this is the case if

$$\lim_{u \rightarrow 0} \frac{1}{u^3} \Gamma(\vec{k}, u, \vec{v}) = 0$$

It is also interesting to note that from the relation of $\Gamma(\vec{k}, u, \vec{v})$ with a probability density its integrability over u follows and therefore

$$\lim_{u \rightarrow 0} u^{-\delta} \Gamma(\vec{k}, u, \vec{v}) = 0, \quad \delta > 0$$

We summarize the state of affairs as follows:

If $\Gamma(\vec{k}, u, \vec{v})$ is proportional to $|u|^p$ for small u then

$p \leq -1$	is physically impossible
$-1 < p < 0$	leads to a growing $F^{(1)}$
$0 \leq p \leq 3$	results in a decaying $F^{(1)}$ but $\partial F^{(1)}/\partial \tau_0$ does not decrease faster than the $\partial F^{(2)}/\partial \tau_0$ of section 6.
$p > 3$	gives a $\partial F^{(1)}/\partial \tau_0$ relaxing faster than the $\partial F^{(2)}/\partial \tau_0$ of section 6

A special case of the condition $p \geq 0$ is the "absence of parallel motions" as formulated in eq. (7.6)

We now continue the formal description of the complete initial value problem in order to investigate the conditions for kineticity in the next order. The relevant equations are

$$\begin{aligned}
 & \frac{\partial g_2^{(1)}(1,2)}{\partial \tau_0} + \frac{\partial g_2^{(1)}(1,2)}{\partial \tau_1} + \vec{v}_{12} \cdot \frac{\partial g_2^{(1)}(1,2)}{\partial \vec{x}_{12}} - \frac{1}{m} \frac{\partial \phi_{12}}{\partial \vec{x}_{12}} \\
 & \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) \left\{ F^{(1)}(1) F^{(1)}(2) + g_2^{(1)}(1,2) \right\} - \frac{n}{m} \frac{\partial F^{(1)}(1)}{\partial \vec{v}_1} \\
 & \int \frac{\partial \phi_{13}}{\partial \vec{x}_1} g_2^{(1)}(2,3) d\vec{x}_3 - \frac{n}{m} \frac{\partial F^{(1)}(2)}{\partial \vec{v}_2} \cdot \int \frac{\partial \phi_{23}}{\partial \vec{x}_2} g_2^{(1)}(1,3) d\vec{x}_3 = \\
 & = \frac{n}{m} \int \left(\frac{\partial \phi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} + \frac{\partial \phi_{23}}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{v}_2} \right) g_3^{(1)}(1,2,3) d\vec{x}_3
 \end{aligned} \tag{7.7}$$

and

$$\frac{\partial F^{(2)}}{\partial \tau_0} + \frac{\partial F^{(1)}}{\partial \tau_1} + \frac{\partial F^{(1)}}{\partial \tau_2} = \frac{n}{m} \frac{\partial}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \vec{x}_{12}} g_2^{(1)}(1,2) d\vec{x}_2 \tag{7.8}$$

If the conditions mentioned above are satisfied then

$g_2^{(1)}(1,2)$ and $F^{(1)}$ decay in the time scale τ_0 .

Therefore we may put without loss of generality

$$\frac{\partial g_2^{(1)}(1,2)}{\partial \tau_1} = \frac{\partial F^{(1)}}{\partial \tau_1} = 0$$

We may proceed in the same way as in the preceding sections 5 and 6 in order to find the kinetic equation (5.17)

The relaxation phenomena as described by the transient

correlation of eq. (5.9) and the $\partial F^{(2)}/\partial \tau_0$ of

eq. (6.2) are not complete now. We introduce again Fourier

transforms and denote the new contributions by $\{\hat{g}_T(\vec{R}, \vec{u}, \vec{v}, \tau_0)\}$

and $\{\partial F^{(2)}(\vec{v})/\partial \tau_0\}$ respectively.

Using eq. (7.2) and a similar solution for the double

Fourier transform $\hat{g}_3^{(0)}$ namely

$$\hat{g}_3^{(0)}(\vec{R}, \vec{R}', \vec{u}, \vec{u}', \vec{v}, \tau_0) = \hat{g}_3^{(0)}(\vec{R}, \vec{R}', \vec{u}, \vec{u}', \vec{v}, \tau_0=0) \exp\{-i(\vec{R} \cdot \vec{u} + \vec{R}' \cdot \vec{u}')/\tau_0\}$$

and with the aid of the symmetry of correlations with

respect to interchange of particles we find

$$\begin{aligned}
 & \left(\frac{\partial}{\partial \tau_0} + i \vec{k} \cdot \vec{u} \right) \{ \hat{g}_T(\vec{k}, \vec{u}, \vec{v}, \tau_0) \} = \frac{1}{m} \int \hat{\phi}(k') i \vec{k}' \cdot \left(2 \frac{\partial}{\partial \vec{u}} + \frac{\partial}{\partial \vec{v}} \right) \\
 & \left[\hat{g}^{(0)}(\vec{k}-\vec{k}', \vec{u}, \vec{v}, \tau_0=0) \exp \{ -i(\vec{k}-\vec{k}') \cdot \vec{u} \tau_0 \} \right] d^3 k' + 8\pi^3 i \frac{n}{m} \hat{\phi}(k) \vec{k} \cdot \\
 & \cdot \int \exp(i \vec{k} \cdot \vec{u} \tau_0) \left\{ \hat{g}^{(0)}(-\vec{k}, \vec{u}', \vec{v}-\vec{u}, \tau_0=0) \frac{\partial F^{(0)}(\vec{v})}{\partial \vec{v}} \right. \\
 & \left. - \hat{g}^{(0)}(\vec{k}, -\vec{u}', \vec{v}, \tau_0=0) \frac{\partial F^{(0)}(\vec{v}-\vec{u})}{\partial \vec{v}} \right\} d^3 u' \quad (7.9) \\
 & - 8\pi^3 i \frac{n}{m} \int \vec{k}' \hat{\phi}(k') \cdot \left[\left(\frac{\partial}{\partial \vec{u}} + \frac{\partial}{\partial \vec{v}} \right) \left\{ \hat{g}_3^{(0)}(\vec{k}, \vec{k}', \vec{u}, \vec{u}', \vec{v}, \tau_0=0) \right. \right. \\
 & \left. \left. \exp(-i \vec{k} \cdot \vec{u} \tau_0) \right\} - \frac{\partial}{\partial \vec{u}} \left\{ \hat{g}_3^{(0)}(-\vec{k}, \vec{k}', -\vec{u}, \vec{u}', \vec{v}-\vec{u}, \tau_0=0) \right. \right. \\
 & \left. \left. \exp(i \vec{k} \cdot \vec{u} \tau_0) \right\} \right] \exp(-i \vec{k}' \cdot \vec{u}' \tau_0) d^3 k' d^3 u'
 \end{aligned}$$

We write the solution in the form, $\{ \hat{g}_T(\vec{k}, \vec{u}, \vec{v}, \tau_0=0) \} = 0$ being the initial condition,

$$\begin{aligned}
 \{ \hat{g}_T(\vec{k}, \vec{u}, \vec{v}, \tau_0) \} &= \exp(-i \vec{k} \cdot \vec{u} \tau_0) \int_0^{\tau_0} L(\vec{k}, \vec{u}, \vec{v}, \tau_0') \\
 &\quad \exp(i \vec{k} \cdot \vec{u} \tau_0') d\tau_0'
 \end{aligned}$$

where $L(\vec{k}, \vec{u}, \vec{v}, \tau_0)$ is short hand for the right-hand side of eq. (7.9). The equation for $\{ \partial F^{(2)}/\partial \tau_0 \}$ is

$$\left\{ \frac{\partial F^{(2)}(\vec{v})}{\partial \tau_0} \right\} = -8\pi^3 i \frac{n}{m} \frac{\partial}{\partial \vec{v}} \cdot \int \vec{k} \hat{\phi}(k) \{ \hat{g}_T(\vec{k}, \vec{u}, \vec{v}, \tau_0) \} d^3 u d^3 k \quad (7.10)$$

The most dangerous terms in eq. (7.9) are those which contain the $\partial/\partial \vec{u}$ -operator, because this operator introduces a factor τ_0 . We restrict the attention to two typical terms. Integrating by parts with respect to \vec{u} we derive

$$\left\{ \frac{\partial F^{(2)}(\vec{v})}{\partial \tau_0} \right\} = 8\pi^3 i \frac{n}{m^2} \frac{\partial}{\partial \vec{v}} \cdot \int_0^{\tau_0} ds \int d^3k d^3k' d^3u$$

$$\vec{k} \hat{\phi}(k) \hat{\phi}(k') \vec{k}, \vec{k}' s \exp(-i \vec{k} \cdot \vec{u} \tau_0)$$

$$\left[2 \hat{g}^{(0)}(\vec{k} - \vec{k}', \vec{u}, \vec{v}, \tau_0 = 0) \exp\{i \vec{k}' \cdot \vec{u}(\tau_0 - s)\} \right.$$

$$\left. - 8\pi^3 i \int d^3u' \hat{g}_3^{(0)}(\vec{k}, \vec{k}', \vec{u}, \vec{u}', \vec{v}, \tau_0 = 0) \exp\{-i \vec{k}' \cdot \vec{u}'(\tau_0 - s)\} \right]$$

$$+ \quad - \quad - \quad -$$

Introducing

$$\Gamma'(\vec{k} - \vec{k}', u, u', \vec{v}) = \int \hat{g}^{(0)}(\vec{k} - \vec{k}', \vec{u}, \vec{v}, \tau_0 = 0)$$

$$\delta\left(\frac{\vec{k} \cdot \vec{u}}{k} - u\right) \delta\left(\frac{\vec{k}' \cdot \vec{u}}{k'} - u'\right) d^3u$$

$$\Gamma_3(\vec{k}, \vec{k}', u, w, \vec{v}) = \int \hat{g}_3^{(0)}(\vec{k}, \vec{k}', \vec{u}, \vec{u}', \vec{v}, \tau_0 = 0)$$

$$\delta\left(\frac{\vec{k} \cdot \vec{u}}{k} - u\right) \delta\left(\frac{\vec{k}' \cdot \vec{u}'}{k'} - w\right) d^3u d^3u'$$

carrying out the s -integration and substituting

$$\eta = u \tau_0 \quad \eta' = u' \tau_0 \quad \xi = w \tau_0$$

we obtain

$$\left\{ \frac{\partial F^{(2)}(\vec{v})}{\partial \tau_0} \right\} = 8\pi^3 \frac{n}{m^2} \frac{\partial}{\partial \vec{v}} \cdot \int d^3k d^3k' d\eta \vec{k} \hat{\phi}(k) \hat{\phi}(k') \\ \vec{k} \cdot \vec{k}' \exp(-i k \eta) \left[2 \int d\eta' \Gamma'(\vec{k} - \vec{k}', \eta/\tau_0, \eta'/\tau_0, \vec{v}) \alpha(-i k' \eta') \right. \\ \left. - 8\pi^3 n \int d^3\xi \Gamma_3(\vec{k}, \vec{k}', \eta/\tau_0, \xi/\tau_0, \vec{v}) \alpha(i k' \xi) \right] + \dots$$

where

$$\alpha(ix) = \frac{1}{ix} + \frac{1}{x^2} \{1 - \exp(-ix)\}$$

Relaxation into the kinetic regime requires a $F^{(2)}(\vec{v})$ bounded for all τ_0 or

$$\lim_{\tau_0 \rightarrow \infty} \tau_0^{1+\delta} \left\{ \frac{\partial F^{(2)}(\vec{v})}{\partial \tau_0} \right\} = 0, \delta > 0$$

An analysis similar to that leading to our extended principle of absence of parallel motions shows that we have to impose the conditions that if for small u, u', w

$$\Gamma'(\vec{k} - \vec{k}', u, u', \vec{v}) \propto |u|^p |u'|^q$$

$$\Gamma_3(\vec{k}, \vec{k}', u, w, \vec{v}) \propto |u|^{p'} |w|^{q'} \quad (7.11)$$

then $p+q > 1$ and $p'+q' > 1$.

These conditions are stronger than those we found before.

Of course we can formulate also a new and stronger condition for $\partial F^{(2)}/\partial \tau_0$ to decay faster than τ_0^{-4} .

It should be emphasized that our conditions only refer to the validity of the multiple time scale method up to the order in which the kinetic equation is found. On one hand these conditions give certainly no information about the validity of the theory in higher orders, on the other hand it is not impossible that a system relaxes into the kinetic regime in a way which cannot be described correctly by the multiple time scale analysis. (e.g. mixing of different time scales).

8. Conclusions.

We treated a classical many body system interacting with central forces. It appeared to be advantageous to exploit the existence of multiple time scales by means of the extension method of Sandri¹⁾. In this way we derived kinetic equations valid for long times for weakly coupled and long range systems. In the case of weak coupling we investigated the relaxation into the kinetic regime. We found conditions on the initial correlations which are stronger than the "absence of Parallel motions" of Sandri. If these initial correlations satisfy still stronger conditions the relaxation into the kinetic regime obeys a t^{-4} -law.

The theory exposed here does not go beyond the order in which the kinetic equation is obtained.

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Appendix A Proof of eq. (5.25) starting from eq. (5.24).

The integral in eq. (5.24) may be written as a two dimensional integral

$$C = \frac{8\pi^5 n}{m^2} \frac{1}{2\pi} \int k_{\perp}^2 \hat{\phi}^2(k_{\perp}) d^2 k_{\perp} \quad (A.1)$$

The vector \vec{k}_{\perp} lies here in a fixed plane and the integration has to be carried out over this entire plane. From the definition of the Fourier transform $\hat{\phi}(k)$ we can derive

$$\hat{\phi}(k_{\perp}) = \frac{1}{8\pi^3} \int \phi(\sqrt{z^2 + z'^2}) \exp(-i \vec{k}_{\perp} \cdot \vec{z}) dz d^2 z \quad (A.2)$$

where the integration over z runs from $-\infty$ to $+\infty$ and the \vec{z} -integration over the entire \vec{z} -plane.

Now we form the expression

$$\vec{k}_{\perp} \hat{\phi}(k_{\perp}) = \frac{i}{8\pi^3} \int \phi(\sqrt{z^2 + z'^2}) \frac{\partial}{\partial \vec{z}} \exp(-i \vec{k}_{\perp} \cdot \vec{z}) dz d^2 z$$

or by integration by parts

$$\vec{k}_{\perp} \hat{\phi}(k_{\perp}) = -\frac{i}{8\pi^3} \int \frac{\vec{z} \phi'(\sqrt{z^2 + z'^2})}{\sqrt{z^2 + z'^2}} \exp(-i \vec{k}_{\perp} \cdot \vec{z}) dz d^2 z \quad (A.3)$$

Inserting eq. (A.3) into eq. (A.1) we obtain

$$C = -\frac{n}{m^2 16 \pi^2} \int \frac{\vec{z}_1 \cdot \vec{z}_2 \phi'(\sqrt{z_1^2 + z_1'^2}) \phi'(\sqrt{z_2^2 + z_2'^2})}{\sqrt{(z_1^2 + z_1'^2)(z_2^2 + z_2'^2)}} \cdot \exp\{-i \vec{k}_{\perp} \cdot (\vec{z}_1 + \vec{z}_2)\} dz_1 dz_2 d^2 z_1 d^2 z_2 d^2 k_{\perp}$$

Because

$$\int \exp \{ -i \vec{k}_\perp \cdot (\vec{z}_1 + \vec{z}_2) \} d^2 k_\perp = (2\pi)^2 \delta(\vec{z}_1 + \vec{z}_2)$$

$\delta(\vec{z}_1 + \vec{z}_2)$ being a two dimensional delta function, we see immediately that

$$C = \frac{n \pi}{2 m^2} \int_0^\infty dz \int_{-\infty}^{+\infty} dz_1 \int_{-\infty}^{+\infty} dz_2 \frac{z^3 \phi'(\sqrt{z_1^2 + z^2}) \phi'(\sqrt{z_2^2 + z^2})}{\sqrt{z_1^2 + z^2} \sqrt{z_2^2 + z^2}}$$

and this leads directly to eq. (5.25) .

Appendix B. Penrose criterion and Bogoliubov-Lenard-Balescu equation.

If $F(\vec{v})$ is isotropic $\tilde{F}(u)$ does not depend on \vec{x} and is single humped. This follows from

$$\begin{aligned}\frac{\partial \tilde{F}(u)}{\partial u} &= \frac{\partial}{\partial u} \int_0^\infty F(u^2 + v_\perp^2) 2\pi v_\perp dv_\perp \\ &= \int_0^\infty \frac{u}{v_\perp} \frac{\partial F(u^2 + v_\perp^2)}{\partial v_\perp} 2\pi v_\perp dv_\perp \\ &= -2\pi u F(u^2)\end{aligned}$$

Therefore $\tilde{F}(u)$ has only one extremum, a maximum at $u=0$, and is stable according to the Penrose criterion.

We now prove that an isotropic $F(v)$ remains isotropic if it obeys the Bogoliubov-Lenard-Balescu equation given in eq. (5.36) together with eqs. (5.18), (5.37) and (5.38). Performing the \vec{u} -integration we may write

$$\frac{\partial F}{\partial \tau_i} = \frac{\omega_p^4}{8\pi^2 n} \frac{\partial}{\partial \vec{v}} \cdot \int \vec{x} L(\vec{v}, k, \vec{x}) d^3x dk \quad (\text{B.1})$$

with

$$L(\vec{v}, k, \vec{x}) = \frac{\tilde{F}(\vec{x}, \vec{x} \cdot \vec{v}) \vec{x} \cdot \frac{\partial F(\vec{v})}{\partial \vec{v}} - F(\vec{v}) \frac{\partial \tilde{F}(\vec{x}, \vec{x} \cdot \vec{v})}{\partial (\vec{x} \cdot \vec{v})}}{k \left[\left(1 - \frac{\omega_p^2}{k^2} P \right) \frac{\partial \tilde{F}(\vec{x}, u)}{\partial u} \Big|_{u=\vec{x} \cdot \vec{v}}^2 + \frac{\pi^2 \omega_p^4}{k^4} \left\{ \frac{\partial \tilde{F}(\vec{x}, u)}{\partial u} \right\}_{u=\vec{x} \cdot \vec{v}}^2 \right]} \quad (\text{B.2})$$

If $F(\vec{v})$ is isotropic the explicit dependence of L on \vec{x} vanishes and we have

$$L = L(v, \lambda, \kappa)$$

where

$$\lambda = \frac{\vec{x} \cdot \vec{v}}{v}$$

and we obtain from eq. (B.1)

$$\frac{\partial F}{\partial \tau_i} = \frac{\omega_p^4}{4\pi n} \int_{-1}^{+1} d\lambda \int_0^\infty dk \lambda \frac{\partial L(v, \lambda, \kappa)}{\partial v}$$

i.e. $\partial F / \partial \tau_i$ is a function only of the magnitude v of the velocity. Therefore $F(\vec{v})$ remains isotropic if it is isotropic at some fixed time.

The more general statement, that $F(\vec{v})$ remains stable for all directions if it has this property at some fixed time, is not true. We illustrate this by means of the Landau equation (5.23) for which we give a counter-example.

If the statement is not true for the Landau equation it cannot be expected to be true for the eqs. (B.1), (B.2). The difference is admittedly that in the Bogoliubov-Lenard-Balescu equation the denominator goes through a zero when the border of instability is passed. If, however, one starts with a distribution such that $\tilde{F}(\vec{x}, u)$ is stable for all \vec{x} except one specific value \vec{x}_0 for which $\tilde{F}(\vec{x}_0, u)$ is at the border of instability, then the zero of the denominator will be integrable in \vec{k} -space and no essential difference with results obtained from the Landau-equation is to be expected.

Therefore we restrict ourselves to the proof that the Landau-equation does not conserve the property of electrostatic stability.

We write the Landau-equation in the form

$$\frac{\partial F}{\partial \tau} = C \frac{\partial}{\partial \vec{v}} \cdot \int \vec{x} \left\{ \tilde{F}(\vec{x}, \vec{x}, \vec{v}) \vec{x} \cdot \frac{\partial F(\vec{v})}{\partial \vec{v}} - F(\vec{v}) \frac{\partial \tilde{F}(\vec{x}, \vec{x}, \vec{v})}{\partial (\vec{x}, \vec{v})} \right\} d^2 x \quad (\text{B.3})$$

Substitute

$$F(\vec{v}) = \delta(v_x) \delta(v_y) f(v_z)$$

$$\tilde{F}(\vec{x}, u) = \frac{1}{|x_z|} f\left(\frac{u}{x_z}\right) \quad (\text{B.4})$$

in the right-hand side and integrate over v_x and v_y .
The result is

$$\frac{\partial f}{\partial \tau} = -C' \frac{\partial}{\partial v_z} \left(f \frac{\partial f}{\partial v_z} \right) \quad (\text{B.5})$$

where

$$C' = 2C \int \frac{x_x^2 + x_y^2}{|x_z|} d^2 x \quad (\text{B.6})$$

is a positive constant.

Now we consider a situation where $f(v_z)$ is single-humped but has a horizontal inclination point at $v_z = v_0$, say

$$\left. \frac{\partial f}{\partial v_z} \right|_{v_z=v_0} = \left. \frac{\partial^2 f}{\partial v_z^2} \right|_{v_z=v_0} = 0 \quad (\text{B.7})$$

We assume furthermore

$$\int \frac{\partial f(v_z)/\partial v_z}{v_z - v_0} dv_z > 0 \quad (\text{B.8})$$

If

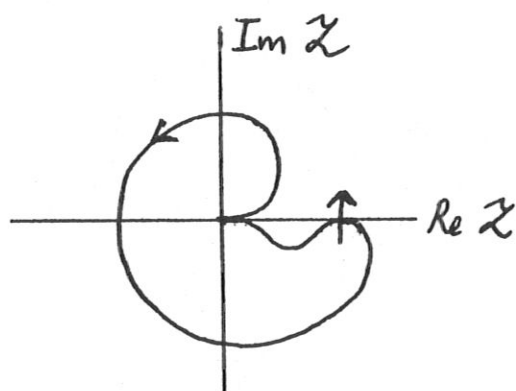
$$\text{a) } \left. \frac{\partial^3 f}{\partial v_z^3} \right|_{v_z=v_0} < 0, \left. \frac{\partial^2 f}{\partial \tau_1 \partial v_z} \right|_{v_z=v_0} > 0$$

or

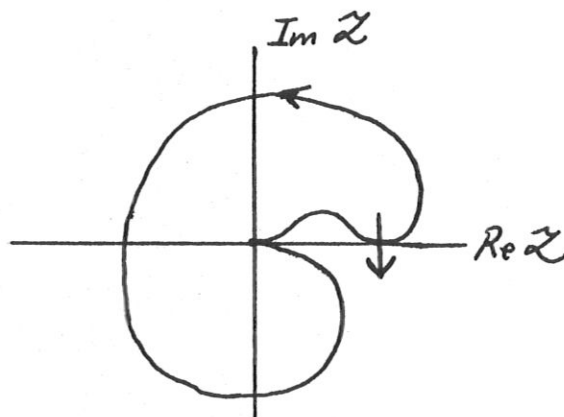
$$\text{b) } \left. \frac{\partial^3 f}{\partial v_z^3} \right|_{v_z=v_0} > 0, \left. \frac{\partial^2 f}{\partial \tau_1 \partial v_z} \right|_{v_z=v_0} < 0 \quad (\text{B.9})$$

then $f(v_z)$ passes the border of instability.

The situations a) and b) correspond to the Nyquist diagrams a) and b), see Ref. 22.



situation a



situation b

From eqa. (B.5), B.7) we find

$$\left. \frac{\partial^2 f}{\partial \tau_1 \partial v_z} \right|_{v_z=v_0} = -C' f(v_0) \left. \frac{\partial^3 f}{\partial v_z^3} \right|_{v_z=v_0} \quad (\text{B.10})$$

Therefore one of the conditions (B.9) is always satisfied and $f(v_z)$ becomes unstable.

Our example (B.4) is, of course, a very singular one. The integral in eq. (B.6) diverges and the distribution is at the border of instability for all $x_z \neq 0$ because $\tilde{F}(\vec{x}, u)$ has a minimum at $u = x_z v_0$ and

$$\int_{-\infty}^{+\infty} \frac{\partial \tilde{F}(\vec{x}, u) / \partial u}{u - x_z v_0} du = \frac{1}{x_z^2} \int_{-\infty}^{+\infty} \frac{\partial f(v_z) / \partial v_z}{v_z - v_0} dv_z > 0$$

as may be seen from eqs. (B.4) and (B.8).

These disadvantages may be removed, however, by taking sharply peaked Maxwellians instead of delta-functions in eq. (B.4). This will not change the result (B.10) appreciably.