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Summary

In the present note we discuss two different dispersion relations derived from the linearized Vlasov equation.
The dispersion relation for the system of n moment equations which is closed by equating to zero the (n+1)-th moment in the n th equation yields unstable solutions for $n > 3$.
On the contrary the dispersion relation obtained by expanding the distribution function in series of orthogonal polynomials has always real roots if the unperturbed distribution is given by a single-humped function, proving that this state is stable. The method is then applied to the initial value problem and Landau damping.

In the present note we are concerned with the discussion of two different dispersion relations which can be derived when the linearized Vlasov equation is approximated by a finite set of differential equations in $(\vec{x}, t)$. First we consider the system of n moment equations, closed by equating to zero the (n+1)-th moment in the n_th equation.

The dispersion relation which can be obtained in this way has complex roots for $n>3$, yielding unstable solutions which should not be possible in thermodynamic equilibrium.

We show that the first three moment equations, which are the usual magnetohydrodynamic equations, correspond to a single-humped stable unperturbed distribution function; the equations of higher order, on the contrary, correspond to an unperturbed state with many streams of charges, yielding unstable oscillations; they cannot, therefore, be used to approximate the Vlasov equation.

Then we generalize the method of Hermite Polynomials, and expand the distribution function in series of general orthogonal polynomials. We derive a closed system of equations for the coefficients c_n of the expansion by equating to zero all the c_n from a certain n on. If the unperturbed state is given by a single-humped distribution, all the roots of the dispersion relation are real, proving that the state is stable. The method is applied to the initial values problem and Landau damping. Some known results can be rapidly derived and clearly discussed with this technique.

<u>Moment equations</u>

Consider the one-dimensional, one-component Vlasov equation:

$$\begin{aligned} &\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}E(x,t)\frac{\partial f}{\partial v} = 0 \\ &\frac{\partial E}{\partial x} = 4\pi e\left(\int_{-\infty}^{\infty} f\,dv - \rho_0\right) \end{aligned} \tag{1}$$

(where e and m are the charge and the mass of the electron respectively, and ρ_0 is the unperturbed electron and ion density). If we linearize it and set

$$\frac{\partial}{\partial t} = -i\omega \quad \text{and} \quad \frac{\partial}{\partial x} = iK ,$$

we obtain

$$(Kv-\omega)f_1 - \frac{4\pi e^2}{mK}\frac{\partial f_0}{\partial v}\int_{-\infty}^{\infty} f_1\,dv = 0 \tag{1a}$$

Multiplication by $(v-u)^n$ $(n = 1,2,\ldots)$ and integration over $\underline{v}$ give then the well known moment equations

$$K p_{n+1} - \omega p_n + \frac{4\pi e^2}{mK} p_0 \, n \, p^0_{n-1} = 0 \quad (n = 0,1,2,\ldots)$$

where
$$p_n = \int_{-\infty}^{\infty} (v-u)^n f_1 \, dv \quad n \neq 1 \;, \; p_1 \equiv \varrho_0 u_1 = \int_{-\infty}^{\infty} v f_1 \, dv \tag{2}$$

$$p^0_n = \int_{-\infty}^{\infty} v^n f_0 \, dv$$

and we have supposed

$$u_0 = \int_{-\infty}^{\infty} v f_0 \, dv = 0$$

The dispersion relation can now be obtained by setting

$$p_{n+r+1} = 0 \quad \text{and} \quad p^0_{n+r} = 0 \quad (r = 0,1,2,\ldots) \tag{3}$$

With the assumption $p^0_{2s+1} = 0$ $\left(s = 0,1,\ldots \leq \frac{n-1}{2}\right)$ we obtain in the case $n = 2N$:

$$D_{2N} = \begin{vmatrix} -\omega & K & & & \\ \frac{4\pi e^2}{mK} p^0_0 & -\omega & K & & \\ & & -\omega & K & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{4\pi e^2}{mK}(2N-1) p^0_{2(N-1)} & & & & -\omega \end{vmatrix} = 0 \tag{4}$$

We see from the above that

$$D_{2N+1} = -\omega D_{2N}$$

$$D_{2N} = \omega^{2N} - \frac{4\pi e^2}{m} \sum_{r=1}^{N} \left(2(N-r)+1\right) p^0_{2(N-r)} K^{2(N-r)} \omega^{2(r-1)} \, . \tag{5}$$

If equation (5) is considered as an equation in ω^2, it always has only one real positive root, the other being complex or negative (because the first coefficient is positive and the others negative). Hence there are only two real roots $\omega_1, -\omega_1$ different from zero, and it follows that the solutions of equ. (2) are always unstable when n is greater than three. As an example we give the roots of the first five dispersion relations:

1st : $\omega = 0$; 2nd : $\omega = \pm \omega_p$; 3rd: $\omega = 0$, $\omega = \pm\omega_p$;

4th: $\omega = \pm\sqrt{\dfrac{\omega_p^2 \pm \sqrt{\omega_p^4 + 12\,\omega_p^2 \Omega^2}}{2}}$; 5th : $\omega = 0$; $\omega = \pm\sqrt{\dfrac{\omega_p^2 \pm \sqrt{\omega_p^4 + 12\omega_p^2 \Omega^2}}{2}}$

where $$\Omega^2 = K^2 \frac{RT}{m} .$$

The fourth and fifth dispersion relations have two imaginary roots. It is interesting to note that in the limit $\Omega = 0$ the equation $D_{2N} = 0$ has only two non-vanishing roots, $\omega = \pm\omega_p$. In the other limiting case, $\omega_p = 0$, all the roots are equal to zero.

The results derived above (that the solutions of equ. (2) are unstable if $n > 3$) can be explained with the following considerations.

First of all, as a consequence of equations (3) we see that the distribution function has the form:

$$f(v) = \sum_{s=0}^{s_0} \frac{(-1)^s}{s!} \delta^{(s)}(v-u)\left(P_s - u\, c_0\, \delta_{s1}\right) \tag{6}$$

if all the moments P_s vanish when $s \geqslant s_0 + 1$.

This can be seen by expanding the distribution function in series of general orthogonal polynomials $P_m(v-u)$ corresponding to a weight function $g\,(v-u)$

$$f(v) = g(v-u) \sum_{m=0}^{\infty} c_m P_m(v-u) . \tag{7}$$

We choose the normalizing condition:

$$\int_{-\infty}^{\infty} g(x)\, P_n(x)\, P_m(x)\, dx = \delta_{nm} .$$

If we set:

$$P_n(x) = \sum_{s=0}^{n} \alpha_{ns} x^s$$

from (7) we get, with definition (2) for the moments:

$$c_n = \int_{-\infty}^{\infty} f P_n dv = \sum_{s=0}^{n} \alpha_{ns} \left(P_s - u c_0 \delta_{s1}\right) .$$

We remember now that

$$\alpha_{ns} = \frac{1}{s!} \left(P_n^{(s)}(x)\right)_{x=0}$$

and that $$\delta^{(s)}(x-y) = (-1)^s g(x) \sum_{n=0}^{\infty} P_n^{(s)}(y) P_n(x) \qquad (8)$$

where $$P_n^{(s)}(y) = 0 \quad \text{if } s > n .$$

(equ. (8) is only apparently unsymmetrical, because for x=y both sides vanish).
We can conclude that

$$f(v) = \sum_{s=0}^{s_0} \frac{(-1)^s}{s!} \delta^{(s)}(v-u)\left(P_s - u c_0 \delta_{s1}\right) \qquad (6)$$

The limit $s_0 \to \infty$ of equ. (6) is essentially a Taylor expansion with coefficients given by the distribution function and its derivatives calculated at the temperature T = 0.

The unperturbed distribution then has the following form:

$$f_0(v) = \sum_{r=0}^{\leq \frac{s_0}{2}} \frac{1}{(2r)!} \delta^{(2r)}(v-u) P_{2r}^{0}$$

because we have supposed that it is an even function.

When n is equal 2 or 3, f_0 is proportional to a delta function; and therefore the solutions of equ. (2) are stable. When n is greater than 3, even order derivatives of the delta function appear, which can be thought of as streams of charges. This can be visualized as in fig. 1

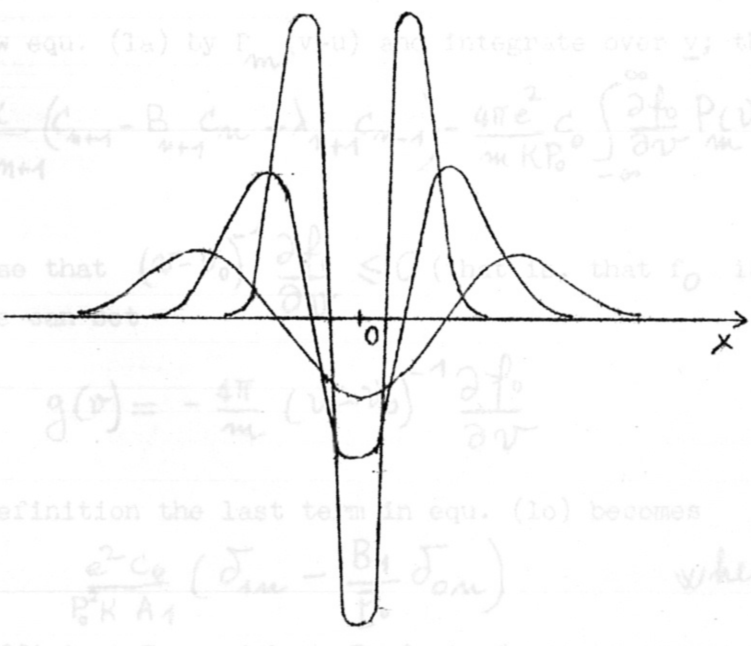

fig. 1

where some of the functions of the sequence used to define $\delta^{(2)}(x)$ are sketched. Such unperturbed states are of course unstable. We conclude that equ. (2) cannot be used to describe stable plasma configurations.

Orthogonal polynomials

In the last section we have seen that it is not possible to approximate the Vlasov equation with a set of moment equations (2) for $n > 3$. In this section we shall write a finite set of equations approximating the Vlasov equation by means of the following expansion of the distribution function in series of general orthogonal polynomials:

$$f_1(v) = g(v-v_0)\sum_{n=0}^{\infty} c_n P_n(v-v_0)$$

$g(x)$ is the weight function and the polynomials P_n obey the recurrence formula (Ref. (1) p. 42)

$$P_n(x) = (A_n x + B_n) P_{n-1}(x) - \lambda_n P_{n-2}(x) \qquad n = 1, 2, \dots \tag{9}$$

where $P_0 = \left(\int_{-\infty}^{\infty} g(x)\,dx\right)^{-1/2}$, $P_{-1} = 0$ and $A_n > 0$, $\lambda_n = \dfrac{A_n}{A_{n-1}}$.

Multiply now equ. (1a) by $P_m(v-u)$ and integrate over v; then we get:

$$(Kv_0-\omega)c_m + \frac{K}{A_{m+1}}\left(c_{m+1} - B_{m+1}c_m + \lambda_{m+1}c_{m-1}\right) - \frac{4\pi e^2}{mKP_0}c_0\int_{-\infty}^{\infty}\frac{\partial f_0}{\partial v}P_m(v-v_0)\,dv = 0 \tag{10}$$

If we suppose that $(v-v_0)^{-1}\dfrac{\partial f_0}{\partial v} \leq 0$ (that is, that f_0 is a single-humped function) we can set

$$g(v) = -\frac{4\pi}{m}(v-v_0)^{-1}\frac{\partial f_0}{\partial v} \tag{11}$$

With this definition the last term in equ. (10) becomes

$$\frac{e^2 c_0}{P_0^3 K A_1}\left(\delta_{1m} - \frac{B_1}{P_0}\delta_{0m}\right) \quad \text{where } A_1 = \frac{e}{P_0\,\omega_p}$$

and the coefficient B_1 vanishes. In fact, from recurrence formula (9), written for $n = 1$,

$$0 = -\frac{4\pi}{m}A_1\int_{-\infty}^{\infty}\frac{(v-v_0)}{(v-v_0)}\frac{\partial f_0}{\partial v}\,dv + \frac{B_1}{P_0}.$$

The dispersion relation obtained from (1o) by setting $c_{n+r} = 0$ ($r=0,1,\ldots$) is then:

$$D_{n+1} = \begin{vmatrix} -A_1\left(\frac{\omega}{K}-v_0\right) & 1 & & & \\ \lambda_2\left(1+\frac{e^2}{P_0^2K^2}\right) & -\left(B_2+\left(\frac{\omega}{K}-v_0\right)A_2\right) & 1 & & \\ & \lambda_3 & -\left(B_3+\left(\frac{\omega}{K}-v_0\right)A_3\right) & & \\ & & & \ddots & \\ & & & \lambda_{n+1} & -\left(B_{n+1}+\left(\frac{\omega}{K}-v_0\right)A_{n+1}\right) \end{vmatrix} = 0 \qquad (12)$$

Due to the structure of the determinant, the polynomials $D_n\left(\frac{\omega}{K}-v_0\right)$ are orthogonal (Ref. (1) p. 374), and therefore the roots of the dispersion relation are real and simple. These last two properties can be proved directly by using the recurrence formula for the polynomials $D_n\left(\frac{\omega}{K}-v_0\right)$:

$$D_{n+1} = -\left[A^*_{n+1}\left(\frac{\omega}{K}-v_0\right)+B_{n+1}\right]D_n - \lambda_{n+1}D_{n-1} - \frac{e^2 A_{n+1}}{P_0^2K^2A_1}\delta_{1n} \qquad (13)$$

$$A^*_s = A_s \quad s \neq 1 \,, \quad A^*_1 = \frac{A_1}{D_0}\,. \qquad D_{-1}=0\,,\ D_0 = \mathrm{const}$$

Suppose that the zeros of D_{n-1} are real and simple, and that they separate the zeros of D_n; then the zeros of D_n have the same property with respect to D_{n+1}, due to the fact that D_{n+1} goes to infinity with the same sign as D_{n-1} and that in the zeros of D_n the sign of D_{n+1} is opposite to the sign of D_{n-1}, as the recurrence formula show (see fig. 2). Moreover it can be seen that this property holds for D_2 and D_3.

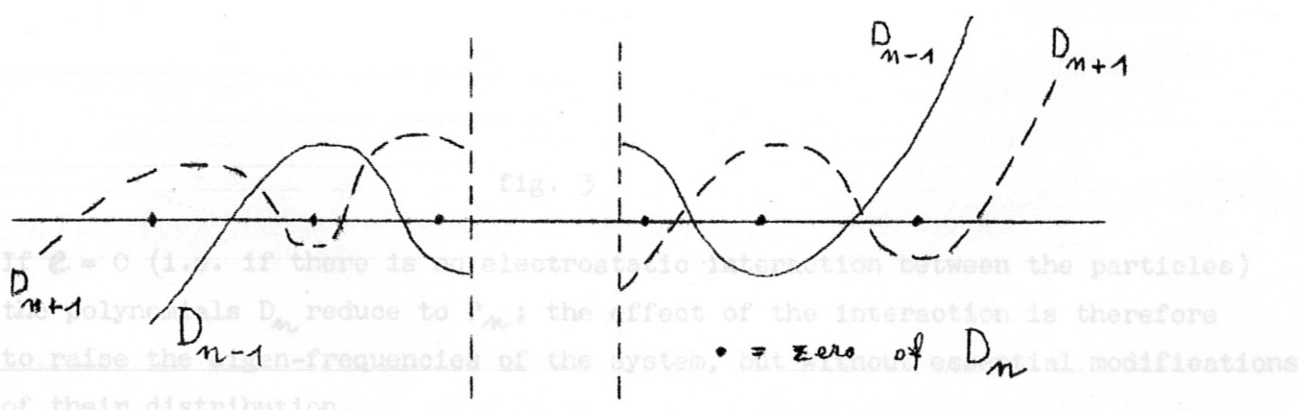

fig. 2

A consequence of the reality of the roots is that all unperturbed functions such that $(v-v_0)^{-1}\frac{\partial f_0}{\partial v}$ can be used as weight function (that is, all single-humped distribution functions) are stable.

Now we shall establish some properties of the distribution of the zeros of the dispersion relation which will be necessary in the discussion of the initial values problem and Landau damping (section 3).

We write the polynomials D_n in the following way:

$$D_n = \left(1+\frac{e^2}{P_0^2K^2}\right)P_n + e^2\left(\frac{\omega}{K}-v_0\right)\frac{A_1}{K^2}Q_{n-1}\,, \quad Q_{-1}=0 \tag{14}$$

where the polynomial Q_n is obtained from D_n omitting the first row and the first column.

Note that the polynomials of degree $n-r$ ($r=0,1,..,n$), which can be derived from determinant (12) omitting the first r rows and columns, are orthogonal polynomials for a fixed n as can be deduced from Ref. (1) p. 374.

As a consequence, the zeros of Q_{n-1} separate the zeros of P_n; therefore the zeros of D_n separate the zeros of P_n and are greater (smaller) than the corresponding ones of P_n if $\frac{\omega}{K} > v_0$ ($\frac{\omega}{K} < v_0$) (see fig. 3).

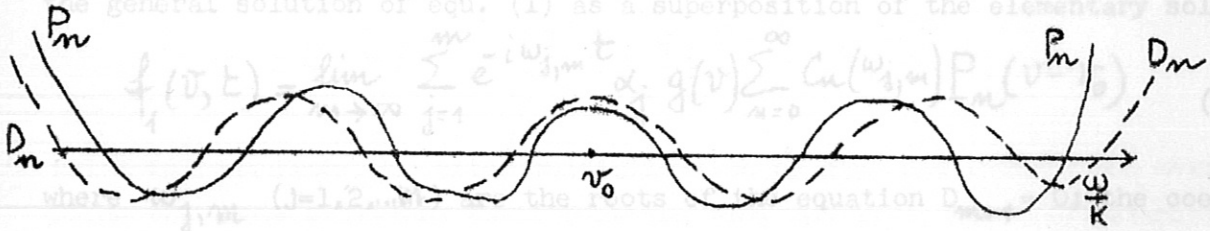

fig. 3

If $e = 0$ (i.e. if there is no electrostatic interaction between the particles) the polynomials D_n reduce to P_n; the effect of the interaction is therefore to raise the eigen-frequencies of the system, but without essential modifications of their distribution.

For example, when the distance between two consecutive zeros of P vanishes asymptotically, the same happens to the zeros of D_n.

A significant and important case is that of the Maxwellian; in this case the polynomials P_n are Hermite polynomials and the asymptotic distribution of the zeros can be deduced from the equation (Ref. (1) p. 198):

$$\lim_{n\to\infty} He_n(x) = e^{-\frac{x^2}{4}}\left(\lim_{n\to\infty}\right) 2^{-\frac{n}{2}} \frac{\Gamma(n+1)}{\Gamma(\frac{n}{2}+1)} \cos\left(\sqrt{n+1/2}\, x - n\frac{\pi}{2}\right), \quad |x|<M$$

A formula of the same type holds for $x \propto \sqrt{n+1/2}$.

The roots of the first five dispersion relations are:

$$\begin{aligned}
&\text{1st: } \omega = 0\\
&\text{2nd: } \omega = \pm\sqrt{\omega_p^2+\Omega^2}\\
&\text{3rd: } \omega = 0, \quad \omega = \pm\sqrt{\omega_p^2+3\Omega^2}\\
&\text{4th: } \omega = \pm\sqrt{\frac{\omega_p^2+6\Omega^2\pm\sqrt{\omega_p^4+24\Omega^4}}{2}}\\
&\text{5th: } \omega = 0, \quad \omega = \pm\sqrt{\frac{\omega_p^2+10\Omega^2\pm\sqrt{\omega_p^4-8\omega_p^2\Omega^2+40\Omega^4}}{2}}
\end{aligned} \tag{15}$$

Landau damping

In this section we shall study the asymptotic behaviour of the density in relation with the initial value of the distribution function. First, we write the general solution of equ. (1) as a superposition of the elementary solutions

$$f_1(v,t) = \lim_{m\to\infty}\sum_{j=1}^{m} e^{-i\omega_{j,m}t}\alpha_j\, g(v)\sum_{n=0}^{\infty} c_n(\omega_{j,m})\, P_n(v-v_0) \tag{16}$$

where $\omega_{j,m}$ (j=1,2,..m) are the roots of the equation $D_{m+1} = 0$; the coefficients c_n can be identified with the polynomials

$$D_n\left(\frac{\omega_{j,m}}{k}-v_0\right),$$

as they have the same recurrence formula and c_0 can be set equal to D_0.

Then, using equ. (14), we substitute equ. (16) with the equation

$$f_1(v,t)=\int_{-\infty}^{\infty}\alpha(\omega)e^{-i\omega t} g(v)\sum_{n=0}^{\infty} P_n\left(\frac{\omega}{k}-v_0\right)P_n(v-v_0)\,d\omega + \frac{e^2A_1}{P_0k^2+e^2}\int_{-\infty}^{\infty}\alpha(\omega)e^{-i\omega t}g(v)\left(\frac{\omega}{k}-v_0\right)\sum_{n=0}^{\infty} Q\left(\frac{\omega}{k}-v_0\right)P_n(v-v_0)\,d\omega \tag{17}$$

where $\alpha(\omega)$ can be thought of as the product of an arbitrary function determined by the initial conditions and a function which gives the distribution of the zeros of the dispersion realtion.

Now we integrate (17) over $\underline{v}$ and, noting that the integral in the last term on the right is equal to zero because the series does not contain P_0 , we find the following expression for the density:

$$d(t) = e^{iKx}\int_{-\infty}^{\infty}\alpha(\omega)e^{-i\omega t}\,d\omega$$

$\alpha(\omega)$ is therefore the Fourier transform of the density.
We can simplify equation (17) if we remember that the series

$$g(v)\sum_{n=0}^{\infty} P_n\left(\frac{\omega}{K}-v_0\right)P_n(v-v_0) \tag{18}$$

is equal to $\delta\left(\frac{\omega}{K}-v\right)$. The expression

$$g(v)\left(\frac{\omega}{K}-v_0\right)\sum_{n=0}^{\infty} Q_{n-1}\left(\frac{\omega}{K}-v_0\right)P_n(v-v_0) \tag{19}$$

can be calculated by substituting equ. (16) into equ. (1a); we obtain:

$$\frac{v-v_0}{K} + \frac{A_1}{P_0^2K^2+e^2}(Kv-\omega)\left(\frac{\omega}{K}-v_0\right)\sum_{n=0}^{\infty} Q_{n-1}\left(\frac{\omega}{K}-v_0\right)P_n(v-v_0) = 0 \tag{20}$$

that is

$$\frac{A_1}{P_0^2K^2+e^2}\left(\frac{\omega}{K}-v_0\right)\sum_{n=0}^{\infty} Q_{n-1}\left(\frac{\omega}{K}-v_0\right)P_n(v-v_0) = \frac{v_0-v}{K}\,P\frac{1}{Kv-\omega} + c\,\delta(Kv-\omega)$$

The constant $\underline{c}$ is determined by the equation

$$c\,g\left(\frac{\omega}{K}\right) + \frac{4\pi}{m}P\int_{-\infty}^{\infty}\frac{f_0'\,dv}{Kv-\omega} = 0$$

which follows from (2o) after multiplication by g(v) and integration over $\underline{v}$. Using the above relations for the series (18) and (19), equ. (17) simplifies to:

$$f_1(v,t) = e^{-iKvt}K\alpha(Kv) + \frac{4\pi e^2}{mK}\left\{f_0'(v)\,P\int_{-\infty}^{\infty}\frac{\alpha(\omega)e^{-i\omega t}\,d\omega}{Kv-\omega} + e^{-iKvt}\alpha(Kv)\,P\int_{-\infty}^{\infty}\frac{f_0'\left(\frac{\omega}{K}\right)d\omega}{Kv-\omega}\right\}. \tag{21}$$

If we set $t=0$ in equ. (21) we get the following equation for $\alpha(Kv)$:

$$f_1(v,t=0) = K\alpha(Kv) - \frac{4\pi e^2}{mK}\left\{f_0'(v)\,P\int_{-\infty}^{\infty}\frac{\alpha(\omega)\,d\omega}{\omega-Kv} + \alpha(Kv)\,P\int_{-\infty}^{\infty}\frac{f_0'\left(\frac{\omega}{K}\right)d\omega}{\omega-Kv}\right\} \tag{22}$$

We take now the Hilbert transform of this equation, and remembering the convolution theorem, we have:

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{f_1(v,t=0)\,dv}{v-\frac{\omega'}{K}} = \frac{K}{\pi}P\int_{-\infty}^{\infty}\frac{\alpha(Kv)\,dv}{v-\frac{\omega'}{K}} - \frac{4\pi e^2}{mK}\left\{\frac{1}{\pi^2}P\int_{-\infty}^{\infty}\frac{f_0'(v)\,dv}{v-\frac{\omega'}{K}}\,P\int_{-\infty}^{\infty}\frac{\alpha(Kv)\,dv}{v-\frac{\omega'}{K}} - f_0'\left(\frac{\omega'}{K}\right)\alpha(\omega')\right\} \tag{23}$$

From equ. (19) and (2o) we get finally:

$$\alpha(\omega) = \frac{K f_1(\frac{\omega}{K}, t=0) + \frac{4\pi^2 e^2}{mK}\left\{ f_0'(\frac{\omega}{K})\, \overline{f_1}(\frac{\omega}{K}, t=0) - f_1(\frac{\omega}{K}, t=0)\, \overline{f_0'}(\frac{\omega}{K}) \right\}}{G^2(\frac{\omega}{K}) + \left(\frac{4\pi^2 e^2}{mK} f_0'(\frac{\omega}{K})\right)^2} \tag{24}$$

where

$$\overline{F}(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{F(y)\,dy}{y - x}$$

and

$$G(\frac{\omega}{K}) = K - \frac{4\pi^2 e^2}{mK} \overline{f_0'}(\frac{\omega}{K})$$

We note that expression

$$G^2(\frac{\omega}{K}) + \left(\frac{4\pi^2 e^2}{mK} f_0'(\frac{\omega}{K})\right)^2$$

is never zero, because f_0 is a single humped function. In fact, if $f_0'(\frac{\omega}{K})$ is equal to zero at the point $\frac{\omega_0}{K}$, $\overline{f_0'}(\frac{\omega_0}{K})$ is negative definite, because $\frac{f_0'(v)}{v - \frac{\omega_0}{K}}$ is never positive.

At this point we shall examine very briefly how far the form of the unperturbed distribution function limits the arbitrariness in the choice of the initial perturbation.

If the distance between two consecutive zeros of the orthogonal polynomials associated with $-\frac{4\pi}{m}(v-v_0)^{-1} f_0'(v)$ goes asymptotically to zero, there is no limitation on the form of $f_1(v, t=0)$. An example is given by the Maxwellian. In general, we can say that this happens for all the functions $f_0(v)$ which have no interval of constancy (Ref. (1) p. 5o). If the unperturbed distribution function is different from zero only in a finite interval, then f_1 too can be different from zero only in the same interval.

If the unperturbed distribution function is a step function, the function $\alpha(\kappa v)$ is a linear combination of delta functions, and the other possible cases are easily deduced from these.

We can now devine some conclusions about the asymptotic behaviour of the density.

When the unperturbed distribution function is a step function, the function $\alpha(\omega)$ is a linear combination of delta functions, and therefore the density is not damped.

In general we can say that there is no damping only when the function $\alpha(\omega)$ contains delta functions, or other generalized functions. An interesting example is provided by the function $\alpha(\omega) \propto \cos(\frac{\omega^2}{a})$. The Fourier transform of this function is defined as (Ref. (2) p. 49)

$$2\lim_{\varepsilon\to 0}\int_0^{\infty} e^{-\varepsilon\omega^2}\cos\left(\frac{\omega^2}{a}\right)\cos\omega t\,d\omega \quad ;$$

hence the density comes out to be proportional to

$$\sin\left(\frac{a t^2+\pi}{4}\right) .$$

Except these rather special cases, the density goes to zero as t goes to infinity. More exactly, it goes at least as t^{-n-1}, if the n-th derivative of $\alpha(\omega)$ exist; that is, from equ. (24), if $f_0'(v)$ and $f_1(v,t=o)$ are n times differentiable.

Finally, it is obvious that no damping can be expected from a set of n equations like equ. (8). In fact the general solution is, in this case, a linear superposition of n periodic solutions. In addition it is evident from equ. (15) that, in the case of the Maxwellian distribution, the distribution of the eigenfrequencies is not appropriate to approximate a damping even for a short time, because for small n, the frequencies are dense around $\omega = o$ and not around $\omega = \omega_p$

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