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On the Superkinetic Equation
for
Spatially-Homogeneous Plasmas.

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Abstract

A derivation from the Liouville equation of the equations describing the irreversible evolution of the N-momenta distribution function for a classical spatially-homogeneous plasma $(n r_D^3 \gg 1)$ is given. Collective effects are included. This "superkinetic" description reduces to the weak-coupling "master equation" of Brout and Prigogine when collective effects are neglected. The results are given in terms of a closed, coupled set of equations analogous to those obtained by Bogoliubov for the kinetic description of a plasma. The conditions under which this system reduces to Bogoliubov's, and hence to the kinetic equation of Balescu and Lenard, are derived. The origins of irreversibility in plasmas are discussed in the light of this and other recent work, and the properties of the stationary solutions of the superkinetic equations are analyzed.

I. Introduction

In the study of the logical relationship between the Liouville equation of statistical mechanics and the equations describing the irreversible evolution of classical N-body systems, an important link is provided by the so-called "superkinetic" description. In the case of weakly-coupled systems without significant collective effects (screening, etc.) this description has been given by a single equation[1,2,4] referred to as the "weak-coupling master equation", which describes the time evolution of the distribution function for the whole set of N-momenta. Since with a few additional assumptions the kinetic equation of Landau can be derived from this superkinetic equation[2,3], such a description provides a route alternative to the one normally used [5,6,7,8,9] (involving the BBGKY hierarchy)in passing from the Liouville equation to the kinetic regime. Further insight is thus provided into the nature of irreversible processes in gases.

To date no general superkinetic description for a plasma has been made available. As mentioned above the weak-coupling master equation of Brout and Prigogine does not include collective effects. The purpose of the present paper is to propose for a plasma a description of irreversible behavior at the superkinetic level, taking account of those collective phenomena appropriate to the macroscopically homogeneous case.

The existing superkinetic equation for weakly-coupled systems is of fundamental importance in itself through the fact that, although its derivation nowhere requires the assumption of molecular chaos or statistical independence of the particles, the equation is time irreversible. Thus, as emphasized by Prigogine and Balescu[2], the phenomenon of irreversibility is seen to be independent of any such assumption.

Recent work[5,8,6,7,9] has made it clear that irreversibility arises rather through the appearance in many-body systems of different phenomena occurring on very different time scales, some fast, some slow. Before the "slow" processes can become fully established, the "fast" processes reach their time-asymptotic states, determining in effect the "initial" conditions for the slow processes, and the evolution no longer can reverse itself. (This holds, of course, only for systems with a sufficiently large number of particles, so that the Poincaré recurrence time is essentially infinite and plays no role.)

The appearance of a multiplicity of time scales is a quite general feature of many-body systems.[4,5] In an ordinary simple dilute gas with short-range forces, for example, these different time scales can easily be visualized: the average interaction time, the average time between interactions, the hydrodynamic time, the diffusion time or time to reach thermodynamic equilibrium. Although it is not a main point of the present paper, use will be made of a technique recently developed [6,7,9] which takes advantage of the natural appearance of this multiplicity of time scales in treating the time-asymptotic behavior of the governing equations for many-body systems. The connections between this technique and the ideas of Enskog in treating the Boltzmann equation on the one hand, and those of Bogoliubov in treating the Hierarchy on the other, have been demonstrated in two previous papers.[10,9]

In the present article we are concerned with spatially-homogeneous plasmas in which collective phenomena are important. We shall later assume therefore that the number of particles in a Debye sphere is very large. For simplicity we assume no magnetic field to be present. Quantum effects are neglected;[11] we describe our "plasma" in terms of a single charged species (the electrons). The effect of the ions is included only through the assumption of a neutralizing background of charge.

We shall be dealing in what follows with the limit of very large systems. This is a critical point of the theory. In order first to gain some insight into the meaning of this limit, we shall discuss in the next Section the familiar BBGKY Hierarchy and the normalization of its solutions for large systems.

II. The BBGKY Hierarchy and Normalization.

Since we have adopted the classical point of view, we start from Liouville's fundamental law of conservation of probability density in phase space:

$$1)\quad \frac{D\hat{F}^N}{D\hat{t}} \equiv \frac{\partial \hat{F}^N}{\partial \hat{t}} + \sum_{i=1}^{N} \hat{\underline{v}}_i \cdot \frac{\partial \hat{F}^N}{\partial \hat{\underline{x}}_i} - \frac{1}{m}\sum_{i<j}^{N} \frac{\partial \hat{\mathcal{U}}_{ij}}{\partial \underline{x}_i}\cdot\left(\frac{\partial}{\partial \underline{v}_i} - \frac{\partial}{\partial \underline{v}_j}\right)\hat{F}^N - \frac{1}{m}\sum_{i=1}^{N} \hat{\underline{W}}_i \cdot \frac{\partial}{\partial \hat{\underline{v}}_i}\hat{F}^N = 0$$

where $\hat{F}^N(\hat{\underline{x}}_1 \ldots \hat{\underline{x}}_N, \underline{v}_1 \ldots \underline{v}_N)$ is symmetric to interchange among the N particles of any particle i with any other j, and so normalized that in a box of volume V

$$2)\quad 1 = \int \hat{F}^N \frac{d\hat{\underline{x}}_1 \ldots d\hat{\underline{x}}_N}{V^N} d\hat{\underline{v}}_1 \ldots d\hat{\underline{v}}_N$$

The Liouville equation (1) guarantees that this normalization is preserved for all time. In equation (1), as we have written it, the ions do not appear because of the assumptions given above, and $\underline{W}_i$ represents the effect of any external force field and (in principle) the effects of the walls.[12] $\hat{\mathcal{U}}_{ij}$ is the spherically symmetric Coulomb interaction potential

$$3)\quad \hat{\mathcal{U}}_{ij} = \frac{e^2}{|\hat{\underline{x}}_i - \hat{\underline{x}}_j|}$$

while **e** is the electronic charge and **m** the electron mass.

It is convenient for our purposes to write (1) (2) and (3) and all subsequent equations in dimensionless form. Thus the variables $\hat{\underline{x}}_i, \hat{\underline{v}}_i, \hat{t}$ are replaced by $\underline{x}_i, \underline{v}_i, t$, measured in units of $r_0, (\Theta/m)^{1/2}$, and $r_0/(\Theta/m)^{1/2}$, respectively. (The mean energy per electron is $3\Theta/2$.) $\hat{\mathcal{U}}_{ij}$ and $\hat{\underline{W}}_i$ are replaced by $\mathcal{U}_{ij}$ and $\underline{W}_i$, measured in units of e^2/r_0 and e^2/r_0^2, respectively, while $\hat{F}^N$ is replaced by F^N measured in units $(\Theta/m)^{-3N/2}$. The results are:

$$(1a)\quad \frac{\partial F^N}{\partial t} + K^N F^N - \frac{e^2}{r_0\Theta}\left(I^N F^N + \mathcal{W}^N F^N\right) = 0$$

$$(2a)\quad 1 = \left(\frac{r_0^3}{V}\right)^N \int F^N (d\underline{x})^N (d\underline{v})^N$$

$$(3a)\qquad \mathcal{U}_{ij} = \frac{r_0}{|\hat{\underline{x}}_i - \hat{\underline{x}}_j|} \equiv \frac{r_0}{|\hat{\underline{x}}_{ij}|} = \frac{1}{|\underline{x}_{ij}|}$$

where we have introduced the notation

$$(d\xi)^N = \prod_{i=1}^{N} d\xi_i$$

$$\mathcal{K}^N = \sum_{i=1}^{N} \underline{v}_i \cdot \frac{\partial}{\partial \underline{x}_i}$$

$$I^N = \sum_{i<j}^{N} \frac{\partial \mathcal{U}_{ij}}{\partial \underline{x}_i} \cdot \left(\frac{\partial}{\partial \underline{v}_i} - \frac{\partial}{\partial \underline{v}_j}\right) \equiv \sum_{i<j}^{N} I_{ij}$$

(If for a plasma we identify r_0 as the Debye length, $r_D = (\theta/4\pi n e^2)^{1/2}$ where $n \equiv N/V$, we see that the parameter $e^2/r_D\theta$ is identically $(4\pi n r_D^3)^{-1}$, i.e. proportional to the inverse of the number of electrons in the Debye sphere).

Since we are interested in bulk effects in homogeneous plasmas, we neglect in what follows external fields and the effect of walls, and put $\underline{W}_i \equiv 0$. The neglect of edge effects is reasonable so long as $r_D^3 / V \ll 1$. We shall discuss this approximation further below.

The well-known BBGKY Hierarchy provides an infinite set of coupled equations for the reduced functions

$$(4)\qquad F^s(\underline{x}_1 \ldots \underline{x}_s, \underline{v}_1 \ldots \underline{v}_s) \equiv \left(\frac{r_D^3}{V}\right)^{N-s} \int F^N d\underline{x}_{s+1} \ldots d\underline{x}_N \, d\underline{v}_{s+1} \ldots d\underline{v}_N \equiv \int \left(\frac{r_D^3 d\underline{x}}{V}\right)(d\underline{v})^{N-s} F^N$$

The Hierarchy is obtained by integration of equation (1a) for each s with respect to the N-s "unwanted" phase variables, and by taking advantage of the symmetries and normalizability of F^N. The result can be written in dimensionless form:

$$(5)\qquad \frac{\partial F^s}{\partial t} + K^s F^s - \left(\frac{e^2}{r_D\theta}\right) I^s F^s = \left(\frac{e^2}{r_D\theta}\right)\left(\frac{N-s}{V} r_D^3\right) L_s F^{s+1}$$

where

(6) $$L_s \equiv \sum_{i=1}^{s} L_{i,s+1}$$

and

(7) $$L_{ij} \equiv \int dv_j \cdot dx_j \frac{\partial U_{ij}}{\partial x_i} \cdot \frac{\partial}{\partial v_i}$$

F^s is symmetric to interchange of particles in the set s.

The importance of the reduced functions F^s, and hence of the BBGKY Hierarchy, lies in the fact that the first member of the set, F^1, is the kinetic function (the single-particle distribution function) while the last member is the probability density in $6N$-dimensional phase space, satisfying the Liouville equation. Moreover, in the Hierarchy maximum advantage has already been taken of the symmetries of the function F^N, and the parameter N appears explicitly. Study of the properties of the Hierarchy has provided considerable insight into the connection between the Liouville equation and kinetic theory. [5,6,7,8,9]

The normalization used in (4) in defining the reduced functions F^s is the same as that employed by Bogoliubov[5], since we have obviously $F^N \equiv (V/r_D^3)^N D$, where D is the density in phase space in his notation. In the limit of large systems, $V/r_D^3 \to \infty$, F^N remains finite, while $D \to 0$. The functions F^s are to be interpreted as joint probability densities in phase space.

However, the Hierarchy is traditionally treated only in the so-called "bulk limit" $V \to \infty$, $N \to \infty$, $N/V = n$ (fixed), so that in each equation the factor $(N-s)/V$ is replaced simply by n. (We shall call this the "limit Hierarchy.") The limit solutions obtained then do not, in general, identically satisfy the normalization conditions originally supposed; the normalization is preserved only in the limit $N \to \infty$.

For example, in the case of a plasma, the classical equilibrium solution with Debye screening gives

$$\int F^2 \left(\frac{r_D^3 dx_1}{V}\right)\left(\frac{r_D^3 dx_2}{V}\right) dv_1 \, dv_2 = 1 - \frac{1}{N}$$

instead of 1.

An interesting interpretation of this point has been suggested by Prof. Schlüter.* We observe that one produces the limit Hierarchy exactly by substituting for F^s in (5) the renormalized reduced functions $\overset{*}{F}{}^s \equiv (N!/(N-s)!\,N^s)F^s$. The new functions are exactly the solutions of the limit Hierarchy and go over into the original F^s as $N \to \infty$. Moreover, for example,

$$\int \overset{*}{F}{}^2 \left(\frac{r_D^3 dx_1}{V}\right)\left(\frac{r_D^3 dx_2}{V}\right) dv_1\, dv_2 = 1 - \frac{1}{N}$$

The $\overset{*}{F}{}^s$, then, are density functions; in fact the "correlation" function $\overset{*}{F}{}^2 - F^1F^1$ is proportional to the charge density relative to a singled-out election; and the value of its integral, $(-1/N)$ insures complete screening.

Since in the Hierarchy the behavior of the s-body cluster is coupled to a "typical" $s+1$ -body cluster, the problem of solving the Hierarchy is one of achieving a reasonable closure of the system of equations. Closure is usually achieved by assuming (but only in the initial conditions) a certain amount of statistical independence of the particles, and, in the case of a plasma, a related cluster expansion in terms of the two-body correlations.[5] We see that such an expansion is best interpreted in terms of the $\overset{*}{F}{}^s$, which are the solutions actually obtained from the limit Hierarchy. The true joint probabilibites can then be interpreted from the definitions relating $\overset{*}{F}{}^s$ and F^s. For example, one has, while $\overset{*}{F}{}^1 = F^1$, $\overset{*}{F}{}^2 = \frac{N-1}{N}F^2$. If we expect a solution of the form $\overset{*}{F}{}^2 = F^1F^1 + \varepsilon g$, then we should write for F^2

$$F^2 = F^1F^1 + \varepsilon\hat{\gamma} + O(1/N)$$

and note that this implies

$$\varepsilon \int \frac{r_D^3 dx_1}{V} dv_1\, \hat{\gamma} = O(1/N)$$

instead of zero, as in the usual treatment.

This point of view enables us to interpret the limit of large systems slightly differently from the usual formal "bulk limit", and in a manner more appropriate to plasmas. We must remember that even though the limit Hierarchy can be obtained "exactly" through renormalization of (4) and (5), it still implies the approximation $(r_D^3/V) \equiv \lambda$ (say) where $\lambda \to 0$, through the neglect of the effect of the walls. On the other hand, in a plasma we assume, as already mentioned, Nr_D^3/V very large, $O(1/\varepsilon)$, say. Then $\frac{1}{N} = O(\varepsilon\lambda)$

* The author is also indebted to D. Pfirsch and P. Schram for several illuminating discussions on this subject.

i.e., a cross term in the two separate expansions. We can avoid interference between the two expansions only if we assume λ sufficiently small compared to ε (in the limit, transcendentally small). It is however most convenient in what follows simply to keep track of the error terms in the manner indicated above. In our treatment of the superkinetic description we shall then not require functions analogous to the $\overset{*}{F}{}^{s}$, and the symbols $\sum_{j=1}^{N}$ etc., will have meaning also in the limit of large systems.

III. The Generalized Hierarchy

In our study of the superkinetic description of plasmas we are interested primarily in the distributions of a large number of bodies in momentum space, with relatively less importance to be attached to configuration space. We consider therefore the generalized set of reduced functions:

(8) $$f^{s,\nu}(\underline{v}_1\ldots\underline{v}_s,\underline{x}_1\ldots\underline{x}_\nu) \equiv \left(\frac{r_D^3}{V}\right)^{N-\nu}\int F^N d\underline{x}_{\nu+1}\ldots d\underline{x}_N\, d\underline{v}_{s+1}\ldots d\underline{v}_N$$

or

$$f^{s,\nu}(\{s\}|\{\nu\}) \equiv \int F^s\left(\frac{r_D^3\, d\underline{x}}{V}\right)^{s-\nu}, \quad \nu\leq s$$

The set $\{\nu\}$ is always contained in the set $\{s\}$; the number s may be quite large. Starting either with (1a) or with (5) one obtains the generalized hierarchy, valid in the limit $(r_D^3/V)\rightarrow 0$,

$$\frac{\partial f^{s,\nu}}{\partial t} + K^\nu f^{s,\nu} - \frac{e^2}{r_D\theta} I^\nu f^{s,\nu} =$$

(9) $$= \frac{e^2}{r_D\theta}\left\{\sum_i^{\{\nu\}}\sum_j^{\{s-\nu\}}\int \frac{r_D^3\, d\underline{x}_j}{V} I_{ij} f^{s,\nu+1}_{(\{s\}|\{\nu\},j)} + \sum_{i<j}^{\{s-\nu\}}\iint\left(\frac{r_D^3\, d\underline{x}_i}{V}\right)\left(\frac{r_D^3\, d\underline{x}_j}{V}\right) I_{ij} f^{s,\nu+2}_{(\{s\}|\{\nu\},i,j)}\right\}$$

$$+ \frac{e^2}{r_D\theta}\left(\frac{N-s}{V}\right) r_D^3 \left\{\sum_i^{\{\nu\}} L_{i,s+1}\, f^{s+1,\nu+1}_{(\{s\},s+1|\{\nu\},s+1)} + \sum_i^{\{s-\nu\}}\int \frac{r_D^3\, d\underline{x}_i}{V} L_{i,s+1}\, f^{s+1,\nu+2}_{(\{s\},s+1|\{\nu\},i,s+1)}\right\}$$

Study of the properties of the functions $f^{s,\nu}$ and their related hierarchy (9) should be expected to provide insight into the relationship between the Liouville equation and the superkinetic as well as kinetic descriptions of a plasma. We note that if $s=\nu$,eq.(9) reduces to the usual Hierarchy (5).

For the superkinetic equation we are interested first of all in the distribution function for all N momenta, $f^{N,0}(\underline{v}_1 \ldots \underline{v}_N)$. It is then sufficient to investigate (9) for the case s=N:

$$\frac{\partial f^{N,\nu}}{\partial t} + K^{\nu} f^{N,\nu} - \frac{e^2}{r_D \theta} I^{\nu} f^{N,\nu} =$$

(10) $$= \left(\frac{e^2}{r_D\theta}\right)\left\{ \sum_i^{\{\nu\}} \sum_j^{\{N-\nu\}} \int \frac{r_D^3 d\underline{x}_j}{V} I_{ij} f^{N,\nu+1}_{(\{\nu\},j)} + \sum_{i<j}^{\{N-\nu\}} \int \left(\frac{r_D^3 d\underline{x}_i}{V}\right)\left(\frac{r_D^3 d\underline{x}_j}{V}\right) I_{ij} f^{N,\nu+2}_{(\{\nu\},i,j)} \right\}$$

This hierarchy for the functions $f^{N,\nu}$ was first given by Higgins.[13] Each function $f^{N,\nu}$ is a function of all N momenta; however, for $\nu \neq 0$ it is symmetric to interchange of particles only within the sets $\{\nu\}$ or $\{N-\nu\}$, and not to interchange between the sets.

We note that the superkinetic function $f^{N,0}$ is coupled only to $f^{N,2}$, while $f^{N,2}$ is coupled to $f^{N,3}$ and $f^{N,4}$, etc. (The relationship between this hierarchy and the Fourier description of Prigogine and Balescu is readily apparent[13].) The evolution of $f^{N,0}$ in momentum space is determined by the development of the two-body correlations in configuration space:

(11) $$\frac{\partial f^{N,0}}{\partial t} = \left(\frac{e^2}{r_D\theta}\right) \sum_{i<j}^{N} \int \left(\frac{r_D^3 d\underline{x}_i}{V}\right)\left(\frac{r_D^3 d\underline{x}_j}{V}\right) I_{ij} f^{N,2}_{(i,j)}$$

We note for future use that whenever $f^{N,2}_{(\underline{x}_i,\underline{x}_j)} = f^{N,2}_{(\underline{x}_i-\underline{x}_j)}$ we have the identity

$$\int \left(\frac{r_D^3 d\underline{x}_i}{V}\right)\left(\frac{r_D^3 d\underline{x}_j}{V}\right) I_{ij} f^{N,2} \equiv \frac{r_D^3}{V} \int d\underline{x}_{ij} I_{ij} f^{N,2}_{(\underline{x}_{ij})}$$

Moreover, from the symmetry of $\mathcal{U}_{ij}$, the spatially-homogeneous parts of the functions $f^{N,\nu}$ make no contribution to the integrals in (10) or (11).

IV. Approximate Closure - The "Cluster" Expansion.

We seek an approximate closure of the hierarchy (10) under limiting conditions appropriate for a plasma. We have already assumed r_D^3/V to be very small in order to neglect edge effects (see above discussion); we shall have further use for this approximation below.In the bulk of the plasma we expect the average interaction between particles to be weak. We express this by the assumption

(12) $$\frac{e^2}{r_D\theta} \equiv \varepsilon \ll 1$$

which, as already pointed out, implies

(13) $$4\pi n r_D^3 = \frac{1}{\varepsilon} \gg 1$$

We see that if $\frac{r_D^3}{V}$ is very small, the inequality (13) requires N to be very large (of order $V/r_D^3\varepsilon$), and we therefore expect the Poincaré recurrence time to be longer than any other time of interest.

As implied by (13), although the average interaction between pairs is weak, the number of particles simultaneously interacting in a plasma can be very large. Thus the terms appearing in the sums on the rhs of (10) can act in concert; we shall demonstrate that it is exactly these terms which in equilibrium provide the usual Debye screening. Thus we assume, for example, $\sum_{j=1}^{N} = O(N)$, $\sum_{i<j}^{N} = O(N^2)$ for the purposes of ordering terms in our expansion. Moreover, $f^{N,\nu}$ can approach a product of N functions (as for example, in equilibrium), so we assume $\frac{\partial}{\partial t} = O(N)$ when operating on $f^{N,\nu}$. Thus in an expansion in the small parameter ε , we may neglect in lowest order (at least for each ν sufficiently small compared to N) the interaction term on the lhs in (10), but we must retain the terms on the right.

With this ordering in mind we make the observation, based on the symmetry of U_{ij} already mentioned, that to zeroth order in ε and r_D^3/V, even including the collective terms, the hierarchy (10) is solved for each $\nu \ll N$ by the assumption

$$f^{N,\nu} = f^{N,0} + O(\varepsilon) + O\left(\frac{r_D^3}{V}\right) + O\left(\varepsilon \frac{r_D^3}{V}\right) \tag{14}$$

provided this is compatible with the initial conditions. This means physically that the interactions between particles cannot produce spatial correlations (or, more simply, inhomogeneities) of a magnitude larger than $O(\varepsilon)$.

We thus limit ourselves to initial conditions of the type (14) and seek solutions of (10) in the form

$$f^{N,\nu} = f^{N,0} + \varepsilon \gamma^{N,\nu} + O\left(\frac{1}{N}\right) \tag{15}$$

where the functions $\gamma^{N,\nu}$ thus defined must have the property

$$\varepsilon \int \left(\frac{r_D^3 d\underline{x}}{V}\right)^{\nu} \gamma^{N,\nu} = O\left(\frac{1}{N}\right) \tag{16}$$

and satisfy the hierarchy (correct to $O(\varepsilon)$):

$$\frac{\partial \gamma^{N,\nu}}{\partial t} + K^{\nu}\gamma^{N,\nu} = I^{\nu} f^{N,0} + \varepsilon \sum_{i}^{\{\nu\}} \sum_{j}^{\{N-\nu\}} \frac{r_D^3}{V} \int d\underline{x}_j I_{ij} \gamma^{N,\nu+1}$$
$$+ \varepsilon \frac{r_D^6}{V^2} \left[\sum_{i<j}^{\{N-\nu\}} \int d\underline{x}_i d\underline{x}_j I_{ij} \gamma^{N,\nu+2}_{(\{\nu\},i,j)} - \sum_{i<j}^{N} \int d\underline{x}_i d\underline{x}_j I_{ij} \gamma^{N,2}_{(i,j)} \right] \tag{17}$$

From (11) we note that $f^{N,0}$ is determined by $\gamma^{N,2}$:

$$\frac{\partial f^{N,0}}{\partial t} = \varepsilon \left(\varepsilon \frac{r_D^6}{V^2} \sum_{i<j}^{N} \int d\underline{x}_i d\underline{x}_j I_{ij} \gamma^{N,2}_{(\underline{x}_i,\underline{x}_j)} \right) \tag{18}$$

To achieve a closure of the system given by (17) and (18) we require a general expression relating $\gamma^{N,\nu}$ to $\gamma^{N,2}$ (and, in general, $f^{N,0}$). This relationship should hold for each ν (sufficiently small compared to N) and produce, when substituted in (17), a unique relationship between $\gamma^{N,2}$ and $f^{N,0}$ independent of ν, to the order of ε in question.

In his treatment of the BBGKY Hierarchy for the spatially-homogeneous plasma, Bogoliubov[5] noticed at this point that a cluster expansion of the type (see previous discussion)

$$F^{\nu} = \prod_{i=1}^{\nu} F^{1}_{(i)} + \varepsilon \sum_{i<j}^{\{\nu\}} \prod_{\substack{k=1 \\ k\neq i,j}}^{\nu} F^{1}_{(k)} \hat{\gamma}^{2,2}_{(i,j)} + O(\varepsilon^2) + O(1/N) \, ,$$

where the two-body correlation function $\hat{\gamma}^{2,2}_{(i,j)} \equiv \hat{\gamma}^{2,2}(\underline{x}_i, \underline{x}_j, \underline{v}_i, \underline{v}_j)$ is defined by

$$F^{2}_{(i,j)} = F^{1}_{(i)} F^{1}_{(j)} + \varepsilon \hat{\gamma}^{2,2}_{(i,j)} + O(1/N)$$

solves the (limit) Hierarchy, producing a unique integral equation for $\hat{\gamma}^{2,2}$, independent of ν .

In our present problem, if we limit ourselves to the case of a spatially-homogeneous plasma, we notice that an analogous expansion in the form

$$\text{(19)} \qquad \varepsilon\gamma^{N,\nu} = \sum_{\alpha<\beta}^{\{\nu\}} \varepsilon\gamma^{N,2}(\underline{x}_\alpha, \underline{x}_\beta) + O(\varepsilon^2) + O\left(\frac{1}{N}\right) \equiv \sum_{\alpha<\beta}^{\{\nu\}} \varepsilon\gamma_{\alpha\beta} + O(\varepsilon^2) + O\left(\frac{1}{N}\right)$$

also solves the entire "γ-hierarchy" (17), producing a unique equation for $\gamma^{N,2}$ in terms of $f^{N,0}$.

To see this, we note first that in the spatially-homogeneous case the functions $\gamma^{N,2}_{ij}$ can only depend on the difference coordinates $\underline{x}_{ij} \equiv \underline{x}_i - \underline{x}_j$ and with (19) each pair (i,j) in the set $\{\nu\}$ occurs once. Thus the operators K^{ν} can be written in that case

$$K^{\nu} = \sum_{\alpha<\beta}^{\{\nu\}} \underline{v}_{\alpha\beta} \cdot \frac{\partial}{\partial \underline{x}_{\alpha\beta}} \equiv \sum_{\alpha<\beta}^{\{\nu\}} K_{\alpha\beta}$$

and also, generally,

$$I^{\nu} = \sum_{\alpha<\beta}^{\{\nu\}} I_{\alpha\beta}$$

Moreover, the general requirement, according to definition,

$$\text{(20)} \qquad \varepsilon\gamma^{N,2} = \int \left(\frac{r_D^3 dx}{V} \right)^{\nu-2} \varepsilon\gamma^{N,\nu} + O\left(\frac{1}{N}\right)$$

is satisfied in the spatially-homogeneous case by (19) since from (16) we have for spatial-homogeneity

(21) $$\varepsilon \frac{r_D^3}{V} \int d\underline{x}_{\alpha\beta}\, \gamma_{\alpha\beta} = O\left(\frac{1}{N}\right)$$

Substitution of (19) in (17) gives

$$\sum_{\alpha<\beta}^{\{\nu\}} \left\{ \frac{\partial \gamma_{\alpha\beta}}{\partial t} + K_{\alpha\beta}\gamma_{\alpha\beta} - I_{\alpha\beta} f^{N,0} \right\} =$$

$$= \varepsilon \frac{r_D^3}{V} \sum_i^{\{\nu\}} \sum_j^{\{N-\nu\}} \sum_{\alpha<\beta}^{\{\nu,j\}} \int d\underline{x}_j\, I_{ij}\, \gamma_{\alpha\beta}$$

(22) $$+ \varepsilon \frac{r_D^6}{V^2} \left\{ \sum_{i<j}^{\{N-\nu\}} \sum_{\alpha<\beta}^{\{\nu,i,j\}} \int d\underline{x}_i\, d\underline{x}_j\, I_{ij}\, \gamma_{\alpha\beta} - \sum_{i<j}^{N} \int d\underline{x}_i\, d\underline{x}_j\, I_{ij}\, \gamma_{ij} \right\}$$

We now consider the terms on the right side of (22) in order. The first term vanishes unless $\beta = j$ from the symmetry of I_{ij} and thus reduces to

$$\varepsilon \frac{r_D^3}{V} \sum_j^{\{N-\nu\}} \sum_i^{\{\nu\}} \sum_\alpha^{\{\nu\}} \int d\underline{x}_j\, I_{ij}\, \gamma_{\alpha j}$$

Similarly, the first integral in the bracket vanishes unless $\alpha = i$ and $\beta = j$.

Thus using $\sum_{i<j}^{\{N\}} = \sum_{i<j}^{\{\nu\}} + \sum_i^{\{\nu\}} \sum_j^{\{N-\nu\}} + \sum_{i<j}^{\{N-\nu\}}$, and

neglecting terms of order $\left(\varepsilon r_D^3/V\right)\nu^2 = O\left(\frac{\nu^2}{N}\right)$, we obtain

(23) $$\text{rhs}\,(22) \simeq \varepsilon \frac{r_D^3}{V} \sum_j^{\{N-\nu\}} \left\{ \sum_i^{\{\nu\}} \sum_\alpha^{\{\nu\}} \int d\underline{x}_j\, I_{ij}\, \gamma_{\alpha j} - \sum_i^{\{\nu\}} \int d\underline{x}_j\, \frac{r_D^3\, d\underline{x}_i}{V}\, I_{ij}\, \gamma_{ij} \right\}$$

Finally, since $\gamma_{ij} = \gamma(\underline{x}_i - \underline{x}_j)$, $\int \frac{r_D^3 dx_i}{V} dx_j\, I_{ij}\gamma_{ij} = \int dx_j\, I_{ij}\gamma_{ij}$,

and we obtain cancellation in (23) when $\alpha = i$. For each ν equations (22) can therefore be written

$$\sum_{\alpha<\beta}^{\{\nu\}} \left\{ \frac{\partial \gamma_{\alpha\beta}}{\partial t} + K_{\alpha\beta}\gamma_{\alpha\beta} - I_{\alpha\beta} f^{N,0} \right\} =$$

(24)
$$= \sum_{\alpha<\beta}^{\{\nu\}} \left\{ \varepsilon \frac{r_D^3}{V} \sum_{j=1}^{N} \int d\underline{x}_j \left(I_{\alpha j}\gamma_{\beta j} + I_{\beta j}\gamma_{\alpha j} \right) \right\} + O\left(\varepsilon \frac{r_D^3}{V} \nu^2\right) + O(\varepsilon \nu^2) + O\left(\frac{r_D^3}{V}\right)$$

where we define $\gamma_{\alpha\alpha} \equiv 0$.

Hence, with the error indicated, substitution of the cluster-type expansion (19) yields for each member of the γ-hierarchy the <u>same</u> equation for $\gamma_{\alpha\beta}$. We have therefore obtained the desired approximate closure of the set of equations (17) and (18) appropriate for a spatially-homogeneous plasma with collective effects.

V. Time - Asymptotic Analysis.

The equation (18) and (24) form a closed set from which we wish to obtain the superkinetic description of a plasma. Although the analysis can be extended to higher orders, as the equations stand they are valid only to first order in ε for $f^{N,0}$ and zeroth order in ε for $\gamma_{\alpha\beta}$. Thus, in terms of an expansion in ε:

(25)
$$f^{N,0} = \varphi^0 + \varepsilon\varphi^1 + \varepsilon^2\varphi^2 + \cdots$$
$$f^{N,\nu} = \varphi^0 + \varepsilon\left(\varphi^1 + \sum_{\alpha<\beta}^{\{\nu\}} \gamma_{\alpha\beta}\right) + \cdots$$

we may treat with our present equations only the coefficients φ^0, φ^1 and $\gamma_{\alpha\beta}$. We note that in their present form the equations are still time-reversible.

We notice, however, that our equations actually contain very different time scales. It is clear, for example, that $f^{N,0}$ evolves much more slowly than $\gamma_{\alpha\beta}$; indeed, the zeroth-order approximation for equation (18) is simply $(\partial f^{N,0}/\partial t)^{0} = 0$ If, however, $\varphi^{0} \equiv (f^{N,c})^{(0)}$ is treated strictly as independent of time, the first correction, φ' , can readily be seen to grow in time without bound, rendering the expansion invalid after a short time. The physical reason for this "secular" behavior has been discussed in the introduction. To avoid this problem, we must adopt a method for making time-asymptotic analysis in the presence of multiplicity of time scales.

The method we shall use has been discussed extensively elsewhere [6,7,9,10]. We adopt the formalism of Sandri [7] and treat the coefficients of our expansion as extended functions of the independent time variables $\tau_0, \tau_1, \ldots$ where in reduction to physical space,*

$$\tau_\mu = \varepsilon^{\mu} t \tag{26}$$

This leads, in addition to (25), to an expansion of the time derivatives in the form

$$\frac{\partial f^{N,0}}{\partial t} = \frac{\partial \varphi^{0}}{\partial \tau_0} + \varepsilon\left(\frac{\partial \varphi^{0}}{\partial \tau_1} + \frac{\partial \varphi'}{\partial \tau_0}\right) + \cdots \tag{27}$$

$$\frac{\partial \gamma_{\alpha\beta}}{\partial t} = \frac{\partial \gamma_{\alpha\beta}}{\partial \tau_0} + \cdots$$

The relation of this expansion to the ideas of Enskog and Bogoliubov can be made quite explicit.[9,10]

* It should be noted that the reduction equations (26) given here, and our simple treatment of the initial conditions, are not in general adequate to go on to higher orders in the expansion.[18] More refined analysis is required to obtain corrections to the kinetic or superkinetic results,[19] but this will cause no trouble here.

By treating our functions in terms of several time scales we have, of course, introduced an arbitrariness in the expansion. The essence of the method as given by Frieman[6] and Sandri[7] is to remove this arbitrariness by demanding that the coefficients of the expansion remain bounded for long times.

In our present problem, using (25) and (27), we may write our equations as:

(28) $$\frac{\partial \varphi^0}{\partial \tau_0} = 0$$

(29) $$\frac{\partial \varphi^0}{\partial \tau_1} + \frac{\partial \varphi^1}{\partial \tau_0} = \varepsilon \frac{r_D^3}{V} \sum_{i<j}^{N} \int d\underline{x}_{ij}\, I_{ij}\, \gamma_{ij}(\tau_0, \tau_1, \dots)$$

(30) $$\frac{\partial \gamma_{\alpha\beta}}{\partial \tau_0} + K_{\alpha\beta}\gamma_{\alpha\beta} = I_{\alpha\beta}\varphi^0 + \frac{\varepsilon r_D^3}{V}\sum_{j=1}^{N}\int d\underline{x}_j \left(I_{\alpha j}\gamma_{\beta j} + I_{\beta j}\gamma_{\alpha j}\right)$$

In order to avoid secular behavior, we must have $\partial\varphi^1/\partial\tau_0 \to 0$ for large τ_0 ; thus the requirement that φ^1 remain bounded yields the equation *

(31) $$\frac{\partial \varphi^0}{\partial \tau_1} = \varepsilon \frac{r_D^3}{V} \sum_{i<j}^{N} \int d\underline{x}_{ij}\, I_{ij}\, \gamma_{ij}(\tau_0 = \infty, \tau_1, \dots)$$

We therefore need the solution of (30) for large τ_0 in order to complete the superkinetic description of the plasma:

(32) $$\gamma_{\alpha\beta}(\infty) \equiv \gamma_{\alpha\beta}(\tau_0 = \infty, \tau_1, \dots) = \lim_{\tau_0 \to \infty} \Big[e^{-K_{\alpha\beta}\tau_0}\gamma_{\alpha\beta}(\tau_0 = 0, \tau_1, \dots) \Big] + \int_0^\infty e^{-K_{\alpha\beta}\lambda} d\lambda \left\{ I_{\alpha\beta}\varphi^0(\tau_1) + \frac{\varepsilon r_D^3}{V}\sum_{j=1}^{N}\int d\underline{x}_j \left(I_{\alpha j}\gamma_{\beta j}(\infty) + I_{\beta j}\gamma_{\alpha j}(\infty)\right)\right\}$$

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* It may be also noted in passing that (29), together with (31), yields an equation for the transient behavior of

The equations (31) and (32), taken together, form our proposed superkinetic description of a plasma; valid on the long-time scale $\tau_1 = \varepsilon t$. These equations are no longer time reversible; our replacement of $\gamma_{\alpha\beta}$ by its asymptotic value for large τ_0 has destroyed the time-reversal invariance of the system.

VI. Transformed Equations.

It seems natural to assume[4] the initial spatial-correlations $\gamma_{\alpha\beta}(0)$ to have a finite range in configuration space. Therefore, except for the possibility of singular behavior in small relative velocities, $\underline{v}_{\alpha\beta}$,[7,19] the initial value of $\gamma_{\alpha\beta}$ plays no role for large τ_0 and can be omitted in (32). (See, however, footnote,p.14)

We may rewrite (31) and (32) in terms of the Fourier transforms of $\gamma_{\alpha\beta}$ and $\mathcal{U}_{\alpha\beta}$. We introduce

(33) $$\Gamma_{\alpha\beta}(\underline{k}, \tau_1, \underline{v}_1 \ldots \underline{v}_N) = \int d\underline{x}_{\alpha\beta}\, e^{-i\underline{k}\cdot\underline{x}_{\alpha\beta}}\, \gamma_{\alpha\beta}(\underline{x}_{\alpha\beta}, \ldots)$$

and

(34) $$\phi(k) = \int d\underline{x}_{\alpha\beta}\, e^{-i\underline{k}\cdot\underline{x}_{\alpha\beta}}\, \mathcal{U}(|\underline{x}_{\alpha\beta}|)$$

The indices on $\Gamma_{\alpha\beta}$ are still necessary because the velocity variables $\underline{v}_\alpha$ and $\underline{v}_\beta$ are not interchangeable with the remaining N-2 variables. From the reality of $\gamma_{\alpha\beta}$ we have the property

(35) $$\Gamma^*_{\alpha\beta}(\underline{k}) = \Gamma_{\alpha\beta}(-\underline{k})$$

We then obtain in Fourier representation

(36) $$\frac{\partial \varphi^0}{\partial \tau_1} = \frac{1}{(2\pi)^3}\left(\varepsilon \frac{r_D^3}{V}\right)\sum_{\alpha<\beta}^{N} \int d\underline{k}\, \phi(k)\, \underline{k}\cdot \underline{D}_{\alpha\beta}\Gamma^I_{\alpha\beta}(\underline{k})$$

(37) $$\Gamma_{\alpha\beta}(\underline{k}) = \lim_{p\to 0^+} \frac{\phi(k)}{\underline{k}\cdot\underline{v}_{\alpha\beta} - ip}\, \underline{k}\cdot\left\{\underline{D}_{\alpha\beta}\varphi^0 + \frac{\varepsilon r_D^3}{V}\sum_{j=1}^{N}\left(\underline{D}_{\alpha j}\Gamma^*_{\beta j} - \underline{D}_{\beta j}\Gamma_{\alpha j}\right)\right\}$$

where $\Gamma^{I}_{\alpha\beta}$ is the imaginary part of the solution of (37), and we have introduced the notation

(38) $$\underline{D}_{\alpha\beta} \equiv \left(\frac{\partial}{\partial \underline{v}_\alpha} - \frac{\partial}{\partial \underline{v}_\beta}\right)$$

Using the Dirac identity

(39) $$\lim_{p\to 0^+} \frac{1}{x-ip} = \mathcal{P}\frac{1}{x} + \pi i\,\delta(x)$$

where $\mathcal{P}$ indicates the Cauchy principal value, one immediately recognizes the weak-coupling master equation in the form given by Prigogine and Balescu[2,4], if the sum on the right representing collective effects is dropped:

(40) $$\frac{\partial \varphi^{o}_{wc}}{\partial(\varepsilon\tau_1)} = \frac{\pi}{(2\pi)^3}\frac{r_D^3}{V}\sum_{\alpha<\beta}^{N}\int d\underline{k}\;\phi^2(k)\,\underline{k}\underline{k} : \underline{D}_{\alpha\beta}\,\delta(\underline{k}\cdot\underline{v}_{\alpha\beta})\,\underline{D}_{\alpha\beta}\,\varphi^{o}_{wc}$$

However, when collective effects are included a closed-form solution of (37), giving $\Gamma_{\alpha\beta}$ as a functional of φ^{o}, is not readily available (except in the case of stationary solutions, discussed below).

VII. Stationary Solutions and Equilibrium.

Making use of the identity (39), we decompose the equation for $\Gamma_{\alpha\beta}$ into its real and imaginary parts: (See also, Equs (12) and (13))

(37a) $$\Gamma^{I}_{\alpha\beta} = \phi(k)\underline{k}\cdot\Big\{\pi\delta(\underline{k}\cdot\underline{v}_{\alpha\beta})\underline{D}_{\alpha\beta}\varphi^{o} + \pi\delta(\underline{k}\cdot\underline{v}_{\alpha\beta})\frac{1}{4\pi N}\sum_{j=1}^{N}\left(\underline{D}_{\alpha j}\Gamma^{R}_{\beta j} - \underline{D}_{\beta j}\Gamma^{R}_{\alpha j}\right)$$
$$-\frac{\mathcal{P}}{\underline{k}\cdot\underline{v}_{\alpha\beta}}\frac{1}{4\pi N}\sum_{j=1}^{N}\left(\underline{D}_{\alpha j}\Gamma^{I}_{\beta j} + \underline{D}_{\beta j}\Gamma^{I}_{\alpha j}\right)\Big\}$$

$$\Gamma^R_{\alpha\beta} = \phi(k)\underline{k}\cdot\left\{\frac{P}{\underline{k}\cdot\underline{v}_{\alpha\beta}}\underline{D}_{\alpha\beta}\varphi^0 + \frac{P}{\underline{k}\cdot\underline{v}_{\alpha\beta}}\frac{1}{4\pi N}\sum_{j=1}^{N}\left(\underline{D}_{\alpha j}\cdot\Gamma^R_{\beta j} - \underline{D}_{\beta j}\cdot\Gamma^R_{\alpha j}\right)\right.$$

(37b)

$$\left. + \pi\delta(\underline{k}\cdot\underline{v}_{\alpha\beta})\frac{1}{4\pi N}\sum_{j=1}^{N}\left(\underline{D}_{\alpha j}\cdot\Gamma^I_{\beta j} + \underline{D}_{\beta j}\cdot\Gamma^I_{\alpha j}\right)\right\}$$

Inspection of these equations shows that a stationary solution of (36) and (37) is given by

(41) $$\varphi^0 = C^N e^{-K} \equiv \prod_{i=1}^{N} M(v_i^2) \quad ; \quad K \equiv \sum_{i=1}^{N} v_i^2/2$$

(42) $$\Gamma^R_{\alpha\beta} = G(k)\prod_{i=1}^{N} M(v_i^2)$$

where $M(v_i^2) \equiv C e^{-v_i^2/2}$ is the Maxwellian distribution, since then it follows from (37a) that

(43) $$\Gamma^I_{\alpha\beta} = 0$$

This is of course the familiar canonical distribution for thermodynamic equilibrium of a classical plasma; substitution of (41) and (42) in (37b) shows that

(44) $$G(k) = \frac{-\phi(k)}{1+\phi(k)/4\pi}$$

Since $\phi(k) = 4\pi/k^2 =$
$= 4\pi/\hat{k}^2 r_D^2$ from (3a) and (34), this solution represents the familiar Debye screening in the classical approximation.

However, since our system (36) and (37) is linear, it is clear that further stationary solutions of the type

(45) $$\varphi^0 = \varphi^0(K)$$

$$\Gamma^R_{\alpha\beta} = \Gamma(k, K)$$

can be constructed by superposition. We require only that Γ and φ^0 be normalizable. (See Appendix.)

Investigating solutions of this type, we find indeed from (37a) that $\Gamma^{I}_{\alpha\beta} = 0$, and, moreover, from (37b):

$$\Gamma(k,K) = \phi(k)\left\{\varphi^{0\prime}(K) + \frac{1}{4\pi}\frac{\partial\Gamma}{\partial K}\right\} \tag{46}$$

where $(\ )'$ denotes differentiation with respect to the argument K. But the general solution of (46) is

$$\Gamma = \Gamma(k,0)\,e^{k^2K} - 4\pi e^{k^2K}\int_0^K dK^1 e^{-k^2K^1}\varphi^{0\prime}(K^1) \tag{47}$$

where $\phi(k)$ has been replaced by its value $4\pi/k^2$. Imposing the requirement of normalizability of Γ in k and K space leads to the conclusion that

$$\Gamma(k,0) = 4\pi\int_0^\infty dK^1 e^{-k^2K^1}\varphi^{0\prime}(K^1) \tag{48}$$

so that we may write finally

$$\Gamma(k,K) = 4\pi\int_0^\infty d\lambda\, e^{-k^2\lambda}\varphi^{0\prime}(\lambda+K) \tag{49}$$

which clearly exists whenever φ^0 is normalizable.

The existence of this broad class of stationary solutions corresponds to the result of Prigogine[4] that any function of the total kinetic energy, K, is a stationary solution of the weak-coupling master equation (eq.(40)). Indeed, Prigogine has given an H-theorem in that case, showing that $\frac{dH}{dt} \equiv \frac{d}{dt}\int (d\underline{v})^N \varphi^0_{wc}\ln\varphi^0_{wc} \leqq 0$, and that $(dH/dt) = 0$ only if $\varphi^0_{wc} = \varphi^0_{wc}(K)$.
It seems likely, of course, that an H- theorem must hold also when collective effects are included, although a complete proof is not yet available.

The exact form of the stationary solutions $\varphi^0(K)$ is here arbitrary, (aside from normalizability) and will depend on the initial conditions and the type of system to be described, as in ordinary ensemble theory[20]. However, for the bulk properties of a plasma we should not expect to find a significant dependence on the particular form of $\varphi^0(K)$. That this is indeed so is shown in the Appendix, in which we find that the single-particle distribution function itself calculated from a very general class of $\varphi^0(K)$, is always the Maxwellian distribution in the limit $N \to \infty$, and, moreover, $\int (d\underline{v})^N \Gamma(k, K)$ approaches the familiar Debye result (44) in the same limit.

Moreover, consistent with this result we note here a simple but important general property of eq. (49). The spatial average of $\gamma_{\alpha\beta}$ is given, of course, by $\Gamma_{\alpha\beta}(0, K)$. We see that our general stationary solution (49) yields in every case

$$\Gamma(0, K) = -4\pi \varphi^0(K) \tag{50}$$

so that $(\varepsilon r_D^3/V)\,\Gamma(0, K) = -\varphi^0/N$, in accordance with our expectations, as discussed in Section II. (Compare also eqns.(16) and (21)). This result implies in dimensional variables

$$-ne \int (d\hat{\underline{v}})^N \int d\hat{\underline{x}}_{\alpha\beta} \, (\varepsilon \hat{\gamma}_{\alpha\beta}) = +e \tag{51}$$

a property which is shared, of course, by the usual Debye screening solution. That is to say, each of the stationary solutions (49) implies complete screening, with the appropriate amount of charge (absence of negative charge) distributed symmetrically about any singled-out electron.

Indeed, performing the inverse transformation of eq.(49) we find the general expression for this charge distribution in the form

$$-ne\varepsilon\gamma_{\alpha\beta}(r, K) = \frac{-ne\varepsilon}{2\sqrt{\pi}} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \, e^{-r^2/4\lambda} \, \varphi^{0\prime}(\lambda + K) \tag{52}$$

where we have used r for $|x_{\alpha\beta}|$. (If $\varphi^0 = C^N e^{-K}$ one obtains exactly the Debye solution $\gamma_{\alpha\beta} = -C^N e^{-K}(e^{-r}/r)$.)* As in the Debye case, the general expression (52) is singular at $r = 0$; however, it has recently been shown that the corresponding divergence at small distances can be removed (for an "electron

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* CF.Ryshik and Gradstein, Tables, No.3284

plasma") by proper inclusion of the neglected "Boltzmann" term in the Bogoliubov equation for the two-body correlations at small r. We need not concern ourselves with this divergence here.

VIII. Reduction to Kinetic Regime

In a spatially-homogeneous gas the single-particle distribution function is independent of position. One may therefore obtain F^1 by appropriate integration of $f^{N,0}$, since $f^{1,0} = F^1$ in that case. We therefore integrate (36) with respect to (N-1) and (37) with respect to (N-2) momenta:

$$\text{(53)} \qquad \frac{\partial \overset{(0)}{F}{}^{1}(\underline{v}_\alpha)}{\partial \tau_1} = \frac{\partial \overset{(0)}{f}{}^{1,0}}{\partial \tau_1} = \frac{1}{(2\pi)^3} \cdot \frac{1}{4\pi N} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} \int d\underline{v}_\beta \int d\underline{k}\, \phi(k)\, \underline{k} \cdot \frac{\partial}{\partial \underline{v}_\alpha} g^{I}_{\alpha\beta}(\underline{k})$$

where

$$\text{(54)} \qquad g_{\alpha\beta}(\underline{k}) \equiv \int \Big(\prod_{\substack{i=1 \\ i \neq \alpha,\beta}}^{N} d\underline{v}_i \Big) \Gamma_{\alpha\beta}(\underline{k}) = \lim_{p \to 0^+} \frac{\phi(k)\,\underline{k}}{\underline{k} \cdot \underline{v}_{\alpha\beta} - ip} \cdot \Big\{ \underline{D}_{\alpha\beta} \overset{(0)}{f}{}^{2,0}(\underline{v}_\alpha, \underline{v}_\beta)$$
$$+ \frac{1}{4\pi N} \sum_{j=1}^{N} \int d\underline{v}_j \Big(\frac{\partial}{\partial \underline{v}_\alpha} \big(g^{3,2}_{\beta j \alpha}\big)^* - \frac{\partial}{\partial v_\beta} g^{3,2}_{\alpha j \beta} \Big) \Big\}$$

and

$$\text{(55)} \qquad g^{3,2}_{\alpha\beta j} \equiv \int \Big(\prod_{i=1}^{N} d\underline{v}_i \Big) \Gamma_{\alpha\beta}$$

In (53) we observe that the terms in the sum on the right are the same for each β, so the sum is simply proportional to $N-1 \approx N$. We may therefore write

$$(56) \qquad \frac{\partial \overset{(0)}{F}{}^{1}(\underline{v}_\alpha)}{\partial \tau_1} = \frac{1}{(2\pi)^3 4\pi} \int d\underline{v}_\beta \int d\underline{k}\, \phi(k)\, \underline{k} \cdot \frac{\partial}{\partial \underline{v}_\alpha} g^{I}_{\alpha\beta}$$

Similarly, the collective behavior becomes explicit also in (54), so that $(1/4\pi N)\sum_{j=1}^{N}$ can be replaced by $(4\pi)^{-1}$.

The function $f^{2,0}$ is the zeroth-approximation to the space average of the two-body distribution function F^2. We assume, therefore, to this order

$$(57) \qquad \overset{(0)}{f}{}^{2,0}(\underline{v}_\alpha, \underline{v}_\beta) = \overset{(0)}{F}{}^{1}(\underline{v}_\alpha)\, \overset{(0)}{F}{}^{1}(\underline{v}_\beta)$$

The functions $\varepsilon g^{3,2}_{\alpha\beta j}(k)$ are the Fourier transforms with respect to $\underline{x}_{\alpha\beta}$ of the functions

$$(58) \qquad \varepsilon \gamma^{3,2}(\underline{v}_\alpha, \underline{v}_\beta, \underline{v}_j \,|\, \underline{x}_{\alpha\beta}) \equiv f^{3,2} - f^{3,0}$$

The latter are symmetric to interchange of greek indices but not to interchange between latin and greek indices, and are clearly obtained from the symmetric functions $\varepsilon\gamma^{3,3} \equiv f^{3,3} - f^{3,0}$ by integration:

$$(59) \qquad \gamma^{3,2}_{\alpha\beta j} \equiv \frac{r_D^3}{V} \int d\underline{x}_j\, \gamma^{3,3}_{\alpha\beta j}$$

Following Bogoliubov, we introduce the cluster expansion

$$(60) \qquad \varepsilon\gamma^{3,3} = \varepsilon\left(\overset{(0)}{F}{}^{1}(\underline{v}_\alpha)\, \gamma^{2,2}_{\beta\gamma} + \overset{(0)}{F}{}^{1}(\underline{v}_\beta)\, \gamma^{2,2}_{\alpha\gamma} + \overset{(0)}{F}{}^{1}(\underline{v}_\gamma)\, \gamma^{2,2}_{\beta\alpha} \right) + O(\varepsilon^2)$$

where $\varepsilon\gamma^{2,2} \equiv f^{2,2} - f^{2,0}$. [$\hat{\gamma}^{2,2}$ (Section IV) and $\gamma^{2,2}$ are identical if $f^{2,0} = f'f'$ to second order in ε .]

We see from (21) that in the same approximation

$$\gamma^{3,2}_{\alpha\beta j} = \overset{(0)}{F}{}^{1}(\underline{v}_j)\, \gamma^{2,2}_{\alpha\beta} \tag{61}$$

$$g^{3,2}_{\alpha\beta j} = \overset{(0)}{F}{}^{1}(\underline{v}_j)\, g_{\alpha\beta}(\underline{k})$$

where the latter relationship follows from the definitions of $g_{\alpha\beta}(\underline{k})$ and $\gamma^{2,2}_{\alpha\beta}$.

Thus, having imposed the additional assumptions (57) and (60) of a cluster-like expansion in terms of the two-body correlation functions, we obtain finally the integral equation of Bogoliubov for the functions g:

$$g_{\alpha\beta} = \lim_{p\to 0^+} \frac{\phi(k)}{\underline{k}\cdot\underline{v}_{\alpha\beta} - ip}\, \underline{k}\cdot\Big\{ \underline{D}_{\alpha\beta} \overset{(0)}{F}{}^{1}(\underline{v}_\alpha) \overset{(0)}{F}{}^{1}(\underline{v}_\beta) + \frac{1}{4\pi} \frac{\partial \overset{(0)}{F}{}^{1}}{\partial \underline{v}_\alpha} \int d\underline{v}_j\, \overset{x}{g}(\underline{v}_j, \underline{v}_\beta) - \frac{1}{4\pi} \frac{\partial \overset{(0)}{F}{}^{1}}{\partial \underline{v}_\beta} \int d\underline{v}_j\, g(\underline{v}_\alpha, \underline{v}_j) \Big\} \tag{62}$$

This equation has been solved independently by Balescu[16] and by Lenard[17] to yield in combination with (56) the kinetic equation for a spatially-homogeneous plasma. We see that the superkinetic description (36) and (37) implies the kinetic description when additional approximations are imposed.

By giving an H-theorem for these equations Lenard has shown that the stationary solution of (56) with (62) is the classical equilibrium Debye-screening solution, with $\overset{(0)}{F}{}^{1}$ the Maxwellian distribution. (Compare also, the Appendix).

IX. Conclusions.

Starting from the Liouville equation, we have been able to derive, under conditions appropriate for a spatially-homogeneous plasma in which collective effects are important, a closed set of equations describing the irreversible behavior of the distribution function in 3N-momentum space. In the derivation it is necessary to assume only that the initial correlations in configuration space are small, of the same order as the average two-particle interaction strength. No assumption of statistical independence of the particles in momentum space is necessary. We conclude that irreversibility of the equations arises because of the presence in the system of both fast and slow processes; the fast processes relax so rapidly as to effectively determine the "initial" conditions of the slower phenomena in such a way that the evolution cannot reverse itself.

We have further shown that there exists a broad class of normalizable stationary solutions of these "superkinetic" equations, each member corresponding to thermodynamic equilibrium and providing complete screening around any singled out charged particle.

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Appendix

Although the (positive) functions $\varphi^{o}(K)$ are otherwise arbitrary, they are subject to the following two normalization conditions, written here in K -space and in dimensionless form:

$$(A.1) \qquad 1 = \frac{(2\pi)^{3N/2}}{\Gamma_0\left(\frac{3N}{2}\right)} \int_0^\infty \varphi^{o}(K)\, K^{\frac{3N}{2}-1}\, dK \qquad \text{(number)}$$

$$(A.2) \qquad \langle K \rangle = \frac{3N}{2} = \frac{(2\pi)^{\frac{3N}{2}}}{\Gamma_0\left(\frac{3N}{2}\right)} \int_0^\infty \varphi^{o}(K)\, K^{\frac{3N}{2}}\, dK \qquad \text{(kinetic energy)}$$

where $\Gamma_0(x)$ is the Gamma function, $\Gamma_0(x+1) = x!$. Moreover the relative energy dispersion is given by

$$(A.3) \qquad \frac{\langle (K - \langle K \rangle)^2 \rangle}{\langle K \rangle^2} = \frac{(2\pi)^{\frac{3N}{2}}}{\left(\frac{3N}{2}\right)^2 \Gamma_0\left(\frac{3N}{2}\right)} \int_0^\infty \varphi^{o}(K)\, K^{\frac{3N}{2}-1} \left(K^2 - \left(\frac{3N}{2}\right)^2\right) dK$$

There exists an extremely general class of φ^{o}'s* for which, as N becomes large, the integrand in (A.1) or (A.2) becomes very sharply peaked, the important contribution to the integral coming in the neighborhood $K = \frac{3N}{2}$. This class of φ^{o}'s then has the property that the relative energy dispersion (eq.(A.3) goes to zero as $N \to \infty$.

We note, in fact, that two very different distributions, (one corresponding to a macrocanonical ensemble, the other to a microcanonical ensemble) namely, $\varphi^{o} = (2\pi)^{-3N/2} e^{-K}$ and

$$\varphi^{o} = \left\{ \Gamma_0\left(\tfrac{3N}{2}\right) \Big/ (2\pi)^{\frac{3N}{2}} \left(\tfrac{3N}{2}\right)^{\frac{3N}{2}-1} \right\} \delta\left(K - \tfrac{3N}{2}\right)$$

*"Pathological" cases are, of course, excluded from this statement. For example, a sum of two delta-functions, centered on two engergy surfaces far from $K = 3N/2$ can satisfy (A.1) and (A.2), but the energy dispersion in (A.3) will not in general vanish as $N \to \infty$. In that case $F_1(v^2)$ converges to two separate Maxwellians of different "temperatures."

both satisfy (A.1) and (A.2), both give (at least in the limit $N \to \infty$) zero for the quantity in (A.3), and both give the same (Maxwellian) result for $F'(v^2)$ in the limit $N \to \infty$.

Let us, therefore, consider $F'(v_1^2)$ computed from this general class of $\varphi^0(K)$. Defining $K_{-1} \equiv K - v_1^2/2$ we have

(A.3) $$F'(v_1^2) = \frac{(2\pi)^{\frac{3N}{2}-\frac{3}{2}}}{\Gamma_0\left(\frac{3N}{2}-\frac{3}{2}\right)} \int_0^\infty K_{-1}^{\frac{3N}{2}-\frac{5}{2}} \varphi^0\left(K_{-1} + \frac{v_1^2}{2}\right) dK_{-1}$$

If we let Δ be the width of the sharp peak in (A.1) or (A.2) at $K = 3N/2$, we have

(A.4) $$\varphi^0\left(\frac{3N}{2}\right)\cdot\Delta \approx \frac{\Gamma_0\left(\frac{3N}{2}\right)}{(2\pi)^{\frac{3N}{2}}\left(\frac{3N}{2}\right)^{\frac{3N}{2}-1}}$$

and, therefore,

(A.5) $$F'(v_1^2) \approx \left(2\pi\cdot\frac{3N}{2}\right)^{-3/2} \frac{\Gamma_0\left(\frac{3N}{2}\right)}{\Gamma_0\left(\frac{3N}{2}-\frac{3}{2}\right)} \left(1 - \frac{v_1^2/2}{3N/2}\right)^{\frac{3N}{2}-\frac{5}{2}}$$

With Stirling's formula, we have in the limit $N \to \infty$
$\left[\Gamma\left(\frac{3N}{2}\right)/\Gamma\left(\frac{3N}{2}-\frac{3}{2}\right)\right] \sim (3N/2)^{3/2}$,
so that

(A.6) $$F'(v_1^2) \xrightarrow[N\to\infty]{} (2\pi)^{-3/2} e^{-v_1^2/2} \equiv M(v_1^2)$$

where we have of course used $\lim_{a\to\infty}\left(1+\frac{x}{a}\right)^a = e^x$.

Thus, for the "typical" single particle, the Maxwellian distribution is seen here to be the natural consequence for very large systems of nearly all those stationary solutions that depend only on the total kinetic energy.

This result suggests further that the spatial correlations for any $\varphi^0(K)$ should also be essentially the same as the classical Debye result.

In fact, we find

(A.7) $$\langle\Gamma(k)\rangle \equiv \int (dv)^N \Gamma(k, K) =$$

$$= -4\pi\left\{1 - k^2\frac{(2\pi)^{3N/2}}{\Gamma_0\left(\frac{3N}{2}\right)}\int_0^{\infty} d\lambda\, e^{-k^2\lambda}\int_0^{\infty} K^{\frac{3N}{2}-1}\varphi^0(\lambda+K)\,dK\right\}$$

where we have used equ.(49) integrated once by parts and where we have also used (A.1). Once again, using the smallness of the energy dispersion in the K integration, we find

$$\langle\Gamma(k)\rangle \approx -4\pi\left(1 - k^2\int_0^{\infty} d\lambda\, e^{-(k^2+1)\lambda}\right) \tag{A.8}$$

$$= -4\pi\frac{1}{k^2+1}$$

in agreement with the Debye solution (44).

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