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The two-loop superstring five-point amplitude and S-duality

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The low-energy limit of the massless two-loop five-point amplitudes for both type IIA and type IIB superstrings is computed with the pure spinor formalism and its overall coefficient determined from first principles. For the type IIB theory, the five-graviton amplitude is found to be proportional to its tree-level counterpart at the corresponding order in α'. Their ratio ties in with expectations based on S-duality since it matches the same modular function $E_{5/2}$ which relates the two-loop and tree-level four-graviton amplitudes. For R-symmetry violating states, the ratio between tree-level and two-loop amplitudes at the same α'-order carries an additional factor of $-3/5$. Its S-duality origin can be traced back to a modular form derived from $E_{5/2}$.

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1. Introduction

In this paper, we determine the low-energy limit of the five-point closed-string amplitudes among massless type IIA and type IIB states using the pure spinor (PS) formalism [1,2]. The precise elaboration of overall coefficients confirms the predictions [3,4] based on the non-perturbative S-duality of the type IIB effective action [5]. This complements previous S-duality analyses of the five-point amplitudes at one-loop [6] as well as the four-point amplitudes at two- [7,8] and three-loops [9].

S-duality constrains curvature couplings of the schematic form $D^{2k}R^n$ (and their supersymmetric completions) to depend on the scalar fields through modular invariant functions and thereby relates different loop orders in perturbation theory. The subsequent two-loop analysis probes the moduli-dependent coefficient of the D^4R^4 and D^2R^5 interactions which was identified as the non-holomorphic Eisenstein series $E_{5/2}$ in ten dimensions [3,4]. Its perturbative terms relate the tree-level and two-loop contributions of the corresponding graviton amplitudes and their R-symmetry conserving superpartners.

Likewise, R-symmetry violating closed-string amplitudes at different loop-orders (involving, for instance, four gravitons and one dilaton) are interlocked by modular forms [10]. Given that R-symmetry violating four-point amplitudes vanish, the five-point amplitudes in this work furnish the simplest perturbative fingerprints of their modular properties. Specifically, the tree-level and two-loop results at the α'-order under discussion are expected to orginate from a certain modular derivative of $E_{5/2}$.

We verify the expected ratios by explicit computation at the five-point level, i.e. by extracting the type IIB components involving five gravitons as well as four gravitons and one dilaton from the supersymmetric two-loop low-energy limit. This is the first perturbative check at genus two for the S-duality properties of the five-point interaction D^2R^5 and its R-symmetry violating counterparts.

Since the main objective of this work requires precise control over normalizations, section 2 contains a detailed account on the conventions used (closely following [8,9]). In sections 3 and 4, well-known amplitudes at tree-level and one-loop are recomputed using the conventions of section 2 – not only to review their end result but also to verify the reliability of the PS setup in keeping track of their overall normalizations. The novel result on the two-loop five-point amplitude is derived in section 5. Finally, section 6 is devoted to the S-duality analysis of the above results and is suitable for self-contained reading.

2. Review of conventions

In this section the conventions used in the rest of the paper are presented. They closely follow the conventions used in [8,9] but deviations were taken when deemed appropriate.

2.1. World-sheet fields

The world-sheet action for the left-moving sector in the non-minimal pure spinor formalism is [2]

$$S = \frac{1}{2\pi\alpha'}\int d^2z\,\big(\partial x^m\overline{\partial}x_m + \alpha' p_\alpha\overline{\partial}\theta^\alpha - \alpha' w_\alpha\overline{\partial}\lambda^\alpha - \alpha'\overline{w}^\alpha\overline{\partial}\overline{\lambda}_\alpha + \alpha' s^\alpha\overline{\partial}r_\alpha\big)\,, \tag{2.1}$$

where $m = 0, 1\ldots, 9$ and $\alpha = 1, \ldots, 16$ are the vector and spinorial indices of the ten-dimensional Lorentz group, and α' denotes the inverse string tension. In addition, λ^α and $\overline{\lambda}_\alpha$ are bosonic pure spinors and r_α is a constrained fermionic variable,

$$(\lambda\gamma^m\lambda) = 0, \quad (\overline{\lambda}\gamma^m\overline{\lambda}) = 0, \quad (\overline{\lambda}\gamma^m r) = 0. \tag{2.2}$$

The Green–Schwarz constraint $d_\alpha(z)$ and the supersymmetric momentum $\Pi^m(z)$ are defined by

$$d_\alpha(z) = p_\alpha - \frac{1}{\alpha'}(\gamma^m\theta)_\alpha\partial x_m - \frac{1}{4\alpha'}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta), \quad \Pi^m(z) = \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta)\,, \tag{2.3}$$

while the BRST charge and the energy-momentum tensor,

$$Q = \oint(\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha), \quad T(z) = -\frac{1}{\alpha'}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + w_\alpha\partial\lambda^\alpha + \overline{w}^\alpha\partial\overline{\lambda}_\alpha - s^\alpha\partial r_\alpha\,, \tag{2.4}$$

are related by $\{Q, b(z)\} = T(z)$, with the following expression for the b-ghost [2]

$$b = s^\alpha\partial\overline{\lambda}_\alpha + \frac{1}{4(\lambda\overline{\lambda})}\big[2\Pi^m(\overline{\lambda}\gamma_m d) - N_{mn}(\overline{\lambda}\gamma^{mn}\partial\theta) - J_\lambda(\overline{\lambda}\partial\theta) - (\overline{\lambda}\partial^2\theta)\big] \tag{2.5}$$

$$+ \frac{(\overline{\lambda}\gamma^{mnp}r)}{192(\lambda\overline{\lambda})^2}\Big[\frac{\alpha'}{2}(d\gamma_{mnp}d) + 24N_{mn}\Pi_p\Big] - \frac{\alpha'}{2}\frac{(r\gamma_{mnp}r)}{16(\lambda\overline{\lambda})^3}\Big[(\overline{\lambda}\gamma^m d)N^{np} - \frac{(\overline{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{8(\lambda\overline{\lambda})}\Big]\,.$$

2.2. Scalar Green function and OPEs

The regularized scalar Green function $G(z, w)$ is written in terms of the prime form $E(z, w)$ and the global holomorphic one-forms $\omega_I(z)$ as [11]

$$G(z, w) = -\frac{\alpha'}{2}\ln\big|E(z, w)\big|^2 + \alpha'\pi\sum_{I,J=1}^{g}(\mathrm{Im}\int_z^w\omega_I)\,(\mathrm{Im}\,\Omega)^{-1}_{IJ}\,(\mathrm{Im}\int_z^w\omega_J), \tag{2.6}$$

and satisfies

$$\frac{2}{\alpha'}\partial_z\overline{\partial}_{\overline{z}}G(z,w) = -2\pi\delta^{(2)}(z-w) + \pi\sum_{I,J=1}^{g}\omega_I(z)(\operatorname{Im}\Omega)^{-1}_{IJ}\overline{\omega}_J(\overline{z}) \tag{2.7}$$
$$\frac{2}{\alpha'}\partial_z\overline{\partial}_{\overline{w}}G(z,w) = 2\pi\delta^{(2)}(z-w) - \pi\sum_{I,J=1}^{g}\omega_I(z)(\operatorname{Im}\Omega)^{-1}_{IJ}\overline{\omega}_J(\overline{w})\,,$$

where Ω_{IJ} is the genus-g period matrix to be defined in section 2.5. Furthermore,

$$\eta(z_i,z_j) \equiv \eta_{ij} \equiv -\frac{2}{\alpha'}\frac{\partial}{\partial z_i}G(z_i,z_j). \tag{2.8}$$

The genus-g OPEs are [12,13]

$$\begin{aligned}
x^m(z,\overline{z})\,x_n(w,\overline{w}) &\sim \delta^m_n G(z,w), & p_\alpha(z)\,\theta^\beta(w) &\sim \delta^\beta_\alpha\eta(z,w),\\
d_\alpha(z)d_\beta(w) &\sim -\frac{2}{\alpha'}\gamma^m_{\alpha\beta}\Pi_m\eta(z,w), & d_\alpha(z)f(x(w),\theta(w)) &\sim D_\alpha f\eta(z,w),\\
d_\alpha(z)\Pi^m(w) &\sim \gamma^m_{\alpha\beta}\partial\theta^\beta\eta(z,w), & \Pi^m(z)f(x(w),\theta(w)) &\sim -\frac{\alpha'}{2}k^m f\eta(z,w)\,,
\end{aligned} \tag{2.9}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha k_m$ is the supersymmetric derivative and $f(x,\theta)$ represents a generic superfield.

It follows from (2.9) and (2.3) that

$$\Pi^m(z)\overline{\Pi}^n(\overline{w}) \sim \frac{\alpha'}{2}\eta^{mn}\Big(2\pi\delta^{(2)}(z-w) - \pi\sum_{I,J=1}^{g}\omega_I(z)(\operatorname{Im}\Omega)^{-1}_{IJ}\overline{\omega}_J(\overline{w})\Big). \tag{2.10}$$

Left- and right-movers can be kept separated in the evaluation of the amplitude by expanding $\Pi^m(z) = \hat{\Pi}^m(z) + \sum_{I=1}^{g}\Pi^m_I\omega_I(z)$ and computing the holomorphic square with

$$\Pi^m_I\overline{\Pi}^n_J = -\frac{\alpha'}{2}\eta^{mn}\pi\,(\operatorname{Im}\Omega)^{-1}_{IJ}. \tag{2.11}$$

Using this prescription, contributions containing a single Π^m_I or $\overline{\Pi}^m_I$ vanish.

2.3. SYM superfields and massless vertex operators

The closed-string massless vertex operators are related to the holomorphic square of the open string vertex operators

$$V = \lambda^\alpha A_\alpha(x,\theta), \quad U = \partial\theta^\alpha A_\alpha(x,\theta) + \Pi^m A_m(x,\theta) + \frac{\alpha'}{2}d_\alpha W^\alpha(x,\theta) + \frac{\alpha'}{4}N_{mn}F^{mn}(x,\theta)\,, \tag{2.12}$$

where $A_\alpha(x,\theta), A^m(x,\theta), W^\alpha(x,\theta)$ and $F^{mn}(x,\theta)$ are the super-Yang–Mills (SYM) superfields in ten dimensions. Their equations of motion [14]

$$
\begin{aligned}
D_\alpha A_\beta + D_\beta A_\alpha &= \gamma^m_{\alpha\beta} A_m, \qquad & D_\alpha A_m &= (\gamma_m W)_\alpha + k_m A_\alpha \\
D_\alpha F_{mn} &= 2k_{[m}(\gamma_{n]}W)_\alpha, \qquad & D_\alpha W^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}
\end{aligned}
\tag{2.13}
$$

are solved by the θ-expansions in [15] involving gluon polarization vectors and gaugino wave functions. More precisely, the closed-string vertex operators are given by

$$
|V(z)|^2 \equiv V(\theta)\otimes\tilde V(\overline\theta)e^{k\cdot x}, \quad |U(z)|^2 \equiv U(\theta)\otimes\tilde U(\overline\theta)e^{k\cdot x}\,,
\tag{2.14}
$$

where $V(\theta)$ and $U(\theta)$ are defined from (2.12) by stripping off the plane-wave factor, e.g. $U(z) = U(\theta)e^{k\cdot x}$. Furthermore, each massless vertex is normalized with a coefficient κ (see e.g. [7]) so the n-point amplitude prescription contains an overall factor of κ^n. As shown in appendix A, unitarity relates it to the other string parameters (such as the coupling constant $e^{-2\lambda}$) via $\kappa^2 e^{-2\lambda} = \pi/\alpha'^2$.

2.4. Integration on pure spinor space

The zero-mode measures for the non-minimal pure spinor variables in a genus-g surface have length dimension zero and are given by [8]

$$
\begin{aligned}
&[d\lambda]\,T_{\alpha_1\ldots\alpha_5} = c_\lambda\,\epsilon_{\alpha_1\ldots\alpha_{16}} d\lambda^{\alpha_6}\ldots d\lambda^{\alpha_{16}} \qquad && [dw] = c_w\,T_{\alpha_1\ldots\alpha_5}\epsilon^{\alpha_1\ldots\alpha_{16}} dw_{\alpha_6}\ldots dw_{\alpha_{16}} \\
&[d\overline\lambda]\,\overline T^{\alpha_1\ldots\alpha_5} = c_{\overline\lambda}\,\epsilon^{\alpha_1\ldots\alpha_{16}} d\overline\lambda_{\alpha_6}\ldots d\overline\lambda_{\alpha_{16}} && [d\overline w]\,T_{\alpha_1\ldots\alpha_5} = c_{\overline w}\,\epsilon_{\alpha_1\ldots\alpha_{16}} d\overline w^{\alpha_6}\ldots d\overline w^{\alpha_{16}} \\
&[dr] = c_r\,\overline T^{\alpha_1\ldots\alpha_5}\epsilon_{\alpha_1\ldots\alpha_{16}}\partial_r^{\alpha_6}\ldots\partial_r^{\alpha_{16}} && [ds^I] = c_s\,T_{\alpha_1\ldots\alpha_5}\epsilon^{\alpha_1\ldots\alpha_{16}}\partial^{s^I}_{\alpha_6}\ldots\partial^{s^I}_{\alpha_{16}} \\
&[d\theta] = c_\theta\,d^{16}\theta && [dd^I] = c_d\,d^{16}d^I.
\end{aligned}
\tag{2.15}
$$

The normalizations are [8]

$$
\begin{aligned}
c_\lambda &= \Big(\frac{\alpha'}{2}\Big)^{-2}\frac{1}{11!}\Big(\frac{A_g}{4\pi^2}\Big)^{11/2} \qquad & c_w &= \Big(\frac{\alpha'}{2}\Big)^{2}\frac{(2\pi)^{-11}}{11!\,5!}\,Z_g^{-11/g} \\
c_{\overline\lambda} &= \Big(\frac{\alpha'}{2}\Big)^{2}\frac{2^6}{11!}\Big(\frac{A_g}{4\pi^2}\Big)^{11/2} & c_{\overline w} &= \Big(\frac{\alpha'}{2}\Big)^{-2}\frac{(\lambda\overline\lambda)^3}{11!\,(2\pi)^{11}}Z_g^{-11/g} \\
c_r &= \Big(\frac{\alpha'}{2}\Big)^{-2}\frac{R}{11!5!}\Big(\frac{2\pi}{A_g}\Big)^{11/2} & c_s &= \Big(\frac{\alpha'}{2}\Big)^{2}\frac{(2\pi)^{11/2}R^{-1}}{2^6 11!\,5!\,(\lambda\overline\lambda)^3}Z_g^{11/g} \\
c_\theta &= \Big(\frac{\alpha'}{2}\Big)^{4}\Big(\frac{2\pi}{A_g}\Big)^{16/2} & c_d &= \Big(\frac{\alpha'}{2}\Big)^{-4}(2\pi)^{16/2}\,Z_g^{16/g}\,,
\end{aligned}
\tag{2.16}
$$

where $A_g = \int d^2z\sqrt{h}$ denotes the area of the genus-g Riemann surface with metric h,

$$Z_g = \frac{1}{\sqrt{\det(2\,\mathrm{Im}\,\Omega)}}, \qquad g \geq 1\,, \tag{2.17}$$

and R is an arbitrary parameter capturing the freedom to normalize the string tree-level amplitudes[1]. As discussed in [8], the final expressions for multiloop amplitudes are independent of the area A_g. The tensors $T_{\alpha_1\ldots\alpha_5}$ and $\overline{T}^{\alpha_1\ldots\alpha_5}$ appearing in (2.15) are totally antisymmetric due to the pure spinor constraint (2.2),

$$\begin{aligned} T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} &= (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\,, \\ \overline{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} &= (\overline{\lambda}\gamma^m)^{\alpha_1}(\overline{\lambda}\gamma^n)^{\alpha_2}(\overline{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} \end{aligned} \tag{2.18}$$

and satisfy $T\cdot\overline{T} = 5!\,2^6(\lambda\overline{\lambda})^3$.

One can show using the results of [16] that the integration over an arbitrary number of pure spinors λ^α and $\overline{\lambda}_\beta$ is given by

$$\int [d\lambda][d\overline{\lambda}]e^{-(\lambda\overline{\lambda})}(\lambda\overline{\lambda})^m\lambda^{\alpha_1}\cdots\lambda^{\alpha_n}\overline{\lambda}_{\beta_1}\cdots\overline{\lambda}_{\beta_n} = \Big(\frac{A_g}{2\pi}\Big)^{11}\frac{\Gamma(8+m+n)}{302400}\mathcal{T}^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}\,, \tag{2.19}$$

where $\mathcal{T}^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}$ are the γ-matrix traceless tensors discussed in [9]. From $\mathcal{T}^{\alpha_1\ldots\alpha_p}_{\alpha_1\ldots\alpha_p} = 1$ it follows that [8]

$$\int [d\lambda][d\overline{\lambda}](\lambda\overline{\lambda})^n e^{-(\lambda\overline{\lambda})} = \Big(\frac{A_g}{2\pi}\Big)^{11}\frac{\Gamma(8+n)}{7!\,60}\,. \tag{2.20}$$

For an arbitrary superfield $M(\lambda,\overline{\lambda},\theta,r)$ we define [8]

$$\langle M(\lambda,\overline{\lambda},\theta,r)\rangle_{(p,g)} \equiv \int [d\theta][dr][d\lambda][d\overline{\lambda}]\,\frac{e^{-(\lambda\overline{\lambda})-(r\theta)}}{(\lambda\overline{\lambda})^{3-p}}\,M(\lambda,\overline{\lambda},\theta,r)\,, \tag{2.21}$$

and therefore the pure spinor measure $(\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma^t\theta)(\theta\gamma_{rst}\theta) \equiv (\lambda^3\theta^5)$ is mapped to

$$\langle(\lambda^3\theta^5)\rangle_{(p,g)} = N_{(p,g)}\langle\langle(\lambda^3\theta^5)\rangle\rangle\,, \quad N_{(p,g)} \equiv 2^7\frac{R}{P}\Big(\frac{2\pi}{A_g}\Big)^{5/2}\Big(\frac{\alpha'}{2}\Big)^2\frac{\Gamma(8+p)}{7!}\,, \tag{2.22}$$

and the identity factor $\frac{\langle\langle(\lambda^3\theta^5)\rangle\rangle}{P} = 1$ keeps track of the normalization convention [1]

$$\langle\langle(\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma^t\theta)(\theta\gamma_{rst}\theta)\rangle\rangle = P. \tag{2.23}$$

---

[1] In previous works [8,9] the choice $R = \sqrt{2}/(2^{16}\pi)$ was made to match the tree-level conventions of [7]. In this work we deviate from that motivation and the choice (2.24) will lead to tree-level amplitudes (3.4) with unit overall coefficient.

The choice $P = 2880$ is convenient in view of the factorization properties of pure spinor superspace kinematic factors and has been observed in [17] to imply tree-level normalizations compatible with RNS computations. Unless otherwise noted we use,

$$R^2 = \frac{\pi^5}{2^5}, \quad P = 2880\ . \tag{2.24}$$

2.4.1. Abbreviations and (anti-)symmetrization combinatorics

The (anti)symmetrization over n indices includes a factor of $1/n!$, the generalized Kronecker delta is $\delta^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n} \equiv \delta^{[\alpha_1}_{\beta_1}\cdots\delta^{\alpha_n]}_{\beta_n}$ and satisfies $\delta^{\alpha_1\ldots\alpha_n}_{\alpha_1\ldots\alpha_n} = \binom{d}{n}$ where $d = 10$ ($d = 16$) for vector (spinor) indices. The integration over θ is given by $\int d^{16}\theta\, \theta^{\alpha_1}\cdots\theta^{\alpha_{16}} = \epsilon^{\alpha_1\ldots\alpha_{16}}$ and $\epsilon^{\alpha_1\ldots\alpha_{11}\gamma_1\ldots\gamma_5}\epsilon_{\alpha_1\ldots\alpha_{11}\beta_1\ldots\beta_5} = 11!5!\,\delta^{\gamma_1\ldots\gamma_5}_{\beta_1\ldots\beta_5}$.

Partitions of d_α zero-modes are denoted by $(p_1, p_2, \ldots, p_g)_d$, signaling the presence of p_I factors of d^I_α for $I = 1, 2, \ldots, g$. Accordingly, contributions from the b-ghost will be labeled by their partition of d_α zero-modes as $B^{m_1\ldots m_r}_{(p_1,p_2,\ldots,p_g)}$, where the vector indices take into account that those contributions need not be Lorentz scalars. Furthermore, we define

$$(\epsilon\cdot T\cdot d^I) \equiv \epsilon^{\alpha_1\ldots\alpha_{16}} T_{\alpha_1\ldots\alpha_5} d^I_{\alpha_6}\cdots d^I_{\alpha_{16}}\ , \quad (\overline{\lambda} r d^I d^J) \equiv (\overline{\lambda}\gamma^{mnp} r)(d^I\gamma_{mnp} d^J) \tag{2.25}$$

$$D^{m_1 m_2\ldots m_r}_{(11+p_1,11+p_2,\ldots,11+p_g)} \equiv \int \prod_{I=1}^{g} [dd^I] \frac{(\epsilon\cdot T\cdot d^I)}{11!\,5!} B^{m_1\ldots m_r}_{(p_1,p_2,\ldots,p_g)}\ . \tag{2.26}$$

2.4.2. Frequent zero-mode integrals

Some integrals which are frequently used in the next sections are summarized here,

$$\int [dd^I](\epsilon\cdot T\cdot d^I)\, d^I_{\alpha_1} d^I_{\alpha_2} d^I_{\alpha_3} d^I_{\alpha_4} d^I_{\alpha_5} = 11!\,5!\, c_d\, T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \tag{2.27}$$

$$\int [dd^I](\epsilon\cdot T\cdot d^I)\, d^I_{\alpha_1} d^I_{\alpha_2} d^I_{\alpha_3} (d^I\gamma^{mnp} d^I) = 11!\,5!\,96\, c_d\, (\lambda\gamma^{[m})_{\alpha_1}(\lambda\gamma^{n})_{\alpha_2}(\lambda\gamma^{p]})_{\alpha_3}\ ,$$

$$\Big|\int \prod_{I=1}^{g} [dw^I][d\overline{w}^I][ds^I]\, e^{-(w^I\overline{w}^I)-(d^I s^I)}\Big|^2 = \Big(\frac{\alpha'}{2}\Big)^{4g} \frac{1}{(2\pi)^{16g} 2^{2g} Z_g^{22}} \Big|\prod_{I=1}^{g} \frac{(\epsilon\cdot T\cdot d^I)}{(11!\,5!)}\Big|^2\ .$$

To prove the third integral one uses [16,8]

$$\int \prod_{I=1}^{g} [ds^I]\, e^{-(d^I s^I)} = \Big(\frac{\alpha'}{2}\Big)^{2g} \frac{(2\pi)^{11g/2} Z_g^{11}}{R^g 2^{6g} (\lambda\overline{\lambda})^{3g}} \prod_{I=1}^{g} \frac{(\epsilon\cdot T\cdot d^I)}{(11!\,5!)}\ ,$$

$$\int \prod_{I=1}^{g} [dw^I][d\overline{w}^I]\, e^{-(w^I\overline{w}^I)} = \frac{(\lambda\overline{\lambda})^{3g}}{(2\pi)^{11g}} Z_g^{-22}\ . \tag{2.28}$$

2.5. Riemann surfaces and moduli space

A holomorphic field with conformal weight one in a genus-g Riemann surface Σ can be expanded in a basis of holomorphic one-forms as $\phi(z) = \hat{\phi}(z) + \sum_{I=1}^{g} \omega_I(z)\phi^I$, and ϕ^I are the *zero-modes* of $\phi(z)$. If $\{a_I, b_J\}$ are the generators of the $H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$ homology group, the holomorphic one-forms can be chosen such that for $I, J = 1, 2, \ldots, g$

$$\int_{a_I} \omega_J(z)\, dz = \delta_{IJ}, \quad \int_{b_I} \omega_J(z)\, dz = \Omega_{IJ}, \quad \int d^2z\, \omega_I\, \overline{\omega}_J = 2\, \mathrm{Im}\, \Omega_{IJ}\,, \tag{2.29}$$

where Ω_{IJ} is the symmetric period matrix with $g(g+1)/2$ complex degrees of freedom and $d^2z = i dz \wedge d\overline{z} = 2\, d\mathrm{Re}(z) d\mathrm{Im}(z)$ [11]. We also define

$$\int_{\Sigma_n} \equiv \int \prod_{i=1}^{n} d^2 z_i\,, \quad \Delta_{ij} \equiv \epsilon^{IJ} \omega_I(z_i)\omega_J(z_j) = \omega_1(z_i)\omega_2(z_j) - \omega_1(z_j)\omega_2(z_i)\,. \tag{2.30}$$

The moduli space $\mathcal{M}_g$ is defined as the space of inequivalent complex structures τ_i on the Riemann surface of genus g and has complex dimension $3g - 3$, for $g > 1$. For genus two and three, the dimension of the moduli space is the same as the dimension of the period matrices ($3g - 3 = g(g+1)/2$ for $g = 2, 3$) and the amplitudes can be parameterized by the period matrix instead of the moduli coordinates; more explicitly for genus two [11],

$$\int d^2y\, \omega_I(y)\, \omega_J(y) \mu_i(y) = \frac{\delta\Omega_{IJ}}{\delta\tau_i}, \quad \int_{\mathcal{M}_2} d^2\tau \left| \epsilon_{i_1 i_2 i_3} \frac{\delta\Omega_{11}}{\delta\tau_{i_1}} \frac{\delta\Omega_{12}}{\delta\tau_{i_2}} \frac{\delta\Omega_{22}}{\delta\tau_{i_3}} \right|^2 = \int_{\mathcal{F}_2} d^2\Omega\,, \tag{2.31}$$

where $d^2\tau \equiv \prod_{j=1}^{3g-3} d^2\tau_j$, $d^2\Omega \equiv \prod_{I\leq J}^{g} d^2\Omega_{IJ}$ and $\mathcal{F}_g$ denotes the fundamental domain of $Sp(2g, \mathbb{Z})/\mathbb{Z}_2$. To avoid cluttering, the domains $\mathcal{M}_g$ and $\mathcal{F}_g$ will be henceforth omitted.

The $Sp(2g, \mathbb{Z})$-invariant measure for the genus-g moduli space and its volume are [18][2]

$$d\mu_g \equiv \frac{d^2\Omega}{(\det \mathrm{Im}\, \Omega)^{g+1}}, \quad \int d\mu_g = 2 \prod_{k=1}^{g} \left(\frac{2^k}{\pi^k} \Gamma(k) \zeta_{2k} \right). \tag{2.32}$$

In particular,

$$\int d\mu_1 = \frac{2\pi}{3}, \quad \int d\mu_2 = \frac{4\pi^3}{3^3\, 5}, \quad \int d\mu_3 = \frac{2^6 \pi^6}{3^6\, 5^2\, 7}\,. \tag{2.33}$$

[2] The definition of $d^2\Omega$ here is $2^{g(g+1)/2}$ bigger than in the original formula of [18].

2.6. The amplitude prescription

The multiloop n-point closed-string amplitude prescription was given in [2]

$$M_n^{(1)} = S_1 \kappa^n \int d^2\tau \int_{\Sigma_{n-1}} \Big| \langle\langle \mathcal{N}^{(1)} (b,\mu)\, V^1(0) U^2(z_2) \cdots U^n(z_n) \rangle\rangle \Big|^2 \,, \tag{2.34}$$

$$M_n^{(g)} = S_g \kappa^n e^{(2g-2)\lambda} \int \prod_{j=1}^{3g-3} d^2\tau_j \int_{\Sigma_n} \Big| \langle\langle \mathcal{N}^{(g)} (b,\mu_j)\, U^1(z_1) \cdots U^n(z_n) \rangle\rangle \Big|^2 \,, \quad g \geq 2 \,.$$

The symmetry factors for the one- and two-loop amplitudes are $S_1 = 1/2$ [19,20] and $S_2 = 1/2$ [21]. Furthermore, $S_g = 1$ for[3] $g > 2$. The b-ghost insertion is

$$(b, \mu_j) = \frac{1}{2\pi} \int d^2 y\, b(y) \mu_j(y), \quad j = 1, \ldots, 3g-3 \,, \tag{2.35}$$

where μ_j denotes the Beltrami differential for the modulus parameter τ_j, and $\mathcal{N}^{(g)}$ is the BRST regulator [2]

$$\mathcal{N}^{(g)} \equiv \exp\Big(-(\lambda\overline{\lambda}) - (r\theta) + \sum_{I=1}^{g} \big[(w^I \overline{w}^I) + (s^I d^I) \big] \Big) \,. \tag{2.36}$$

The bracket $\langle\langle \ldots \rangle\rangle$ in (2.34) denotes the path integral which integrates out the non-zero modes through OPEs and additionally contains the zero-mode integration measure

$$\langle \ldots \rangle = \int [d\theta][dr][d\lambda][d\overline{\lambda}] \prod_{I=1}^{g} [dd^I][ds^I][d\overline{w}^I][dw^I] \ldots \tag{2.37}$$

After the integration over $[dd^I][ds^I][dw^I][d\overline{w}^I]$ has been performed, the remaining variables $\lambda^\alpha, \overline{\lambda}_\beta, \theta^\delta$ and r_α have conformal weight zero and therefore are the same ones which need to be integrated in the prescription of the tree-level amplitudes. Using the Theorem 1 from [9] all correlators at this stage of the computation reduce to pure spinor superspace expressions whose component expansions can be straightforwardly computed[4] [23,24] from the θ-expansions in [15]. In particular, the last correlator to evaluate is a combination of the zero-mode integration of tree-level pure spinor variables (2.22) and x^m [8]

$$N^2_{(p,g)} \langle \prod_{j=1}^{n} e^{k^j \cdot x^j} \rangle = (2\pi)^{10} \delta^{10}(k) \Big(\frac{\alpha'}{2} \Big)^{-1} \frac{2^9 R^2}{\pi^5 P^2} \Big(\frac{\Gamma(8+p)}{7!} \Big)^2 \mathcal{I}_n^{(g)} \,, \tag{2.38}$$

[3] We thank Edward Witten for emphasizing this point to us.

[4] Note that r_α variables are converted to D_α derivatives using $r_\alpha e^{-(r\theta)} = D_\alpha e^{-(r\theta)}$ [22].

where $\delta^{10}(k) \equiv \delta^{10}(\sum_i k_i^m)$ and $\mathcal{I}_n^{(g)}$ is the n-particle Koba–Nielsen factor

$$\mathcal{I}_n^{(g)} \equiv \exp\Big(\sum_{i<j}^{n} s_{ij} G_{ij}\Big), \qquad s_{ij} \equiv k_i \cdot k_j \,. \tag{2.39}$$

Some products which appear in later sections are

$$\begin{aligned} N^2_{(3,g)} \langle \prod_{j=1}^{n} e^{k^j \cdot x^j} \rangle &= (2\pi)^{10} \delta^{10}(k) \Big(\frac{\alpha'}{2}\Big)^{-1} \mathcal{I}_n^{(g)} \\ N^2_{(2,g)} \langle \prod_{j=1}^{n} e^{k^j \cdot x^j} \rangle &= (2\pi)^{10} \delta^{10}(k) \Big(\frac{\alpha'}{2}\Big)^{-1} \frac{1}{2^2 5^2} \mathcal{I}_n^{(g)} \\ N^2_{(0,g)} \langle \prod_{j=1}^{n} e^{k^j \cdot x^j} \rangle &= (2\pi)^{10} \delta^{10}(k) \Big(\frac{\alpha'}{2}\Big)^{-1} \frac{1}{2^8 3^4 5^2} \mathcal{I}_n^{(g)} \,. \end{aligned} \tag{2.40}$$

Given the above conventions in (2.47), the length dimension $[\ldots]$ of the closed-string n-point amplitude is independent of the genus; $[M_n^{(g)}] = n(2 + [\kappa])$. Since $[\kappa] = -2$ (see appendix A) the amplitudes are dimensionless. Furthermore, in most of the calculations below overall minus signs will not be rigorously tracked.

For four-point amplitudes it is convenient to use the following shorthand notation for symmetric polynomials in Mandelstam invariants (2.39)

$$\sigma_k \equiv \Big(\frac{\alpha'}{2}\Big)^k (s_{12}^k + s_{13}^k + s_{14}^k) \,. \tag{2.41}$$

2.7. Multiparticle fields

The five-point amplitudes at genus $g = 1, 2$ discussed in this work reconcile zero-mode saturation with one OPE among the vertex operators of both left- and right-movers. The systematics of OPEs has been studied using multiparticle fields in [25], starting with:

$$V_1(z_1) U_2(z_2) = \frac{|z_{12}|^{-\frac{\alpha'}{2} s_{12}}}{z_{21}} \Big(\frac{\alpha'}{2}\Big) \big[V_{12} + Q(\ldots)\big] \tag{2.42}$$

$$\begin{aligned} U_1(z_1) U_2(z_2) = \frac{|z_{12}|^{-\frac{\alpha'}{2} s_{12}}}{z_{21}} \Big(\frac{\alpha'}{2}\Big) \big[&\partial\theta^\alpha A^{12}_\alpha + \Pi_m A^m_{12} + \Big(\frac{\alpha'}{2}\Big) d_\alpha W^\alpha_{12} + \frac{\alpha'}{4} N_{mn} F^{mn}_{12}\big] \\ &+ \partial_{1,2}(\ldots) \,. \end{aligned} \tag{2.43}$$

The suppressed BRST-exact terms in (2.42) and worldsheet derivatives in (2.43) drop out from the subsequent computations. The two-particle superfields of interest in this work are

$$\begin{aligned} A^\alpha_{12} &\equiv -\frac{1}{2} \big[A^1_\alpha (k^1 \cdot A^2) + A^1_m (\gamma^m W^2)_\alpha - (1 \leftrightarrow 2)\big] \\ W^\alpha_{12} &\equiv \frac{1}{4} (\gamma^{mn} W^2)^\alpha F^1_{mn} + W^\alpha_2 (k^2 \cdot A^1) - (1 \leftrightarrow 2) \\ F^{mn}_{12} &\equiv F^{mn}_2 (k^2 \cdot A^1) + F_2^{[m}{}_p F_1^{n]p} + k_{12}^{[m} (W_1 \gamma^{n]} W_2) - (1 \leftrightarrow 2) \end{aligned} \tag{2.44}$$

with a similar definition for A^m_{12}, and

$$V_{12} \equiv \lambda^\alpha A^{12}_\alpha \ , \qquad QV_{12} = s_{12} V_1 V_2 \ . \tag{2.45}$$

Generalizations to $p \geq 3$ particles, in particular the $V_{12\ldots p}$ mentioned in the context of tree amplitudes, can be found in [25]. In a notation where $A, B, C, \ldots$ denote multiparticle labels such as $A = 12\ldots p$, the simplest class of one-loop kinematic factors are given by

$$T_{A,B,C} \equiv \frac{1}{3}\big[(\lambda\gamma_m W_A)(\lambda\gamma_n W_B)F^{mn}_C + (C \leftrightarrow A, B)\big] \, . \tag{2.46}$$

They were firstly studied in the context of multiparticle open-string amplitudes at one-loop [26] and identified as box-numerators in one-loop amplitudes of ten-dimensional SYM [27].

2.8. Length dimensions

For convenience, the length dimensions of various fields and constants used throughout this work are summarized here,

$$\begin{gathered}
[\alpha'] = 2, \ \ [x^m] = 1, \ \ [k^m] = -1, \ \ [\kappa] = -2, \ \ [G(z,w)] = 2, \ [\eta_{ij}] = 0 \\
[\theta^\alpha, \lambda^\alpha, \overline{w}^\alpha, s^\alpha] = \frac{1}{2}, \quad [p_\alpha, w_\alpha, \overline{\lambda}_\alpha, r_\alpha] = -\frac{1}{2}, \quad [Q] = [b] = [T] = 0, \\
[A^{12\ldots p}_\alpha] = \frac{3}{2} - p, \ \ [A^{12\ldots p}_m] = 1 - p, \ \ [W^\alpha_{12\ldots p}] = \frac{1}{2} - p, \ \ [F^{12\ldots p}_{mn}] = -p, \\
[V(z)] = [U(z)] = 1, \quad [A^{\rm YM}(1,2,\ldots,n)] = n-4, \quad [\delta^{10}(k)] = 10, \quad [M^{(g)}_n] = 0 \ .
\end{gathered} \tag{2.47}$$

3. Tree-level closed-string amplitudes

In this section the tree-level amplitudes involving $n = 3, 4, 5$ closed-string states are reviewed and recomputed using the normalization conventions of section 2. This ensures that the S-duality discussion of section 6 uses amplitudes computed with a uniform set of conventions (which differ from [8,9]). For earlier references, see [28,29,30,31].

3.1. The amplitude prescription

The prescription to compute the n-point tree-level amplitude in the PS formalism is [2],

$$M^{(0)}_n = \kappa^n e^{-2\lambda} \int \prod_{i=2}^{n-2} d^2 z_i |\langle\langle \mathcal{N}^{(0)} V_1(0) U_2(z_2) \ldots U_{n-2}(z_{n-2}) V_{n-1}(1) V_n(\infty)\rangle\rangle|^2 \, , \tag{3.1}$$

where $\mathcal{N}^{(0)} = e^{-(\lambda\overline{\lambda})-r\theta}$ is the zero-mode regulator at genus zero. As explained below (2.36), $\langle\langle\ldots\rangle\rangle$ denotes the path integral which reduces to the integration over the zero-modes of tree-level variables after the non-zero modes are integrated out through OPEs. The pure spinor computation of the n-point tree-level correlator can be found in [32],

$$\begin{aligned}\langle\mathcal{K}^{(0)}(z_2,\ldots,z_{n-2})\rangle &\equiv \langle\langle V_1(z_1)U_2(z_2)\cdots U_{n-2}(z_{n-2})V_{n-1}(z_{n-1})V_n(z_n)\rangle\rangle \\ &= \Big(\frac{\alpha'}{2}\Big)^{n-3}\sum_{p=1}^{n-2}\frac{\langle V_{12\ldots p}\, V_{n-1,n-2,\ldots,p+1}V_n\rangle}{(z_{12}z_{23}\cdots z_{p-1,p})(z_{n-1,n-2}\cdots z_{p+2,p+1})}+\mathcal{P}(2,\ldots,n-2)\,,\end{aligned} \tag{3.2}$$

where $\mathcal{P}(2,\ldots,n-2)$ instructs to sum over all permutations of $2,\ldots,n-2$. The Möbius symmetry of the genus-zero worldsheet has been fixed by setting $\{z_1,z_{n-1},z_n\}=\{0,1,\infty\}$. The correlator (3.2) was later identified as a superposition of SYM tree amplitudes [32] (see [33] for their pure spinor superspace representation),

$$\begin{aligned}\langle\mathcal{K}^{(0)}(z_2,\ldots,z_{n-2})\rangle = \Big(\frac{\alpha'}{2}\Big)^{n-3}\frac{s_{12}}{z_{12}}\Big(\frac{s_{13}}{z_{13}}+\frac{s_{23}}{z_{23}}\Big)\cdots\Big(\frac{s_{12}}{z_{12}}+\cdots+\frac{s_{1,n-2}}{z_{1,n-2}}\Big) \\ \times A^{\rm YM}(1,2,\ldots,n-1,n)+\mathcal{P}(2,\ldots,n-2)\;.\end{aligned} \tag{3.3}$$

The multiparticle superfields V_{12} and $V_{12\ldots p}$ in (3.2) are defined in (2.45) and [25], respectively. Therefore, the prescription (3.1) yields,

$$\begin{aligned}M_n^{(0)} &= \kappa^n e^{-2\lambda}\int\prod_{i=2}^{n-2}d^2z_i|\langle\mathcal{K}^{(0)}(z_2,\ldots,z_{n-2})\rangle_{(3,0)}|^2\,\Big\langle\prod_{j=1}^{n}e^{k^j\cdot x^j}\Big\rangle \\ &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^{-1}\kappa^n e^{-2\lambda}\int\prod_{i=2}^{n-2}d^2z_i|\langle\mathcal{K}^{(0)}(z_2,\ldots,z_{n-2})\rangle|^2\mathcal{I}_n^{(0)}\;,\end{aligned} \tag{3.4}$$

where we used (2.22) and (2.40). Note that the Koba-Nielsen factor (2.39) simplifies to $\mathcal{I}_n^{(0)}=\prod_{i<j}^n|z_{ij}|^{-\alpha' s_{ij}}$ at genus zero.

3.2. The three-point amplitude

Using the formula (3.4) and taking $\mathcal{I}_3^{(0)}=1$ into account, the three-point amplitude can be written down immediately

$$M_3^{(0)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^{-1}\kappa^3 e^{-2\lambda}\mathcal{K}_3^{(0)}\,, \tag{3.5}$$

where $\mathcal{K}_3^{(0)}\equiv|\langle V_1V_2V_3\rangle|^2=|A^{\rm YM}(1,2,3)|^2$ and (note $[\mathcal{K}_3^{(0)}]=-2$)

$$\langle V_1V_2V_3\rangle = (e^1\cdot e^2)(k^2\cdot e^3)+e^1_m(\chi_2\gamma^m\chi_3)+{\rm cyc}(1,2,3)\,. \tag{3.6}$$

The component expressions are derived from the θ-expansions of [15] and involve transverse polarization vectors e^i of the gluon as well as chiral spinor wave functions χ_i of the gluino.

3.3. The four-point amplitude

Similarly, using the formula (3.4) the four-point amplitude becomes

$$M_4^{(0)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^{-1}\kappa^4 e^{-2\lambda}\int d^2z_2 |\langle\mathcal{K}^{(0)}(z_2)\rangle|^2 \mathcal{I}_4^{(0)} \ , \tag{3.7}$$

where the correlator is [32] (see also [34])

$$\begin{aligned}\langle\mathcal{K}^{(0)}(z_2)\rangle = \langle\langle V_1(z_1)U_2(z_2)V_3(z_3)V_4(z_4)\rangle\rangle &= \Big(\frac{\alpha'}{2}\Big)\Big[\frac{\langle V_{12}V_3V_4\rangle}{z_{12}} + \frac{\langle V_1V_{32}V_4\rangle}{z_{32}}\Big] \\ &= \Big(\frac{\alpha'}{2}\Big)\frac{s_{12}}{z_{12}}A^{\rm YM}(1,2,3,4) \ ,\end{aligned} \tag{3.8}$$

and we used the following representation for the color-ordered tree-level SYM amplitude,

$$A^{\rm YM}(1,2,3,4) = \frac{1}{s_{12}}\langle V_{12}V_3V_4\rangle + \frac{1}{s_{23}}\langle V_1V_{23}V_4\rangle \ . \tag{3.9}$$

Furthermore, using the explicit form $\mathcal{I}_4^{(0)} = |z_2|^{-\alpha' s_{12}}|1-z_2|^{-\alpha' s_{23}}$ of the Koba–Nielsen factor at $\{z_1, z_3, z_4\} = \{0, 1, \infty\}$, the integral in (3.7) boils down to [7]

$$\int d^2z_2 z_2^{-\frac{\alpha'}{2}s_{12}-1}\overline{z}_2^{-\frac{\alpha'}{2}s_{12}-1}(1-z_2)^{-\frac{\alpha'}{2}s_{23}}(1-\overline{z}_2)^{-\frac{\alpha'}{2}s_{23}} = 2\pi s_{23}^2\Big(\frac{\alpha'}{2}\Big)^2\mathcal{B}_0 \tag{3.10}$$

with

$$\mathcal{B}_0 \equiv \frac{\Gamma(-\frac{\alpha'}{2}s_{12})\Gamma(-\frac{\alpha'}{2}s_{13})\Gamma(-\frac{\alpha'}{2}s_{14})}{\Gamma(1+\frac{\alpha'}{2}s_{12})\Gamma(1+\frac{\alpha'}{2}s_{13})\Gamma(1+\frac{\alpha'}{2}s_{14})} = \frac{3}{\sigma_3} + 2\zeta_3 + \zeta_5\sigma_2 + \frac{2}{3}\zeta_3^2\sigma_3 + \cdots \ . \tag{3.11}$$

Hence, the four-point amplitude (3.7) is given by

$$M_4^{(0)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\kappa^4 e^{-2\lambda}\, 2\pi\mathcal{K}_4^{(0)}\mathcal{B}_0 \, , \tag{3.12}$$

where (note $[\mathcal{K}_4^{(0)}] = -8$)

$$\mathcal{K}_4^{(0)} \equiv |s_{12}s_{23}A^{\rm YM}(1,2,3,4)|^2 = |s_{23}\langle V_{12}V_3V_4\rangle + s_{12}\langle V_1V_{23}V_4\rangle|^2 \, . \tag{3.13}$$

3.3.1. The low-energy limit

From $\mathcal{B}_0 = (2/\alpha')^3/(s_{12}s_{13}s_{14}) + \cdots$ and $A^{\rm YM}(1,2,3,4) = \langle V_{12}V_3V_4\rangle/s_{12} + \langle V_1V_{23}V_4\rangle/s_{23}$ the kinematic factor in the amplitude (3.12) becomes

$$-\Big(\frac{\alpha'}{2}\Big)^3\mathcal{K}_4^{(0)}\mathcal{B}_0 = -\frac{|s_{12}s_{23}A^{\rm YM}(1,2,3,4)|^2}{s_{12}s_{13}s_{14}} = \frac{|\langle V_{12}V_3V_4\rangle|^2}{s_{12}} + \frac{|\langle V_{31}V_2V_4\rangle|^2}{s_{13}} + \frac{|\langle V_{23}V_1V_4\rangle|^2}{s_{23}} \tag{3.14}$$

where we used $\langle V_{12}V_3V_4\rangle + \langle V_{23}V_1V_4\rangle + \langle V_{31}V_2V_4\rangle = 0$ [35]. Therefore the low-energy limit of (3.12) is given by

$$M_4^{(0)} = (2\pi)^{10}\delta^{10}(k)\kappa^4 e^{-2\lambda}\, 2\pi\Big[\frac{|\langle V_{12}V_3V_4\rangle|^2}{s_{12}} + \frac{|\langle V_{31}V_2V_4\rangle|^2}{s_{13}} + \frac{|\langle V_{23}V_1V_4\rangle|^2}{s_{23}}\Big] + \mathcal{O}(\alpha'^3) \ . \tag{3.15}$$

3.4. The five-point amplitude

According to the formula (3.4), the five-point amplitude is given by

$$M_5^{(0)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^{-1}\kappa^5 e^{-2\lambda}\int\prod_{i=2}^{3} d^2 z_i |\langle\mathcal{K}^{(0)}(z_2,z_3)\rangle|^2 \mathcal{I}_5^{(0)}\ , \tag{3.16}$$

where

$$\begin{aligned}\langle\mathcal{K}^{(0)}(z_2,z_3)\rangle &= \langle\langle V_1 U_2(z_2)U_3(z_3)V_4V_5\rangle\rangle \\ &= \Big(\frac{\alpha'}{2}\Big)^2\Big[\frac{\langle V_{123}V_4V_5\rangle}{z_{12}z_{23}} + \frac{\langle V_{12}V_{43}V_5\rangle}{z_{12}z_{43}} + \frac{\langle V_1V_{432}V_5\rangle}{z_{43}z_{32}} + (2\leftrightarrow 3)\Big] \\ &= \Big(\frac{\alpha'}{2}\Big)^2 \frac{s_{12}s_{34}}{z_{12}z_{34}} A^{\rm YM}(1,2,3,4,5) + (2\leftrightarrow 3)\ .\end{aligned} \tag{3.17}$$

After inserting (3.17) into (3.16), the α'-expansion of the resulting integrals can be obtained through the KLT procedure [29] and arranged in the form [36]

$$M_5^{(0)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\kappa^5 e^{-2\lambda}(2\pi)^2\mathcal{K}_5^{(0)} \tag{3.18}$$

$$\mathcal{K}_5^{(0)} \equiv \tilde A_{54}^T\cdot S_0\cdot\Big[1 + 2\zeta_3\Big(\frac{\alpha'}{2}\Big)^3 M_3 + 2\zeta_5\Big(\frac{\alpha'}{2}\Big)^5 M_5 + 2\zeta_3^2\Big(\frac{\alpha'}{2}\Big)^6 M_3^2 + \mathcal{O}(\alpha'^7)\Big]\cdot A_{45}\ , \tag{3.19}$$

where $\tilde A_{54}^T$ and A_{45} are two-component vectors of SYM tree-amplitudes

$$\tilde A_{54} \equiv \begin{pmatrix}\tilde A^{\rm YM}(1,2,3,5,4)\\ \tilde A^{\rm YM}(1,3,2,5,4)\end{pmatrix}\ ,\quad A_{45} \equiv \begin{pmatrix} A^{\rm YM}(1,2,3,4,5)\\ A^{\rm YM}(1,3,2,4,5)\end{pmatrix}\ , \tag{3.20}$$

and S_0 denotes the momentum kernel [37], a convenient basis choice for the Mandelstam invariants in the KLT relations [29]

$$S_0 \equiv \begin{pmatrix} s_{12}(s_{13}+s_{23}) & s_{12}s_{13}\\ s_{12}s_{13} & s_{13}(s_{12}+s_{23})\end{pmatrix}\ . \tag{3.21}$$

The 2×2 matrices M_{2n+1} introduced in [36] describe the momentum dependence of the α'-corrections and should not be confused with the amplitudes $M_n^{(g)}$. Their entries are degree $2n+1$ polynomials in Mandelstam invariants, e.g. (see also [30,38])

$$M_3 \equiv \begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\ ,\qquad \begin{aligned} m_{12} &= -s_{13}s_{24}(s_1+s_2+s_3+s_4+s_5)\\ m_{11} &= s_3[-s_1(s_1+2s_2+s_3)+s_3s_4+s_4^2] + s_1s_5(s_1+s_5)\end{aligned} \tag{3.22}$$

with $m_{21} = m_{12}\big|_{2\leftrightarrow 3}$ and $m_{22} = m_{11}\big|_{2\leftrightarrow 3}$ as well as $s_i \equiv s_{i,i+1}$ subject to $s_5 = s_{15}$. Higher-order analogues such as M_5 relevant for the comparison with the two-loop five-point amplitude are available for download at the website [39]. The overall coefficient of the five-point amplitude (3.18) will be verified by factorization at the lowest order in α' in the appendix A.

4. One-loop closed-string amplitudes

In this section the overall coefficients of the four- and five-point one-loop amplitudes are computed using the conventions of section 2, ensuring that the S-duality analysis of section 6 is unaffected by different conventions in the literature. Although the coefficient of the five-point amplitude can be derived from factorization (see appendix A), its computation from first principles as done in section 4.3 is novel and validates the general method developed in [8]. For earlier references, see [40] for the original four-point derivation, [20,41,7,8] for discussions on its overall coefficient and [42,43,44,45,4,46,26] for related extensions.

4.1. The amplitude prescription

According to (2.34), the n-point closed-string one-loop prescription is

$$M_n^{(1)} = \frac{1}{2}\kappa^n \int d^2\tau \int_{\Sigma_{n-1}} \left|\langle\langle \mathcal{N}^{(1)}(b,\mu) V^1(z_1) U^2(z_2)\cdots U^n(z_n)\rangle\rangle\right|^2 , \tag{4.1}$$

where $\mathcal{N}^{(1)}$ is the genus-one instance of the zero-mode regulator (2.36) and the b-ghost insertion (2.35) reads

$$(b,\mu) = \frac{1}{2\pi}\int d^2y\, b(y)\mu(y)\,, \tag{4.2}$$

where μ is the Beltrami differential for the modulus parameter τ. In terms of the genus-one period matrix Ω, equation (2.31) implies

$$\int d^2\tau \Big| \int d^2y\, \omega_1(y)\omega_1(y)\mu(y)\Big|^2 = \int d^2\Omega\,. \tag{4.3}$$

At genus one, there are $(16)_d$ zero-modes of d_α and $(11)_s$ zero-modes of s^α. Since there are no s^α variables in the vertex operators, and the term $s^\alpha\partial\overline{\lambda}_\alpha$ from the b-ghost (2.5) does not contribute in absence of sources for $\overline{w}^\alpha$, the zero-modes of s^α are entirely saturated by the regulator through the factor $\mathcal{N}^{(1)} \to (s^1 d^1)^{11}$. The remaining $(5)_d$ zero-modes must come from the b-ghost and the external vertices.

There are two canonical b-ghost contributions to saturate the fermionic zero-modes of d_α, with either one or two zero-modes. Expanding $\Pi_m(y) = \Pi^1_m\omega_1(y) + \hat{\Pi}_m(y)$ and $d_\alpha(y) = d^1_\alpha\omega_1(y) + \hat{d}_\alpha(y)$ where $\omega_1(z)dz = dz$ is the genus-one holomorphic one-form, one can show that amplitudes up to (and including) five-points receive zero-mode contributions from only two terms[5] in the b-ghost (2.5)

$$\int d^2\tau |(b,\mu)|^2 = \Big(\frac{\alpha'}{2}\Big)^2 \frac{1}{(2\pi)^2}\frac{1}{192^2}\int d^2\Omega\,\big|B_{(2)} + \Pi^1_m B^m_{(1)} + \cdots\big|^2 , \tag{4.4}$$

[5] Terms containing a single N^{mn} zero-mode vanish upon integration over $[dw][d\overline{w}]$ [9].

where the ellipsis represents terms relevant at $(n \geq 6)$ points and

$$B_{(2)} \equiv \frac{1}{(\lambda\overline{\lambda})^2}(\overline{\lambda}\gamma^{mnp}r)(d^1\gamma_{mnp}d^1), \qquad B^m_{(1)} \equiv \Big(\frac{2}{\alpha'}\Big)\frac{96}{(\lambda\overline{\lambda})}(\overline{\lambda}\gamma^m d^1) \; . \tag{4.5}$$

Since the vertex operators are independent of $w_\alpha, \overline{w}^\alpha$ and s^α, the integration over the zero-modes $[dw^1][d\overline{w}^1][ds^1]$ is readily performed using (2.27) and yields

$$\Big|\int [ds^1][dw^1][d\overline{w}^1]\, e^{-(d^1s^1)-(w^1\overline{w}^1)}\Big|^2 = \Big(\frac{\alpha'}{2}\Big)^4\frac{1}{(2\pi)^{16}2^2 Z_1^{22}}\Big|\frac{(\epsilon\cdot T\cdot d^1)}{(11!\,5!)}\Big|^2 \; . \tag{4.6}$$

Defining

$$D_{(13)} \equiv \int [dd^1]\frac{(\epsilon\cdot T\cdot d^1)}{(11!\,5!)}B_{(2)}, \quad D^m_{(12)} \equiv \int [dd^1]\frac{(\epsilon\cdot T\cdot d^1)}{(11!\,5!)}B^m_{(1)}\,, \tag{4.7}$$

as a special case of (2.26), the amplitude (4.1) becomes

$$M_n^{(1)} = \Big(\frac{\alpha'}{2}\Big)^6\frac{\kappa^n}{(2\pi)^{18}2^{15}\,3^2}\int\frac{d^2\Omega}{Z_1^{22}}\int_{\Sigma_{n-1}}|\langle\langle\mathcal{K}^{(1)}_{[d]}(z_2,\ldots,z_n)\rangle\rangle_{(3,1)}|^2\,\Big\langle\prod_{j=1}^n e^{k^j\cdot x^j}\Big\rangle \; . \tag{4.8}$$

The subscript $[d]$ of the kinematic factor

$$\mathcal{K}^{(1)}_{[d]}(z_2,\ldots,z_n) \equiv (D_{(13)} + \Pi_1^m D^m_{(12)} + \cdots)V_1U_2(z_2)\cdots U_n(z_n) \tag{4.9}$$

emphasizes the remaining integration over the d_α zero-modes. The ellipsis along with $\Pi^m_1 D^m_{(12)}$ refers to b-ghost contributions which do not affect $(n\leq 5)$-point amplitudes.

4.1.1. Scalar and vector building blocks at genus one

The integration over the zero-mode d^1_α in (4.9) can be done using (2.27) and gives

$$\begin{aligned} D_{(13)}V_A(d^1W_B)(d^1W_C)(d^1W_D) &= 96\,c_d T_{A|B,C,D}(\lambda,\overline{\lambda}), \\ D^m_{(12)}V_A(d^1W_B)(d^1W_C)(d^1W_D)(d^1W_E) &= 96\,c_d\Big(\frac{2}{\alpha'}\Big)S^m_{A|B,C,D,E}(\lambda,\overline{\lambda})\,, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} T_{A|B,C,D}(\lambda,\overline{\lambda}) &\equiv \frac{(\overline{\lambda}\gamma_{mnp}r)}{(\lambda\overline{\lambda})^2}V_A(\lambda\gamma^mW_B)(\lambda\gamma^nW_C)(\lambda\gamma^pW_D), \\ S^m_{A|B,C,D,E}(\lambda,\overline{\lambda}) &\equiv \frac{(\overline{\lambda}\gamma^m\gamma^r\lambda)}{(\lambda\overline{\lambda})}V_A(\lambda\gamma^sW_B)(\lambda\gamma^tW_C)(W_D\gamma_{rst}W_E) \end{aligned} \tag{4.11}$$

with multiparticle labels $A, B, \ldots$ (see section 2.7).

At this stage the Theorem 1 from [9] can be used to factorize $(\lambda\overline{\lambda})$ from the expressions in (4.11). For a general kinematic factor one then defines $K_{A|B,\ldots}(\lambda, \lambda) = (\lambda\overline{\lambda})^p K_{A|B,\ldots}$ for some power p as the result of this procedure. Doing this for (4.11) leads to

$$\begin{aligned} \langle T_{A|B,C,D}(\lambda, \overline{\lambda}) \rangle_{(3,1)} &= \langle T_{A|B,C,D} \rangle_{(2,1)}, \\ \langle S^m_{A|B,C,D,E}(\lambda, \overline{\lambda}) \rangle_{(3,1)} &= \langle S^m_{A|B,C,D,E} \rangle_{(3,1)} = 10 \langle S^m_{A|B,C,D,E} \rangle_{(2,1)}\,, \end{aligned} \tag{4.12}$$

where symmetry of $T_{A|B,C,D}$ and $S^m_{A|B,C,D,E}$ in (B, C, D) and (B, C, D, E), respectively, is inherited from (4.10). Note that any appearance of $S^m_{A|B,C,D,E}$ in $(n \geq 5)$-point one-loop amplitudes occurs in the combination[6]

$$T^m_{A|B,C,D,E} \equiv A^m_B T_{A|C,D,E} + A^m_C T_{A|B,D,E} + A^m_D T_{A|B,C,E} + A^m_E T_{A|B,C,D} + 10 S^m_{A|B,C,D,E}\,, \tag{4.13}$$

where the factor of 10 is due to the conversion from $\langle \ldots \rangle_{(3,1)} = 10 \langle \ldots \rangle_{(2,1)}$ in (4.12).

4.2. The four-point amplitude

According to the formula (4.8), the four-point amplitude is given by

$$M^{(1)}_4 = \Big(\frac{\alpha'}{2}\Big)^6 \frac{\kappa^4}{(2\pi)^{18} 2^{15}\, 3^2} \int \frac{d^2\Omega}{Z_1^{22}} \int_{\Sigma_3} |\langle\langle \mathcal{K}^{(1)}_{[d]}(z_2, z_3, z_4) \rangle\rangle_{(3,1)}|^2 \Big\langle \prod_{j=1}^4 e^{k^j \cdot x^j} \Big\rangle\,. \tag{4.14}$$

It is easy to see that $D_{(13)}$ is the only non-vanishing contribution from the b-ghost since the external vertices cannot provide four d_α zero-modes to saturate the $D^m_{(12)}$ integral [2]. The integration over $[dd^1]$ is readily performed via (4.10) followed by (4.12),

$$\langle\langle \mathcal{K}^{(1)}_{[d]}(z_2, z_3, z_4) \rangle\rangle_{(3,1)} = 96 c_d \Big(\frac{\alpha'}{2}\Big)^3 \langle T_{1|2,3,4} \rangle_{(2,1)}\ . \tag{4.15}$$

Note that the right-hand side is independent on the vertex insertion points z_2, z_3 and z_4 because only the zero-modes entered the computation. A straightforward application of (2.40) then implies

$$\begin{aligned} M^{(1)}_4 &= (2\pi)^{10} \delta^{10}(k) \Big(\frac{\alpha'}{2}\Big)^3 \frac{\kappa^4}{2^{14}\, 5^2 \pi^2} |\langle T_{1|2,3,4} \rangle|^2 \int \frac{d^2\Omega}{(\mathrm{Im}\,\Omega)^5} \int_{\Sigma_3} \mathcal{I}^{(1)}_4, \\ &= (2\pi)^{10} \delta^{10}(k) \Big(\frac{\alpha'}{2}\Big)^3 \frac{\kappa^4}{2^8 \pi^2} \mathcal{K}^{(1)}_4 \int \frac{d^2\Omega}{(\mathrm{Im}\,\Omega)^5} \int_{\Sigma_3} \mathcal{I}^{(1)}_4, \end{aligned} \tag{4.16}$$

[6] Writing the term $A^m_1 T_{2|3,4,5}$ is an abuse of notation since when computing its component expansion the variables r_α in the definition of $T_{2|3,4,5}(\lambda, \overline{\lambda})$ become covariant derivatives D_α (see [22]) and must also act upon the superfield A^m_1.

where in the second line we used [47] (note $[\mathcal{K}_4^{(1)}] = -8$)

$$\langle T_{1|2,3,4}\rangle = 40\langle V_1 T_{2,3,4}\rangle, \quad \mathcal{K}_4^{(1)} \equiv |\langle V_1 T_{2,3,4}\rangle|^2\,. \tag{4.17}$$

Note that the tree-level (3.13) and one-loop (4.17) kinematic factors are related by [34]

$$\mathcal{K}_4^{(1)} = \mathcal{K}_4^{(0)}\,, \tag{4.18}$$

a well-known result first obtained by Green and Schwarz [40].

4.2.1. The α'-expansion of the four-point amplitude

The α'-expansion of the four-point amplitude[7] has been extensively studied in a series of papers [41], where the subleading term in

$$\int \frac{d^2\Omega}{(\mathrm{Im}\,\Omega)^5}\int_{\Sigma_3}\mathcal{I}_4^{(1)} = \frac{2^4\pi}{3}\big(1+\frac{\zeta_3}{3}\sigma_3+\cdots\big) \tag{4.19}$$

signals the absence of D^4R^4 interactions at one-loop in ten dimensions. Therefore, plugging the above result in the four-point amplitude (4.16) leads to

$$M_4^{(1)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\frac{\kappa^4}{2^4\,3\pi}\mathcal{K}_4^{(1)} + \mathcal{O}(\alpha'^6)\,. \tag{4.20}$$

4.3. The five-point amplitude

Using the general result (4.8) the five-point amplitude (4.1) becomes

$$M_5^{(1)} = \frac{\kappa^5}{(2\pi)^{18}2^{15}\,3^2}\Big(\frac{\alpha'}{2}\Big)^6\int\frac{d^2\Omega}{Z_1^{22}}\int_{\Sigma_4}|\langle\langle\mathcal{K}_{[d]}^{(1)}(z_2,\ldots,z_5)\rangle\rangle_{(3,1)}|^2\Big\langle\prod_{j=1}^{5}e^{k^j\cdot x^j}\Big\rangle, \tag{4.21}$$

where

$$\mathcal{K}_{[d]}^{(1)}(z_2,\ldots,z_5) = (D_{(13)} + \Pi_m^1 D_{(12)}^m)V_1U_2(z_2)\cdots U_5(z_5)\ . \tag{4.22}$$

[7] In addition to the analytic momentum dependence shown in (4.19), threshold singularities arise from the integration region where $\mathrm{Im}\,\Omega\to\infty$. A careful treatment of these non-analytic terms can be found in [41].

The $[dd^1]$ integration with the operators of (4.7) picks up the terms with four and three d_α zero-modes from the vertices, respectively. Using the multiparticle superfields of [25,48] one arrives at

$$
\begin{aligned}
V_1U_2U_3U_4U_5\Big|_{d^4} &= \Big(\frac{\alpha'}{2}\Big)^4 V_1(d^1W_2)(d^1W_3)(d^1W_4)(d^1W_5) \qquad (4.23)\\
V_1U_2U_3U_4U_5\Big|_{d^3} &= \Big(\frac{\alpha'}{2}\Big)^4 V_{12}(d^1W_3)(d^1W_4)(d^1W_5)\eta_{12} + (2|2,3,4,5)\\
&+ \Big(\frac{\alpha'}{2}\Big)^4 V_1(d^1W_{23})(d^1W_4)(d^1W_5)\eta_{23} + (2,3|2,3,4,5)\\
&+ \Big(\frac{\alpha'}{2}\Big)^3 \Pi^1_m V_1 A_2^m(d^1W_3)(d^1W_4)(d^1W_5) + (2|2,3,4,5)\,,
\end{aligned}
$$

where the notation $(A_1, A_2, \ldots, A_p \,|\, A_1, A_2, \ldots, A_n)$ instructs to sum over all possible ways to choose p elements $A_1, A_2, \ldots, A_p$ from the set $\{A_1, \ldots, A_n\}$, for a total of $\binom{n}{p}$ terms. We have $\omega_1(z)dz = dz$ for the genus-one surface, and $\eta_{ij} = \frac{1}{z_{ij}} + \mathcal{O}(z_{ij})$ defined by (2.8) accounts for the singularity from the OPEs among vertex operators. According to (2.42) and (2.43), they introduce multiparticle superfields W^α_{23} and V_{12} defined in (2.44) and (2.45), respectively.

The integration over the zero-modes of d_α uses the formulas (4.10) and (4.12) to yield

$$
\langle\langle \mathcal{K}^{(1)}_{[d]}(z_2,\ldots,z_5)\rangle\rangle_{(3,1)} = 96\, c_d \Big(\frac{\alpha'}{2}\Big)^3 \langle \mathcal{K}^{(1)}(z_2,\ldots,z_5)\rangle_{(2,1)}\,, \qquad (4.24)
$$

where

$$
\begin{aligned}
\mathcal{K}^{(1)}(z_2,\ldots,z_5) \equiv{}& \Big(\frac{\alpha'}{2}\Big)\big[\eta_{12}T_{12|3,4,5} + (2|2,3,4,5)\big] \qquad (4.25)\\
&+ \Big(\frac{\alpha'}{2}\Big)\big[\eta_{23}T_{1|23,4,5} + (2,3|2,3,4,5)\big] + \Pi^1_m T^m_{1|2,3,4,5}\,,
\end{aligned}
$$

see (4.13) for the definition of $T^m_{1|2,3,4,5}$. Upon discarding $\Pi^1_m \to 0$ and adjoining the Koba-Nielsen factor, this is precisely the open-string correlation function for the five-point pure spinor one-loop amplitude [46,26] (for the RNS derivation, see [42,44]).

Therefore, the closed-string amplitude (4.21) becomes

$$
\begin{aligned}
M^{(1)}_5 &= \frac{\kappa^5}{2^7\pi^2}\Big(\frac{\alpha'}{2}\Big)^4 \int \frac{d^2\Omega}{Z_1^{-10}} \int_{\Sigma_4} \big|\langle \mathcal{K}^{(1)}(z_2,\ldots,z_5)\rangle_{(2,1)}\big|^2 \Big\langle \prod_{j=1}^5 e^{k^j\cdot x^j}\Big\rangle\\
&= (2\pi)^{10}\delta^{10}(k)\frac{\kappa^5}{2^{14}\,5^2\pi^2}\Big(\frac{\alpha'}{2}\Big)^3 \int \frac{d^2\Omega}{(\mathrm{Im}\,\Omega)^5} \int_{\Sigma_4} \big|\langle \mathcal{K}^{(1)}(z_2,\ldots,z_5)\rangle\big|^2 \mathcal{I}^{(1)}_5\,, \qquad (4.26)
\end{aligned}
$$

where we used $Z_1^{-10} = (2\,\mathrm{Im}\,\Omega)^5$ and the identity (2.40) on the second line. Integration by parts identities [4,6] allow to express (4.26) in terms of 37 basis integrals with BRST-invariant kinematic numerators. The α'-expansion of these integrals was analyzed in [4,6] and confirms the absence of D^2R^5 interactions at one-loop in ten dimensions.

4.3.1. The leading-order contribution

The low-energy behavior of the Σ_4 integral over $\big|\langle\mathcal{K}^{(1)}(z_2,\ldots,z_5)\rangle\big|^2\mathcal{I}_5^{(1)}$ in (4.26) is governed by two kinds of contributions [4,6]:

(i) zero-mode contractions $\Pi^1_m\overline{\Pi}^1_n \to -\eta_{mn}\big(\frac{\alpha'}{2}\big)\frac{\pi}{\mathrm{Im}\,\Omega}$ following (2.11) at $g=1$

(ii) kinematic poles[8] from the residue of the pole $\eta_{12}\overline{\eta}_{12}\sim|z_{12}|^{-2}$

$$\eta_{12}\overline{\eta}_{12}\mathcal{I}_n^{(g)} = -\Big(\frac{2}{\alpha'}\Big)\frac{2\pi}{s_{12}}\delta^2(z_1-z_2)\mathcal{I}_n^{(g)} + \mathcal{O}(\alpha'^0)\ . \tag{4.27}$$

Non-diagonal products of Green functions such as $\eta_{12}\overline{\eta}_{13}$ or $\eta_{12}\overline{\eta}_{34}$ do not contribute to the leading order in α'. Hence, we have

$$\int_{\Sigma_4}\big|\langle\mathcal{K}^{(1)}(z_2,\ldots,z_5)\rangle\big|^2\mathcal{I}_5^{(1)} = -\Big(\frac{\alpha'}{2}\Big)\frac{\pi}{\mathrm{Im}\,\Omega}\mathcal{K}_5^{(1)}\int_{\Sigma_4} + \mathcal{O}(\alpha'^2)\ , \tag{4.28}$$

where the kinematic factor $\mathcal{K}_5^{(1)}$ is defined by (note $[\mathcal{K}_5^{(1)}]=-8$)

$$\mathcal{K}_5^{(1)} \equiv \Big[\frac{|\langle T_{12|3,4,5}\rangle|^2}{s_{12}} + (2|2,3,4,5)\Big] + \Big[\frac{|\langle T_{1|23,4,5}\rangle|^2}{s_{23}} + (2,3|2,3,4,5)\Big] + |\langle T^m_{1|2,3,4,5}\rangle|^2\ , \tag{4.29}$$

and the integration over Σ_4 gives $\int_{\Sigma_4} = 2^4\,\mathrm{Im}\,\Omega^4$. This leads to the following result for the one-loop five-point amplitude (recall that $\int d\mu_1 = 2\pi/3$)

$$M_5^{(1)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^4\frac{\kappa^5}{2^9\,5^2\,3}\,\mathcal{K}_5^{(1)} + \mathcal{O}(\alpha'^5)\,, \tag{4.30}$$

In the appendix A the overall coefficient in (4.30) will be validated by factorization.

4.3.2. Components in type IIB and type IIA

The type IIB components of the kinematic factor (4.29) are related to the first α'-correction of the five-point tree-level amplitude (3.18) and (3.19) [23]:

$$\mathcal{K}_5^{(1)}\Big|_{\mathrm{IIB}} = 2^5\,5^2\Big(\frac{\alpha'}{2}\Big)^{-3}\,\mathcal{K}_5^{(0)}\Big|_{\zeta_3}\times\left\{\begin{array}{rl} 1 & :\ \text{five gravitons} \\ -\frac{1}{3} & :\ \text{four gravitons, one dilaton}\end{array}\right. \tag{4.31}$$

[8] Strictly speaking, the identity (4.27) is valid under integration over one of z_1, z_2 and results from the behavior of the Koba-Nielsen factor $\mathcal{I}_n^{(g)}\sim|z_{12}|^{-\alpha' s_{ij}}$ as $z_1\to z_2$:

$$\int d^2z_2\eta_{12}\overline{\eta}_{12}\mathcal{I}_n^{(g)} = 4\pi\int|z_{12}|d|z_{12}|\frac{1}{|z_{12}|^2}|z_{12}|^{-\alpha' s_{12}} + \mathcal{O}(\alpha'^0) = -\frac{4\pi}{\alpha' s_{12}} + \mathcal{O}(\alpha'^0)$$

This only depends on the local properties of the worldsheet and therefore holds at any genus.

The relative factor between the tree-level and one-loop amplitudes at order ${\alpha'}^4$ turns out to depend on the charges of the external states under the R-symmetry of type IIB supergravity, as has already been observed in [6]. Components with the same R-symmetry violation as four gravitons and one dilaton give rise to an additional relative factor of $-\frac{1}{3}$. This will be explained in section 6.3 from an S-duality point of view. Since R-symmetry violating four-point amplitudes vanish [10], the five-point amplitudes in this work provide the simplest context to study the S-duality properties of interactions with R-charge. Also, five-point amplitudes that violate R-charge by more units than caused by a single dilaton insertion vanish at any loop-order.

Type IIA components of the five-point low-energy limit (4.30) cannot be expressed in terms of $A^{\rm YM}$ bilinears. Instead, we have

$$\mathcal{K}_5^{(1)}\Big|_{\rm IIA}^{5\ \rm gravitons} = 2^5\, 5^2 \Big(\frac{\alpha'}{2}\Big)^{-3} \Big[\mathcal{K}_5^{(0)}\Big|_{\zeta_3} - \big|\epsilon^{me^1k^2e^2k^3e^3k^4e^4k^5e^5}\big|^2\Big] \,, \tag{4.32}$$

where the notation $\epsilon^{me^1k^2e^2k^3e^3k^4e^4k^5e^5} \equiv \epsilon_{10}^{mnp_2q_2\ldots p_5q_5} e^1_n k^2_{p_2} e^2_{q_2} \ldots k^5_{p_5} e^5_{q_5}$ has been used and the free vector index m is contracted between the left- and right-moving factors in the holomorphic square. The parity-violating type IIA component with a B-field and four gravitons has been evaluated in [6].

Upon insertion into (4.30), the kinematic factors (4.31) and (4.32) give rise to the following low-energy limits for the five-graviton amplitudes:

$$\begin{aligned} M_5^{(1)}\Big|^{\alpha'^4}_{\rm IIB\ gravitons} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\frac{\kappa^5}{2^4 3}\mathcal{K}_5^{(0)}\Big|_{\zeta_3} \\ M_5^{(1)}\Big|^{\alpha'^4}_{\rm IIA\ gravitons} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\frac{\kappa^5}{2^4 3}\Big[\mathcal{K}_5^{(0)}\Big|_{\zeta_3} - \big|\epsilon^{me^1k^2e^2k^3e^3k^4e^4k^5e^5}\big|^2\Big] \,. \end{aligned} \tag{4.33}$$

According to (4.31), the R-symmetry violating type IIB components (e.g. four gravitons and one dilaton) carry an extra factor of $-\frac{1}{3}$.

5. Two-loop closed-string amplitudes

In this section we compute the low-energy limit of the two-loop five-point amplitude including its overall coefficient from first principles. This includes a recomputation of the four-point amplitude using the conventions of section 2. For previous two-loop four-point results see [49,50,51,7,8].

5.1. The amplitude prescription

The n-point two-loop amplitude prescription (4.1) is given by

$$M_n^{(2)} = \frac{1}{2}\kappa^n e^{2\lambda} \int \prod_{j=1}^{3} d^2\tau_j \int_{\Sigma_n} \left|\langle\langle \mathcal{N}^{(2)}(b,\mu_j) U^1(z_1)\cdots U^n(z_n)\rangle\rangle\right|^2 , \tag{5.1}$$

where the zero-mode regulator $\mathcal{N}^{(2)}$ is defined in (2.36), and the b-ghost insertion was specified in (4.2). At genus two, there are $(16,16)_d$ zero-modes of d_α and $(11,11)_s$ zero-modes of s^α. The latter are entirely saturated by the regulator through the factor $\mathcal{N}^{(2)} \to (s^1 d^1)^{11}(s^2 d^2)^{11}$, see the discussion below (4.3).

In presence of five vertex operators, it is easy to see that the total number of d_α zero-modes from the b-ghosts can be distributed as (p,q) such that $p+q$ is either 5 or 6. These two contributions can be separately computed using the zero-mode expansion

$$(d\gamma_{mnp}d)(z) \to (d^1\gamma_{mnp}d^1)\omega_1(z)\omega_1(z) + 2(d^1\gamma_{mnp}d^2)\omega_1(z)\omega_2(z) + (d^2\gamma_{mnp}d^2)\omega_2(z)\omega_2(z)$$

and the general formulas (2.31) as follows

$$\int \prod_{j=1}^{3} d^2\tau_j \Big|(b,\mu_j)\Big|^2 = \Big(\frac{\alpha'}{2}\Big)^6 \frac{1}{(2\pi)^6 192^6} \int d^2\Omega \left| B_{(3,3)} + \big(\Pi^1_m B^m_{(2,3)} + \Pi^2_m B^m_{(3,2)}\big) + \cdots \right|^2 . \tag{5.2}$$

The shorthands for different b-ghost contributions are defined by[9]

$$\begin{aligned} B_{(3,3)} &\equiv \frac{1}{(\lambda\overline{\lambda})^6}\big[2(\overline{\lambda} r d^1 d^1)(\overline{\lambda} r d^1 d^2)(\overline{\lambda} r d^2 d^2)\big] \\ B^m_{(2,3)} &\equiv \Big(\frac{2}{\alpha'}\Big)\frac{96}{(\lambda\overline{\lambda})^5}\big[2(\overline{\lambda}\gamma^m d^1)(\overline{\lambda} r d^1 d^2)(\overline{\lambda} r d^2 d^2) - (\overline{\lambda}\gamma^m d^2)(\overline{\lambda} r d^1 d^1)(\overline{\lambda} r d^2 d^2)\big] \\ B^m_{(3,2)} &\equiv \Big(\frac{2}{\alpha'}\Big)\frac{96}{(\lambda\overline{\lambda})^5}\big[2(\overline{\lambda}\gamma^m d^2)(\overline{\lambda} r d^1 d^1)(\overline{\lambda} r d^1 d^2) - (\overline{\lambda}\gamma^m d^1)(\overline{\lambda} r d^1 d^1)(\overline{\lambda} r d^2 d^2)\big] \end{aligned} \tag{5.3}$$

with the convention that $(\overline{\lambda} r d^I d^J) \equiv (\overline{\lambda}\gamma^{mnp} r)(d^I\gamma_{mnp}d^J)$. Note that $B_{(3,3)} \to -B_{(3,3)}$ and $B^m_{(2,3)} \leftrightarrow -B^m_{(3,2)}$ under the interchange of zero-mode labels $d^1 \leftrightarrow d^2$. As indicated

[9] The $(5,5)_d$ zero-modes from the b-ghosts and the vertices can in principle be saturated by a b-ghost contribution $B^m_{(1,4)} \sim (d^2d^2)(d^1d^2)(\Pi^m d^2)$. However, the integration over the b-ghost insertions via (2.31) yields a τ_j integrand $\sim \epsilon_{i_1i_2i_3}\frac{\delta\Omega_{22}}{\delta\tau_{i_1}}\frac{\delta\Omega_{12}}{\delta\tau_{i_2}}\big(\Pi^m_1\frac{\delta\Omega_{12}}{\delta\tau_{i_3}} + \Pi^m_2\frac{\delta\Omega_{22}}{\delta\tau_{i_3}}\big)$ whose summands do not depend on all entries of the period matrix and which vanish upon contraction with the antisymmetric $\epsilon_{i_1i_2i_3}$. The same mechanism suppresses $\Pi^2_m B^m_{(2,3)}$ and $\Pi^1_m B^m_{(3,2)}$ from (5.2).

by the ellipsis in (5.2), two-loop amplitudes involving $n \geq 6$ closed-string states allow for additional b-ghost contributions with fewer zero-modes of d_α.

Since the vertex operators are independent of w^I_α, $\overline{w}^\alpha_I$ and s^α_I, the integration over their zero-modes can be performed at an early stage using (2.27),

$$\Big|\int \prod_{I=1}^{2} [dw^I][d\overline{w}^I][ds^I]\, e^{-(w^I\overline{w}^I)-(d^I s^I)}\Big|^2 = \Big(\frac{\alpha'}{2}\Big)^8 \frac{1}{(2\pi)^{32} 2^4 Z_2^{22}} \Big|\prod_{I=1}^{2} \frac{(\epsilon \cdot T \cdot d^I)}{(11!\, 5!)}\Big|^2 \,. \tag{5.4}$$

The tensor structure $(\epsilon \cdot T \cdot d^I)$ is captured by the operators $D_{(14,14)}$ and $D^m_{(14,13)}$ defined in (2.26). They allow to rewrite the amplitude (5.1) as

$$M_n^{(2)} = \Big(\frac{\alpha'}{2}\Big)^{14} \frac{\kappa^n e^{2\lambda}}{(2\pi)^{38} 2^5 192^6} \int \frac{d^2\Omega}{Z_2^{22}} \int_{\Sigma_n} \big|\langle\langle \mathcal{K}^{(2)}_{[d]}(z_1,\ldots,z_n)\rangle\rangle_{(3,2)}\big|^2 \Big\langle \prod_{j=1}^{n} e^{k^j \cdot x^j} \Big\rangle \,, \tag{5.5}$$

where

$$\mathcal{K}^{(2)}_{[d]}(z_1,\ldots,z_n) \equiv \big(D_{(14,14)} + \Pi^1_m D^m_{(13,14)} + \Pi^2_m D^m_{(14,13)} + \cdots\big) U^1(z_1)\cdots U^n(z_n) \,. \tag{5.6}$$

The ellipsis along with $\Pi^2_m D^m_{(14,13)}$ accounts for b-ghost zero-mode contributions which drop out from the subsequent four- and five-point computations.

5.1.1. Scalar and vector building blocks at genus two

We shall now evaluate (5.6) on the part of the vertex operators which contribute zero-modes d^1, d^2. One can show that

$$\begin{aligned} D_{(14,14)}(d^1 W_A)(d^1 W_B)(d^2 W_C)(d^2 W_D) &= 96^2 c_d^2\, T_{A,B|C,D}(\lambda,\overline{\lambda}) \\ D^m_{(13,14)}(d^1 W_A)(d^1 W_B)(d^1 W_C)(d^2 W_D)(d^2 W_E) &= \Big(\frac{2}{\alpha'}\Big) 96^2 c_d^2\, S^m_{A,B,C|D,E}(\lambda,\overline{\lambda}) \\ D^m_{(14,13)}(d^2 W_A)(d^2 W_B)(d^2 W_C)(d^1 W_D)(d^1 W_E) &= \Big(\frac{2}{\alpha'}\Big) 96^2 c_d^2\, S^m_{A,B,C|D,E}(\lambda,\overline{\lambda}) \end{aligned} \tag{5.7}$$

with multiparticle labels $A, B, \ldots$ and scalar building block

$$\begin{aligned} T_{A,B|C,D}(\lambda,\overline{\lambda}) \equiv \frac{2}{(\lambda\overline{\lambda})^6} &(\overline{\lambda}\gamma_{m_1 n_1 p_1} r)(\overline{\lambda}\gamma_{def} r)(\overline{\lambda}\gamma_{m_2 n_2 p_2} r)(\lambda\gamma^{m_1 def m_2}\lambda) \\ &\times (\lambda\gamma^{n_1} W_A)(\lambda\gamma^{p_1} W_B)(\lambda\gamma^{n_2} W_C)(\lambda\gamma^{p_2} W_D) \,. \end{aligned} \tag{5.8}$$

The two zero-mode patterns $(\overline{\lambda}\gamma^m d^1)(\overline{\lambda} r d^1 d^2)(\overline{\lambda} r d^2 d^2)$ and $(\overline{\lambda}\gamma^m d^2)(\overline{\lambda} r d^1 d^1)(\overline{\lambda} r d^2 d^2)$ in the b-ghost contribution $B^m_{(3,2)}$ given by (5.3) lead to distinct tensor structures $S^{(1)\,m}_{A,B,C|D,E}$ and $S^{(2)\,m}_{A,B,C|D,E}$ such that

$$S^m_{A,B,C|D,E} \equiv S^{(1)\,m}_{A,B,C|D,E} + S^{(2)\,m}_{A,B,C|D,E} \tag{5.9}$$

with

$$
\begin{aligned}
S^{(1)\,m}_{A,B,C|D,E}(\lambda,\overline{\lambda}) &\equiv -\frac{2}{(\lambda\overline{\lambda})^5}\,(\overline{\lambda}\gamma^m\gamma^{a_1}\lambda)(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\lambda\gamma^{a_2m_1n_1p_1m_2}\lambda)\\
&\qquad\times(W_A\gamma_{a_1a_2a_3}W_B)(\lambda\gamma^{a_3}W_C)(\lambda\gamma^{n_2}W_D)(\lambda\gamma^{p_2}W_E)\,, \qquad (5.10)\\
S^{(2)\,m}_{A,B,C|D,E}(\lambda,\overline{\lambda}) &\equiv \frac{96}{(\lambda\overline{\lambda})^5}\,(\overline{\lambda}\gamma^m\gamma^{b_1}\lambda)(\overline{\lambda}\gamma_{a_1a_2a_3}r)(\overline{\lambda}\gamma_{b_1b_2b_3}r)\\
&\qquad\times(\lambda\gamma^{a_1}W_A)(\lambda\gamma^{a_2}W_B)(\lambda\gamma^{a_3}W_C)(\lambda\gamma^{b_2}W_D)(\lambda\gamma^{b_3}W_E)\,. \qquad (5.11)
\end{aligned}
$$

Note that the integrals of $D^m_{(13,14)}$ and $D^m_{(14,13)}$ give rise to the same kinematic structure $S^m_{A,B,C|D,E}$ because $D^m_{(13,14)} \leftrightarrow D^m_{(14,13)}$ under the interchange of zero-modes $d^1_\alpha \leftrightarrow d^2_\alpha$.

5.1.2. Kinematic symmetry properties at genus two

The above definitions in (5.7) manifest the symmetry properties

$$
T_{A,B|C,D}(\lambda,\overline{\lambda}) = T_{(A,B)|(C,D)}(\lambda,\overline{\lambda})\ , \qquad S^{(j)\,m}_{A,B,C|D,E}(\lambda,\overline{\lambda}) = S^{(j)\,m}_{(A,B,C)|(D,E)}(\lambda,\overline{\lambda}) \qquad (5.12)
$$

with $j=1,2$. As demonstrated in the appendix C, gamma-matrix manipulations and the pure spinor constraint imply that the kinematic factor (5.8) can be rewritten as

$$
T_{A,B|C,D}(\lambda,\overline{\lambda}) = -\frac{192}{(\lambda\overline{\lambda})^4}(\overline{\lambda}\gamma_{amn}r)(r\gamma_{apq}r)(\lambda\gamma^m W_A)(\lambda\gamma^n W_B)(\lambda\gamma^p W_C)(\lambda\gamma^q W_D) \qquad (5.13)
$$

and satisfies the Jacobi identity,

$$
T_{A,B|C,D}(\lambda,\overline{\lambda}) + T_{A,D|B,C}(\lambda,\overline{\lambda}) + T_{A,C|D,B}(\lambda,\overline{\lambda}) = 0\,. \qquad (5.14)
$$

The symmetry (5.14) assembles the holomorphic one-forms in the antisymmetric combinations $\Delta_{ij} = \epsilon^{IJ}\omega_I(z_i)\omega_J(z_j)$,

$$
\begin{aligned}
&D_{(14,14)}(dW_A)(z_A)(dW_B)(z_B)(dW_C)(z_C)(dW_D)(z_D)\\
&= 96^2 c_d^2\Big(T_{A,B|C,D}(\lambda,\overline{\lambda})\big[\omega_1(z_A)\omega_1(z_B)\omega_2(z_C)\omega_2(z_D) + \omega_2(z_A)\omega_2(z_B)\omega_1(z_C)\omega_1(z_D)\big]\\
&\quad + T_{A,C|B,D}(\lambda,\overline{\lambda})\big[\omega_1(z_A)\omega_1(z_C)\omega_2(z_B)\omega_2(z_D) + \omega_2(z_A)\omega_2(z_C)\omega_1(z_B)\omega_1(z_D)\big]\\
&\quad + T_{A,D|B,C}(\lambda,\overline{\lambda})\big[\omega_1(z_A)\omega_1(z_D)\omega_2(z_B)\omega_2(z_C) + \omega_2(z_A)\omega_2(z_D)\omega_1(z_B)\omega_1(z_C)\big]\Big)\\
&= -96^2 c_d^2\big[T_{A,B|C,D}(\lambda,\overline{\lambda})\Delta_{DA}\Delta_{BC} + T_{D,A|B,C}(\lambda,\overline{\lambda})\Delta_{AB}\Delta_{CD}\big]\ . \qquad (5.15)
\end{aligned}
$$

As will become clear later, the appearance of Δ_{ij} in (5.15) is a crucial requirement for modular invariance of the amplitude.

Furthermore, it is shown in appendix C that $S^{(1)m}_{A,B,C|D,E}$ can be eliminated in favor of $S^{(2)m}_{A,B,C|D,E}$ to yield

$$S^m_{A,B,C|D,E}(\lambda,\overline{\lambda}) = 2S^{(2)\,m}_{A,B,C|D,E}(\lambda,\overline{\lambda})+S^{(2)\,m}_{A,D,E|B,C}(\lambda,\overline{\lambda})+S^{(2)\,m}_{B,D,E|A,C}(\lambda,\overline{\lambda})+S^{(2)\,m}_{C,D,E|A,B}(\lambda,\overline{\lambda}). \tag{5.16}$$

Together with the ten-term identity[10]

$$S^{(2)\,m}_{A,B,C|D,E}(\lambda,\overline{\lambda}) + (D,E|A,B,C,D,E) = 0\,, \tag{5.17}$$

one can show that (5.16) implies a vector generalization of the scalar Jacobi identity (5.14)

$$S^m_{A,B,C|D,E}(\lambda,\overline{\lambda}) = S^m_{C,D,E|A,B}(\lambda,\overline{\lambda}) + S^m_{B,D,E|A,C}(\lambda,\overline{\lambda}) + S^m_{A,D,E|B,C}(\lambda,\overline{\lambda})\ . \tag{5.18}$$

This is instrumental to identify Δ_{ij} in the following permutation sum:

$$\begin{aligned}
&(\Pi^1_m D^m_{(13,14)} + \Pi^2_m D^m_{(14,13)})(dW_A)(z_A)(dW_B)(z_B)(dW_C)(z_C)(dW_D)(z_D)(dW_E)(z_E)\\
&= 96^2 c_d^2\Big(\frac{2}{\alpha'}\Big) S^m_{A,B,C|D,E}(\lambda,\overline{\lambda})\big[\Pi^1_m\omega_1(z_A)\omega_1(z_B)\omega_1(z_C)\omega_2(z_D)\omega_2(z_E)\\
&\qquad\qquad + \Pi^2_m\omega_2(z_A)\omega_2(z_B)\omega_2(z_C)\omega_1(z_D)\omega_1(z_E)\big] + (D,E|A,B,C,D,E)\\
&= 96^2 c_d^2\Big(\frac{2}{\alpha'}\Big) S^m_{A,B,C|D,E}(\lambda,\overline{\lambda}) \sum_{I=1}^{2} \Delta_{EA}\omega_I(z_B)\Delta_{CD}\Pi^I_m + \text{cyc}(A,B,C,D,E)\ .
\end{aligned} \tag{5.19}$$

Applying the Theorem 1 of [9] to the expressions (5.8) and (5.9)

$$T_{A,B|C,D}(\lambda,\overline{\lambda}) \equiv \frac{1}{(\lambda\overline{\lambda})^3} T_{A,B|C,D}, \quad S^m_{A,B,C|D,E}(\lambda,\overline{\lambda}) \equiv \frac{1}{(\lambda\overline{\lambda})^2} S^m_{A,B,C|D,E}, \tag{5.20}$$

leads to

$$\langle T_{A,B|C,D}(\lambda,\overline{\lambda})\rangle_{(3,2)} = \langle T_{A,B|C,D}\rangle_{(0,2)}, \quad \langle S^m_{A,B,C|D,E}(\lambda,\overline{\lambda})\rangle_{(3,2)} = 8\langle S^m_{A,B,C|D,E}\rangle_{(0,2)}\,, \tag{5.21}$$

where the factor 8 comes from $\langle\ldots\rangle_{(1,2)} = 8\langle\ldots\rangle_{(0,2)}$. As we will see, the vector building block contributing to two-loop amplitudes with five or more particles is

$$T^m_{A,B,C|D,E} \equiv A^m_A T_{B,C|D,E} + A^m_B T_{A,C|D,E} + A^m_C T_{A,B|D,E} + 8S^m_{A,B,C|D,E}\ , \tag{5.22}$$

where the factor of 8 is due to the use of (5.21). By (5.14) and (5.18), it obeys the same symmetry properties as $S^m_{A,B,C|D,E}(\lambda,\overline{\lambda})$,

$$T^m_{A,B,C|D,E} = T^m_{(A,B,C)|(D,E)}\,, \quad T^m_{B,D,E|A,C} = T^m_{A,B,C|D,E} - T^m_{A,D,E|B,C} - T^m_{C,D,E|A,B}\,. \tag{5.23}$$

Hence, the manipulations shown in (5.19) carry over to $S^m_{A,B,C|D,E}(\lambda,\overline{\lambda}) \to T^m_{A,B,C|D,E}$.

[10] The identity (5.17) was checked to hold for its bosonic (gluon) components [23], and it is believed to hold at the superfield level using similar manipulations seen in the appendix C.

5.2. The four-point amplitude

According to the general formula (5.5) the four-point amplitude at two loops is given by

$$M_4^{(2)} = \left(\frac{\alpha'}{2}\right)^{14} \frac{\kappa^4 e^{2\lambda}}{(2\pi)^{38} 2^5 192^6} \int \frac{d^2\Omega}{Z_2^{22}} \int_{\Sigma_4} |\langle D_{(14,14)} U_1 U_2 U_3 U_4 \rangle_{(3,2)}|^2 \Big\langle \prod_{j=1}^{4} e^{k^j\cdot x^j} \Big\rangle\,, \qquad (5.24)$$

since the four vertices cannot provide enough d_α zero-modes to saturate the terms with $D^m_{(13,14)}$ and $D^m_{(14,13)}$ in (5.6) [2]. Using the formula (5.7), the Jacobi identity (5.14) and the definitions (5.21) it is straightforward to verify that (see (5.15))

$$\langle D_{(14,14)} U_1 U_2 U_3 U_4 \rangle_{(3,2)} = -96^2 c_d^2 \left(\frac{\alpha'}{2}\right)^4 \big[\langle T_{1,2|3,4}\rangle_{(0,2)} \Delta_{41}\Delta_{23} + \langle T_{1,4|2,3}\rangle_{(0,2)} \Delta_{12}\Delta_{34}\big]\,. \qquad (5.25)$$

Together with (2.40) and (5.21), this implies that

$$M_4^{(2)} = (2\pi)^{10}\delta^{10}(k) \left(\frac{\alpha'}{2}\right)^5 \frac{\kappa^4 e^{2\lambda}}{2^{45}\, 3^6\, 5^2 \pi^6} \int \frac{d^2\Omega}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma_4} |\langle \mathcal{K}^{(2)}(z_1,\ldots,z_4)\rangle|^2\, \mathcal{I}_4^{(2)}\,, \qquad (5.26)$$

where

$$\langle \mathcal{K}^{(2)}(z_1,\ldots,z_4)\rangle \equiv \langle T_{1,2|3,4}\rangle \Delta_{41}\Delta_{23} + \langle T_{1,4|2,3}\rangle \Delta_{12}\Delta_{34}\,. \qquad (5.27)$$

In absence of singularities $|z_{ij}|^{-2}$, using Riemann's bilinear identity (2.29) in the form of

$$\int_{\Sigma_4} \Delta_{12}\Delta_{34}\overline{\Delta}_{12}\overline{\Delta}_{34} = 2^6 (\det \operatorname{Im}\Omega)^2\,, \quad \int_{\Sigma_4} \Delta_{12}\Delta_{34}\overline{\Delta}_{41}\overline{\Delta}_{23} = 2^5 (\det \operatorname{Im}\Omega)^2\,, \qquad (5.28)$$

leads to the following low-energy limit

$$\int_{\Sigma_4} |\langle \mathcal{K}^{(2)}(z_1,\ldots,z_4)\rangle|^2\, \mathcal{I}_4^{(2)} = 2^5 (\det \operatorname{Im}\Omega)^2\, \mathcal{K}_4^{(2)} + \mathcal{O}({\alpha'}^2)\,, \qquad (5.29)$$

where (note $[\mathcal{K}_4^{(2)}] = -12$)

$$\mathcal{K}_4^{(2)} = |\langle T_{1,2|3,4}\rangle|^2 + |\langle T_{1,4|2,3}\rangle|^2 + |\langle T_{1,3|4,2}\rangle|^2\,. \qquad (5.30)$$

Finally, using the volume of the genus-two moduli space $\int d\mu_2 = 2^2\pi^3/(3^3\, 5)$, one arrives at the following low-energy limit

$$\begin{aligned} M_4^{(2)} &= (2\pi)^{10}\delta^{10}(k) \left(\frac{\alpha'}{2}\right)^5 \frac{\kappa^4 e^{2\lambda}}{2^{38}\, 3^9\, 5^3 \pi^3}\, \mathcal{K}_4^{(2)} + \mathcal{O}({\alpha'}^6) \\ &= (2\pi)^{10}\delta^{10}(k) \left(\frac{\alpha'}{2}\right)^5 \frac{\kappa^4 e^{2\lambda}}{2^{10}\, 3^3\, 5\, \pi^3} (s_{12}^2 + s_{13}^2 + s_{14}^2)\, \mathcal{K}_4^{(1)} + \mathcal{O}({\alpha'}^6)\,. \end{aligned} \qquad (5.31)$$

In the second line we used the BRST cohomology manipulation [8,34][11]

$$\langle T_{1,2|3,4}\rangle = 2^{14}3^3 5\, s_{12}\langle V_1 T_{2,3,4}\rangle \tag{5.32}$$

together with the definition (4.17) of the one-loop kinematic factor $\mathcal{K}_4^{(1)}$ (which in turn agrees with the tree-level kinematic factor (3.13)).

An alternative presentation of the four-point two-loop amplitude follows by plugging the result (5.32) in (5.26) and using the definition [7]

$$\mathcal{Y}(z_1,\ldots,z_4) \equiv s_{12}\Delta_{41}\Delta_{23} + s_{14}\Delta_{12}\Delta_{34} \tag{5.33}$$

to obtain

$$M_4^{(2)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^5 \frac{\kappa^4 e^{2\lambda}}{2^{17}\pi^6}\,\mathcal{K}_4^{(1)}\int \frac{d^2\Omega}{(\det \mathrm{Im}\,\Omega)^5}\int_{\Sigma_4} |\mathcal{Y}(z_1,\ldots,z_4)|^2 \mathcal{I}_4^{(2)}\,. \tag{5.34}$$

In the low-energy limit where $\mathcal{I}_4^{(2)} \to 1$, using [7]

$$\int_{\Sigma_4} |\mathcal{Y}(z_1,\ldots,z_4)|^2 = 2^5(s_{12}^2 + s_{13}^2 + s_{14}^2)(\det \mathrm{Im}\,\Omega)^2 \tag{5.35}$$

and $\int d\mu_2 = 2^2\pi^3/(3^3\, 5)$ leads to the same answer (5.31).

5.3. The five-point amplitude

The five-point amplitude following from the general formula (5.5) is given by

$$M_5^{(2)} = \Big(\frac{\alpha'}{2}\Big)^{14} \frac{\kappa^5 e^{2\lambda}}{(2\pi)^{38}\, 2^5\, 192^6}\int \frac{d^2\Omega}{Z_2^{22}}\int_{\Sigma_5} |\langle\langle \mathcal{K}_{[d]}^{(2)}(z_1,\ldots,z_5)\rangle\rangle_{(3,2)}|^2 \Big\langle \prod_{j=1}^5 e^{k^j\cdot x^j}\Big\rangle\,, \tag{5.36}$$

where

$$\mathcal{K}_{[d]}^{(2)}(z_1,\ldots,z_5) = \big(D_{(14,14)} + \Pi_m^1 D^m_{(13,14)} + \Pi_m^2 D^m_{(14,13)}\big) U_1(z_1)U_2(z_2)\cdots U_5(z_5)\ . \tag{5.37}$$

For the first term in (5.37), the external vertices must contribute four $d_\alpha(z)$ variables to saturate the remaining $(2,2)_d$ zero-modes required by the $D_{(14,14)}$ integration. This admits one OPE (2.43) resulting in a two-particle superfield W^α_{ij} from (2.44) accompanied by the singular function $\eta_{ij} \sim z_{ij}^{-1}$ defined in (2.8).

[11] The normalization of $T_{1,2|3,4}$ here is two times bigger than in [8], see definition (5.8).

However, the OPE (2.43) only determines the residue of the simple pole z_{ij}^{-1} and allows for two inequivalent functions of the worldsheet positions; either $\partial_i G_{ij}\omega_I(z_j)$ or $-\partial_j G_{ij}\omega_I(z_i)$. Their difference is regular in z_{ij} and drops out from the low-energy behavior of the amplitude due to the factor of $\delta^2(z_i - z_j)$ in (4.27). Since the ambiguity does not affect the subsequent low-energy analysis, we will use the notation $(dW_{ij})\eta_{ij}$ to leave the subtlety in the exact dependence on z_i, z_j undetermined.

Another possible obstruction to extend the current analysis beyond the low-energy limit might stem from OPE singularities between the b-ghost and the vertex operators (see for instance [52,53]). By arguments similar to [9], these might affect the two-loop five-point amplitude at order D^4R^5.

Similarly, for the last two terms in (5.37), the vertices must provide five d_α variables to saturate either $(2,3)_d$ or $(3,2)_d$ zero-modes. Together with the contributions from the previous paragraph, we arrive at

$$
\begin{aligned}
U_1U_2U_3U_4U_5\big|_{d^4} &= \Big(\frac{\alpha'}{2}\Big)^5 \big[(dW_{12})(dW_3)(dW_4)(dW_5)\,\eta_{12} + (1,2|1,2,3,4,5)\big] \\
&\quad + \Big(\frac{\alpha'}{2}\Big)^4 \sum_{I=1}^{2} \Pi^m_I \omega_I(z_1) A^1_m (dW^2)(dW^3)(dW^4)(dW^5) + (1 \leftrightarrow 2,3,4,5)\,, \\
U_1U_2U_3U_4U_5\big|_{d^5} &= \Big(\frac{\alpha'}{2}\Big)^5 (dW_1)(dW_2)(dW_3)(dW_4)(dW_5)\,.
\end{aligned}
\tag{5.38}
$$

Using the formulas in (5.7) a long but straightforward calculation leads to

$$
\langle\langle \mathcal{K}^{(2)}_{[d]}(z_1,\ldots,z_5)\rangle\rangle_{(3,2)} = \Big(\frac{\alpha'}{2}\Big)^4 96^2 c_d^2\,\langle \mathcal{K}^{(2)}(z_1,\ldots,z_5)\rangle_{(0,2)}\,, \tag{5.39}
$$

where

$$
\begin{aligned}
\mathcal{K}^{(2)}(z_1,\ldots,z_5) &\equiv \big[T^m_{1,2,3|4,5} \sum_{I=1}^{2} \Delta_{51}\omega_I(z_2)\Delta_{34}\Pi^I_m + \mathrm{cyc}(12345)\big] \\
&\quad + \Big(\frac{\alpha'}{2}\Big)\Big[\eta_{12}(T_{12,3|4,5}\Delta_{24}\Delta_{35} + T_{12,4|3,5}\Delta_{23}\Delta_{45}) + (1,2|1,2,3,4,5)\Big]\,,
\end{aligned}
\tag{5.40}
$$

and $T^m_{1,2,3|4,5}$ is defined in (5.22). As detailed in (5.19), the symmetry property (5.23) of $T^m_{1,2,3|4,5}$ is crucial to obtain the first line of (5.40) in terms of two factors of Δ_{ij}. After discarding $\Pi^I_m \to 0$, (5.40) is the low-energy regime of the open-string worldsheet integrand for the five-point two-loop amplitude.

Collecting the above results, the low-energy regime of the two-loop amplitude reads

$$\begin{aligned} M_5^{(2)} &= \Big(\frac{\alpha'}{2}\Big)^6 \frac{\kappa^5 e^{2\lambda}}{2^{37}\,3^2\,\pi^6} \int \frac{d^2\Omega}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma_5} \big|\langle \mathcal{K}^{(2)}(z_1,\ldots,z_5)\rangle_{(0,2)}\big|^2 \Big\langle \prod_{j=1}^5 e^{k^j\cdot x^j}\Big\rangle + \mathcal{O}(\alpha'^7) \\ &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^5 \frac{\kappa^5 e^{2\lambda}}{2^{45}\,3^6\,5^2\,\pi^6} \int \frac{d^2\Omega}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma_5} \big|\langle \mathcal{K}^{(2)}(z_1,\ldots,z_5)\rangle\big|^2 \mathcal{I}_5^{(2)} + \mathcal{O}(\alpha'^7)\,, \end{aligned} \tag{5.41}$$

where in the second line we used (2.40).

5.3.1. The low-energy limit

The low-energy limit of the genus-two integral in (5.41) can be extracted along the same lines as done at genus one. First of all, (2.11) allows to perform contractions among left- and right-moving zero-modes of Π^m which can be integrated over Σ_5 using

$$\begin{aligned} \int_{\Sigma_5} \Delta_{12}\sum_{I=1}^2 \Pi^I_m \omega_I(z_3)\Delta_{45} \times \overline{\Delta}_{12}\sum_{J=1}^2 \overline{\Pi}^J_n \overline{\omega}_J(\overline{z}_3)\overline{\Delta}_{45} &= -2^8\pi\,\Big(\frac{\alpha'}{2}\Big)\eta_{mn}(\det\operatorname{Im}\Omega)^2 \\ \int_{\Sigma_5} \Delta_{12}\sum_{I=1}^2 \Pi^I_m \omega_I(z_3)\Delta_{45} \times \overline{\Delta}_{34}\sum_{J=1}^2 \overline{\Pi}^J_n \overline{\omega}_J(\overline{z}_5)\overline{\Delta}_{12} &= 2^7\pi\,\Big(\frac{\alpha'}{2}\Big)\eta_{mn}(\det\operatorname{Im}\Omega)^2 \\ \int_{\Sigma_5} \Delta_{12}\sum_{I=1}^2 \Pi^I_m \omega_I(z_3)\Delta_{45} \times \overline{\Delta}_{23}\sum_{J=1}^2 \overline{\Pi}^J_n \overline{\omega}_J(\overline{z}_4)\overline{\Delta}_{51} &= -2^6\pi\,\Big(\frac{\alpha'}{2}\Big)\eta_{mn}(\det\operatorname{Im}\Omega)^2 \end{aligned} \tag{5.42}$$

and cyclic permutations. Then, the subset of the terms $\sim \eta_{ij}\overline{\eta}_{pq}$ in (5.41) with "diagonal" labels $i=p$ and $j=q$ contributes according to (4.27), resulting in ten permutations of

$$\int_{\Sigma_4} \big|\langle T_{12,3|4,5}\rangle\Delta_{24}\Delta_{35} + \langle T_{12,4|3,5}\rangle\Delta_{23}\Delta_{45}\big|^2 = 2^5(\det\operatorname{Im}\Omega)^2\big[|\langle T_{12,3|4,5}\rangle|^2 + \operatorname{cyc}(3,4,5)\big]\,,$$

where the integrals are identical to the four-point case (5.29). Permutations of $\eta_{12}\overline{\eta}_{13}$ or $\eta_{12}\overline{\eta}_{34}$ from the holomorphic square in (5.40) do not contribute to the low-energy limit (5.41).

By assembling the two sectors with and without contractions between left- and right-movers, one can show that the leading-order terms of the five-point two-loop amplitude (5.41) are given by

$$\int_{\Sigma_5} |\langle \mathcal{K}^{(2)}(z_1,\ldots,z_5)\rangle|^2 \mathcal{I}_5^{(2)} = 2^6\pi\Big(\frac{\alpha'}{2}\Big)(\det\operatorname{Im}\Omega)^2\,\mathcal{K}_5^{(2)} + \mathcal{O}(\alpha'^2)\,, \tag{5.43}$$

with kinematic factor (note $[\mathcal{K}_5^{(2)}]=-12$)

$$\mathcal{K}_5^{(2)} \equiv \frac{\big|\langle T_{12,3|4,5}\rangle\big|^2}{s_{12}} + \frac{\big|\langle T_{12,4|3,5}\rangle\big|^2}{s_{12}} + \frac{\big|\langle T_{12,5|3,4}\rangle\big|^2}{s_{12}} + \big|\langle T^m_{3,4,5|1,2}\rangle\big|^2 + (1,2|1,2,3,4,5)\,. \tag{5.44}$$

Hence, using $\int d\mu_2 = 2^2\pi^3/(3^3\,5)$ implies the following low-energy limit of (5.41),

$$M_5^{(2)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^6 \frac{\kappa^5 e^{2\lambda}}{2^{37}\,3^9\,5^3\,\pi^2}\,\mathcal{K}_5^{(2)} + \mathcal{O}(\alpha'^7)\ . \tag{5.45}$$

5.3.2. Components in type IIB and type IIA

A long and tedious calculation [23] identifies the type IIB components of the two-loop kinematic factor (5.44) with the α'-correction $\sim \zeta_5$ of the five-point tree-level amplitude (3.18) and (3.19):

$$\mathcal{K}_5^{(2)}\big|_{\rm IIB} = -2^{28}\,3^6\,5^2\Big(\frac{\alpha'}{2}\Big)^{-5}\mathcal{K}_5^{(0)}\Big|_{\zeta_5} \times \begin{cases} 1 & : \ \text{five gravitons} \\ -\frac{3}{5} & : \ \text{four gravitons, one dilaton} \end{cases} \tag{5.46}$$

Similar to the kinematic factor (4.31) in the one-loop low-energy limit, the relative coefficient to the tree-amplitude depends on the total R-symmetry charge of the external states, in lines with S-duality. The components considered in (5.46) extend to a variety of further state combinations of alike R-symmetry charges by linearized supersymmetry.

Similar to the analogous one-loop result (4.32), type IIA components involve additional tensor structures as compared to the $A^{\rm YM}$ bilinears in the tree-amplitude,

$$\mathcal{K}_5^{(2)}\big|^{5\text{ gravitons}}_{\rm IIA} = -2^{28}\,3^6\,5^2\Big(\frac{\alpha'}{2}\Big)^{-5}\Big[\mathcal{K}_5^{(0)}\Big|_{\zeta_5} - \frac{1}{2}\sum_{1\le i<j}^{5} s_{ij}^2\big|\epsilon^{me^1k^2e^2k^3e^3k^4e^4k^5e^5}\big|^2\Big]\ , \tag{5.47}$$

where $\epsilon^{me^1k^2e^2k^3e^3k^4e^4k^5e^5} \equiv \epsilon_{10}^{mnp_2q_2\ldots p_5q_5} e^1_n k^2_{p_2} e^2_{q_2}\ldots k^5_{p_5}e^5_{q_5}$.

Upon insertion into (5.45), the kinematic factors (5.46) and (5.47) give rise to the following low-energy limits for the five-graviton amplitudes:

$$\begin{aligned} M_5^{(2)}\big|^{\alpha'^6}_{\rm IIB\ gravitons} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\frac{\kappa^5 e^{2\lambda}}{2^9\,3^3\,5\,\pi^2}\mathcal{K}_5^{(0)}\Big|_{\zeta_5} \\ M_5^{(2)}\big|^{\alpha'^6}_{\rm IIA\ gravitons} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\frac{\kappa^5 e^{2\lambda}}{2^9\,3^3\,5\,\pi^2}\Big[\mathcal{K}_5^{(0)}\Big|_{\zeta_5} - \frac{1}{2}\sum_{1\le i<j}^{5} s_{ij}^2\big|\epsilon^{me^1k^2e^2k^3e^3k^4e^4k^5e^5}\big|^2\Big] \end{aligned} \tag{5.48}$$

According to (5.46), the R-symmetry violating type IIB components (e.g. four gravitons and one dilaton) carry an extra factor of $-\frac{3}{5}$.

6. S-duality properties

In this section we are going to show that the type IIB five-point amplitudes computed with the non-minimal pure spinor formalism agree with expectations based on S-duality.

6.1. Review of four-point S-duality

In the string frame, the $SL(2,\mathbb{Z})$-duality prediction for the perturbative four-graviton type IIB effective action is given by [54,3,55]

$$S_{\rm IIB}^{\rm 4pt} = \int d^{10}x\sqrt{-g}\,\big[R^4(2\zeta_3 e^{-2\phi}+4\zeta_2)+D^4R^4(2\zeta_5 e^{-2\phi}+\frac{8}{3}\zeta_4 e^{2\phi}) + D^6R^4(4\zeta_3^2 e^{-2\phi}+8\zeta_2\zeta_3+\frac{48}{5}\zeta_2^2 e^{2\phi}+\frac{8}{9}\zeta_6 e^{4\phi})+\cdots\big], \tag{6.1}$$

where the ellipsis refers to terms of higher order $D^{\geq 8}R^4$. A dilaton dependence of the form $e^{(2g-2)\phi}$ is associated with the g-loop order in string perturbation theory. The tensor structure of the covariant derivatives D and Riemann curvature tensors R suppressed in the shorthands R^4, D^4R^4 and D^6R^4 will not be important in the following. The coefficients of the R^4 and D^4R^4 interactions can be identified with the zero-modes of the non-holomorphic Eisenstein series

$$E_{3/2}(\Phi,\overline{\Phi}) \equiv 2\zeta_3 e^{-3\phi/2}+4\zeta_2 e^{\phi/2}+\cdots \tag{6.2}$$

$$E_{5/2}(\Phi,\overline{\Phi}) \equiv 2\zeta_5 e^{-5\phi/2}+\frac{8}{3}\zeta_4 e^{3\phi/2}+\cdots \tag{6.3}$$

depending on the complex axio-dilaton field $\Phi\equiv C_0+ie^{-\phi}$. A relative factor of $e^{\pm\phi/2}$ stems from the transformation between string frame and Einstein frame. The Fourier modes in the ellipsis of (6.2) and (6.3) describe the non-perturbative completion of the type IIB action [54,3] and ensure modular invariance w.r.t. Φ. The prefactor of the D^6R^4 operator in (6.1) was firstly predicted in [55] and descends from a modular-invariant function which is made explicit in [56].

The four-point amplitudes reviewed in the previous sections exhibit the following low-energy behavior (in both type IIB and type IIA theory):

$$\begin{aligned} M_4^{(0)} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\kappa^4 e^{-2\lambda}\,2\pi\mathcal{K}_4^{(0)}\big(\frac{3}{\sigma_3}+2\zeta_3+\zeta_5\sigma_2+\frac{2}{3}\zeta_3^2\sigma_3+\cdots\big)\\ M_4^{(1)} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\frac{\kappa^4}{2^4\,3\pi}\mathcal{K}_4^{(0)}\big(1+\frac{\zeta_3}{3}\sigma_3+\cdots\big)\\ M_4^{(2)} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^3\frac{\kappa^4 e^{2\lambda}}{2^{10}\,3^3\,5\pi^3}\,\mathcal{K}_4^{(0)}\big(\sigma_2+\cdots\big) \end{aligned} \tag{6.4}$$

The loop- and α'-orders are in one-to-one correspondence with the curvature couplings $e^{(2g-2)\phi}D^{2k}R^4$ in the action (6.1). Matching the ratio of the R^4 interactions $\sim \zeta_3$ and $\sim \zeta_2$ with the values computed in (6.4) relates the coupling constants e^{ϕ} and e^{λ},

$$\frac{e^{2\phi}\pi^2}{3\zeta_3} = \frac{e^{2\lambda}}{2^6\, 3\pi^2\zeta_3} \to e^{2\lambda} = 2^6\pi^4\, e^{2\phi} \ . \tag{6.5}$$

Furthermore, one can verify using the conversion factor (6.5) that the ratio of all the interactions match between their predicted values in the action (6.1) and the explicit amplitude computations summarized in (6.4). The first perturbative verification of the expressions in (6.1) was achieved in [54,41] for genus one, in [49,7,57] for genus two and [9] for genus three.

6.2. S-duality at five-points for graviton couplings

We will now check if the above ratios predicted for the four-point amplitudes at different loop orders also hold for their corresponding five-point amplitudes at one- and two-loops. The extension of the type IIB effective action (6.1) beyond the four-point level complements the four-curvature corrections $D^{2k}R^4$ by a tail of operators[12] $D^{2(k-l)}R^{4+l}$ with higher powers of curvature $l = 1, 2, \ldots, k$ required by non-linear supersymmetry.

The result for the two-loop five-point amplitude confirms that the five-field completion $(D^4R^4 + D^2R^5)$ is accompanied uniformly by the zero-modes of $E_{5/2}$ given in (6.3). Similarly, the compatibility of the $E_{3/2}R^4$ interaction with five-point amplitudes was verified through the one-loop analysis in [6]. These checks are based on the α'-expansion of the five-point IIB amplitudes at tree-level, one- and two-loop computed in the previous sections,

$$\begin{aligned}
M_5^{(0)} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\,\kappa^5 e^{-2\lambda}(2\pi)^2\mathcal{K}_5^{(0)} \\
M_5^{(1)}\big|^{\alpha'^4}_{\rm IIB} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\frac{\kappa^5}{2^4 3}\,\mathcal{K}_5^{(0)}\Big|_{\zeta_3} \times \begin{cases} 1 & : \ \text{five gravitons} \\ -\frac{1}{3} & : \ \text{four gravitons, one dilaton} \end{cases} \\
M_5^{(2)}\big|^{\alpha'^6}_{\rm IIB} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\frac{\kappa^5 e^{2\lambda}}{2^9\, 3^3\, 5\, \pi^2}\mathcal{K}_5^{(0)}\Big|_{\zeta_5} \times \begin{cases} 1 & : \ \text{five gravitons} \\ -\frac{3}{5} & : \ \text{four gravitons, one dilaton} \end{cases} ,
\end{aligned} \tag{6.6}$$

[12] In addition, novel couplings of the form $D^{2k}R^{\geq 5}$ without a four-field representative in their supersymmetric completion might arise, e.g. the D^6R^5 interaction identified at one-loop [6].

where the tree-level factor $\mathcal{K}_5^{(0)}$ is given by (3.19) [36]. Hence, the ratios of the corresponding five-point interactions at one-loop are easily checked to agree with the perturbative terms in the Eisenstein series (6.2) and (6.3),

$$\frac{M_5^{(1)}}{M_5^{(0)}}\Big|^{\alpha'^4}_{\text{IIB gravitons}} = \frac{e^{2\lambda}}{2^6\, 3\, \pi^2 \zeta_3} = \frac{2e^{2\phi}\zeta_2}{\zeta_3} \ , \tag{6.7}$$

and similarly at two-loops (recall that $\zeta_2 = \frac{\pi^2}{6}$ and $\zeta_4 = \frac{\pi^4}{90}$),

$$\frac{M_5^{(2)}}{M_5^{(0)}}\Big|^{\alpha'^6}_{\text{IIB gravitons}} = \frac{e^{4\lambda}}{2^{11}\, 3^3\, 5\pi^4 \zeta_5} = \frac{4e^{4\phi}\zeta_4}{3\zeta_5} \ . \tag{6.8}$$

By modular invariance of the Eisenstein series, (6.7) and (6.8) confirm S-duality at the five-point level.

6.3. S-duality at five-points for dilaton couplings

The ratios of tree-level and loop-amplitudes seen in (6.6) depend on the external type IIB states, i.e. trading one of the five gravitons for a dilaton introduces additional factors of $-\frac{1}{3}$ and $-\frac{3}{5}$ into the comparison of low-energy limits. These numbers have a natural explanation from the Einstein frame presentation of the leading terms in (6.1),

$$R^4 E_{3/2}(\Phi, \overline{\Phi}) + (D^4 R^4 + D^2 R^5) E_{5/2}(\Phi, \overline{\Phi}) + \cdots \ , \tag{6.9}$$

see (6.2) and (6.3) for their perturbative contributions.

Processes which violate the R-symmetry of type IIB supergravity (such as the scattering of four gravitons and one dilaton) are associated with operators which transform with modular weight under S-duality [10]. Hence, by modular invariance of the type IIB action, they must accompanied by modular forms of opposite weights. The latter can be obtained from modular invariant functions such as E_s by acting with the modular covariant derivative

$$\mathcal{D}: \ e^{q\phi} \to q \cdot e^{q\phi} \ . \tag{6.10}$$

The modular forms obtained from $E_{3/2}$ and $E_{5/2}$ are characterized by the following perturbative terms (with Fourier-modes in the ellipsis):

$$\mathcal{D}E_{3/2}(\Phi, \overline{\Phi}) = \Big(-\frac{3}{2}\Big) 2\zeta_3 e^{-3\phi/2} + \Big(\frac{1}{2}\Big) 4\zeta_2 e^{\phi/2} + \cdots \tag{6.11}$$

$$\mathcal{D}E_{5/2}(\Phi, \overline{\Phi}) = \Big(-\frac{5}{2}\Big) 2\zeta_5 e^{-5\phi/2} + \Big(\frac{3}{2}\Big) \frac{8}{3} \zeta_4 e^{3\phi/2} + \cdots \ . \tag{6.12}$$

In comparison with (6.2) and (6.3), the ratio between tree-level and higher-genus contributions is deformed by the covariant derivative in (6.10), namely by $-\frac{1}{3}$ and $-\frac{3}{5}$ in cases of $E_{3/2}$ and $E_{5/2}$, respectively. The modular forms in (6.11) and (6.12) multiply the R-symmetry violating counterparts of R^4 and $(D^4R^4 + D^2R^5)$ interactions which in turn describe the dilatonic amplitude components in (6.6). Hence, the covariant derivative in (6.10) holds the key for the S-duality origin of the relative factors between graviton and dilaton amplitudes in (6.6). It would be interesting to extend the analysis to higher orders in α' and to compare the ratios between amplitudes at tree-level and two-loops for higher derivative operators with and without R-symmetry charges, as it was done in [6] at one-loop up to order $(\alpha')^9$.

7. Conclusion

As the main result of this work, we have computed the low-energy limit of the five-point two-loop amplitude among massless type II closed-string states. The superspace representation of the result is given in (5.44) with prefactors made precise in (5.45). The type IIB components involving five gravitons as well as four gravitons and one dilaton were found to match the tree-level amplitude at the corresponding order in α', see (5.46). The determined ratios tie in with the S-duality expectation based on the $E_{5/2}$ coefficient of the $(D^4R^4 + D^2R^5)$ operator in the effective action [3] and its counterpart $\mathcal{D}E_{5/2}$ with modular weight, see (6.12).

The computation was performed using the non-minimal pure spinor formalism [2] where the normalizations can be reliably kept track of and where the b-ghost is explicitly known. However, subtle issues regarding possible OPE singularities between the b-ghost and the vertex operators (see for instance [52]) currently prevent the determination of the five-point two-loop amplitude to all orders in α'. These subtleties did not affect the two-loop low-energy analysis of this work, but it would certainly be desirable to extend the five-point correlator in (5.40) to all orders in the low-energy expansion. Starting from the Zhang-Kawazumi invariant expected at the subleading order in α' [57], the systematics of the low-energy expansion and the threshold corrections deserve to be studied along the lines of the one-loop results in [41]. The α'-expansion of the corresponding open string amplitudes at two-loops calls for a higher-genus generalization of the elliptic multiple zeta values [58] which were studied in the context of planar one-loop amplitudes in [59].

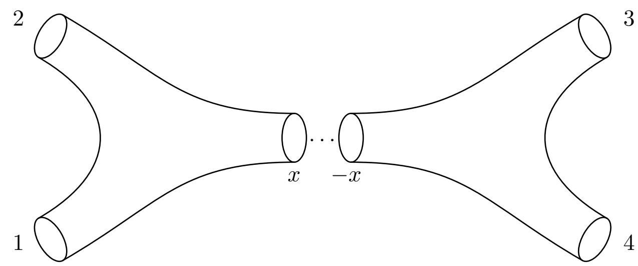

Fig. 1 The factorization of the tree-level four-point amplitude in the massless pole s_{12} in terms of three-point amplitudes. This condition was used to fix the normalization constant $\kappa^2 = \frac{\pi e^{2\lambda}}{\alpha'^2}$ in equation (A.7).

Also, it would be rewarding to cast the kinematic factors into the language of the minimal pure spinor superspace of [1] and to bypass the computational steps required by the extra worldsheet variables of the non-minimal pure spinor formalism. In particular, this concerns the evaluation of covariant derivatives originating from r_α and the tensor manipulations required to arrange the $\overline{\lambda}_\alpha$ into contractions with λ^α. For the three-loop four-point kinematic factors of [9], a much simpler BRST-equivalent representation in terms of (minimal) pure spinor superspace has recently been found [48].

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Appendix A. Worldsheet factorization of the amplitudes

In this appendix we show that five-point multiloop amplitudes computed in main body of this work factorize correctly on their massless poles as required by unitarity. We first fix the overall normalization κ of the vertex operators by imposing unitarity for the four-point amplitude at tree-level. After that there is no freedom left to adjust parameters and we proceed to check the factorization of the higher-loop amplitudes.

A.1. Factorization of the four-point tree-level amplitude

The factorization constraint for the massless pole s_{12} in the four-point amplitude reads

$$M_4^{(0)}\Big|_{s_{12}} = \alpha'^4 \int \frac{d^{10}k}{(2\pi)^{10}} \sum_x \frac{M_3^{(0)}(1,2,x) M_3^{(0)}(-x,3,4)}{k_x^2} \tag{A.1}$$

where the notation $\Big|_{s_{12}}$ projects to the pole in s_{12} and discards regular terms in s_{12}, the sum $\sum_x$ runs over all states x in the supergravity multiplet and $-x$ represents the state x at momentum $-k_x$ and complex conjugate polarization. This is depicted in fig. 1.

On the one hand, recall the low-energy limit (3.15) of the four-point amplitude

$$M_4^{(0)}\Big|_{s_{12}} = (2\pi)^{10}\delta^{10}(k)\kappa^4 e^{-2\lambda}\, 2\pi \frac{|\langle V_{12}V_3V_4\rangle|^2}{s_{12}}\,. \tag{A.2}$$

On the other hand, a short computation using the factorization constraint (A.1) yields

$$\begin{aligned} M_4^{(0)}\Big|_{s_{12}} &= \alpha'^4 \int \frac{d^{10}k}{(2\pi)^{10}} \sum_x \frac{M_3^{(0)}(1,2,x) M_3^{(0)}(-x,3,4)}{k_x^2} \\ &= (2\pi)^{10}\delta^{10}(k)\kappa^6 e^{-4\lambda}\Big(\frac{\alpha'}{2}\Big)^{-2} \alpha'^4 \frac{1}{2s_{12}} |\langle V_{12}V_3V_4\rangle|^2\,, \end{aligned} \tag{A.3}$$

where the three-point amplitude is given by

$$M_3^{(0)}(1,2,x) = (2\pi)^{10}\delta^{10}(k^1+k^2+k^x)\Big(\frac{\alpha'}{2}\Big)^{-1}\kappa^3 e^{-2\lambda}\ |\langle V_1V_2V_x\rangle|^2\ . \tag{A.4}$$

and we used

$$\int d^{10}k \frac{\delta^{10}(k^1+k^2+k^x)\delta^{10}(-k^x+k^3+\cdots+k^n)}{k_x^2} = \frac{1}{2s_{12}}\delta^{10}(k^1+k^2+k^3+\cdots+k^n) \tag{A.5}$$

together with the explicit component sum [23]

$$\sum_x \langle V_1V_2V_x\rangle\langle V_xV_3V_4\rangle = \langle V_{12}V_3V_4\rangle + \mathcal{O}(s_{12})\,. \tag{A.6}$$

Therefore equating (A.2) and (A.3) leads to

$$\kappa^2 e^{-2\lambda} = \frac{\pi}{\alpha'^2}\,. \tag{A.7}$$

A.2. Factorization of the five-point tree-level amplitude

The normalization of the five-point tree amplitude (3.18) will be checked through its factorization on the massless s_{12} pole according to

$$M_5^{(0)}\Big|_{s_{12}} = \alpha'^4 \int \frac{d^{10}k}{(2\pi)^{10}} \sum_x \frac{M_3^{(0)}(1,2,x) M_4^{(0)}(-x,3,4,5)}{k_x^2}\ , \tag{A.8}$$

where the three-point amplitude was recalled in (A.4) and

$$M_4^{(0)}(-x,3,4,5) = (2\pi)^{10}\delta^{10}(-k^x+k^3+k^4+k^5)\,\kappa^4 e^{-2\lambda} \times 2\pi\Big[\frac{|\langle V_x V_3 V_{45}\rangle|^2}{s_{45}} + \frac{|\langle V_x V_4 V_{35}\rangle|^2}{s_{35}} + \frac{|\langle V_x V_5 V_{34}\rangle|^2}{s_{34}}\Big] + \mathcal{O}(\alpha'^3)\ . \tag{A.9}$$

Using (A.7) and

$$\sum_x \langle V_1 V_2 V_x\rangle\langle V_x V_3 V_{45}\rangle = \langle V_{12} V_3 V_{45}\rangle + \mathcal{O}(s_{12}, s_{45}) \tag{A.10}$$

the factorization constraint (A.8) gives

$$M_5^{(0)}\Big|_{s_{12}} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)\,\kappa^5 e^{-2\lambda}(2\pi)^2 \times \Big[\frac{|\langle V_{12} V_3 V_{45}\rangle|^2}{s_{12}s_{45}} + \frac{|\langle V_{12} V_{35} V_4\rangle|^2}{s_{12}s_{35}} + \frac{|\langle V_{12} V_5 V_{34}\rangle|^2}{s_{12}s_{34}}\Big]\Big|_{s_{12}} + \mathcal{O}(\alpha'^3)\,. \tag{A.11}$$

This ties in with a component comparison of the terms with a pole in s_{12},

$$\tilde{A}^T_{54}\cdot S_0\cdot A_{45}\big|_{s_{12}} = \Big[\frac{|\langle V_{12} V_3 V_{45}\rangle|^2}{s_{12}s_{45}} + \frac{|\langle V_{12} V_{35} V_4\rangle|^2}{s_{12}s_{35}} + \frac{|\langle V_{12} V_5 V_{34}\rangle|^2}{s_{12}s_{34}}\Big]\Big|_{s_{12}}\ , \tag{A.12}$$

which confirms the normalization of the five-point closed-string amplitude (3.18).

A.3. Factorization of the five-point one-loop amplitude

From the low-energy limit of the five-point amplitude (4.30), it follows that

$$\begin{aligned} M_5^{(1)}\Big|_{s_{12}} &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^4 \frac{\kappa^5}{2^9\,5^2\,3}\,\frac{|\langle T_{12|3,4,5}\rangle|^2}{s_{12}} + \mathcal{O}(\alpha'^5)\,,\\ &= (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^4 \frac{\kappa^5}{2^3\,3}\,\frac{|\langle V_{12}T_{3,4,5}\rangle|^2}{s_{12}} + \mathcal{O}(\alpha'^5) \end{aligned} \tag{A.13}$$

where in the second line we used $\langle T_{12|3,4,5}\rangle = 40\big[\langle V_{12}T_{3,4,5}\rangle - \frac{s_{12}}{11}\langle A_{12|3,4,5}\rangle\big]$ (as shown in the appendix B) and discarded the contact term since it does not contribute to the s_{12} pole.

One the other hand, given the three-point tree (A.4) and the low-energy limit of (4.16),

$$M_4^{(1)} = (2\pi)^{10}\delta^{10}(k)\frac{\kappa^4}{2^4\,3\,\pi}\Big(\frac{\alpha'}{2}\Big)^3 |\langle V_1 T_{2,3,4}\rangle|^2 + \mathcal{O}(\alpha'^4)\ , \tag{A.14}$$

the factorization constraint

$$M_5^{(1)}\Big|_{s_{12}} = \alpha'^4 \int \frac{d^{10}k}{(2\pi)^{10}} \sum_x \frac{M_3^{(0)}(1,2,x)M_4^{(1)}(-x,3,4,5)}{k_x^2} \tag{A.15}$$

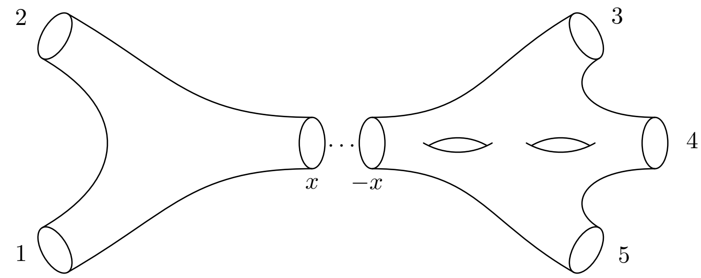

Fig. 2 Factorization channel of the five-point two-loop amplitude into a tree-level three-point and a four-point two-loop amplitude.

together with (A.5) yields

$$M_5^{(1)}\Big|_{s_{12}} = (2\pi)^{10}\delta^{10}(k)\alpha'^4\Big(\frac{\alpha'}{2}\Big)^2\frac{\kappa^7 e^{-2\lambda}}{2^5\, 3\,\pi}\frac{1}{s_{12}}\sum_x |\langle V_1 V_2 V_x\rangle|^2 |\langle V_x T_{3,4,5}\rangle|^2 + \mathcal{O}(\alpha'^7)\ . \quad \text{(A.16)}$$

One can show via a component expansion that the kinematic factors satisfies [23]

$$\sum_x \langle V_1 V_2 V_x\rangle\langle V_x T_{3,4,5}\rangle = \langle V_{12} T_{3,4,5}\rangle + \mathcal{O}(s_{12})\,. \quad \text{(A.17)}$$

By (A.7), one finally arrives at

$$M_5^{(1)}\Big|_{s_{12}} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^4\frac{\kappa^5}{2^3\, 3}\frac{|\langle V_{12} T_{3,4,5}\rangle|^2}{s_{12}} + \mathcal{O}(\alpha'^5)\ , \quad \text{(A.18)}$$

in complete agreement with the expression (A.13).

A.4. Factorization of the five-point two-loop amplitude

In the low-energy limit of the five-point two-loop amplitude (5.45), the terms with a pole in s_{12} are given by

$$M_5^{(2)}\Big|_{s_{12}} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^6\frac{\kappa^5 e^{2\lambda}}{2^{37}\, 3^9\, 5^3\, \pi^2}\Big[\frac{|\langle T_{12,3|4,5}\rangle|^2}{s_{12}} + \frac{|\langle T_{12,4|3,5}\rangle|^2}{s_{12}} + \frac{|\langle T_{12,5|3,4}\rangle|^2}{s_{12}}\Big]\,. \quad \text{(A.19)}$$

The s_{12}-channel factorization constraint in the low-energy limit

$$M_5^{(2)}\Big|_{s_{12}} = \alpha'^4\int\frac{d^{10}k}{(2\pi)^{10}}\sum_x\frac{M_3^{(0)}(1,2,x)M_4^{(2)}(-x,3,4,5)}{k_x^2} \quad \text{(A.20)}$$

for the factorization into a tree-level three-point amplitude (3.5) and a two-loop four-point amplitude (5.31)

$$M_3^{(0)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^{-1}\kappa^3 e^{-2\lambda}\ |\langle V_1V_2V_3\rangle|^2 \tag{A.21}$$
$$M_4^{(2)} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^{5}\frac{\kappa^4 e^{2\lambda}}{2^{38}\,3^9\,5^3\pi^3}\big[|\langle T_{1,2|3,4}\rangle|^2+|\langle T_{1,4|2,3}\rangle|^2+|\langle T_{1,3|4,2}\rangle|^2\big]+\mathcal{O}(\alpha'^6)\,,$$

yields

$$M_5^{(2)}\Big|_{s_{12}} = (2\pi)^{10}\delta^{10}(k)\Big(\frac{\alpha'}{2}\Big)^{4}\frac{\alpha'^4\kappa^7}{2^{39}\,3^9\,5^3\pi^3}\Big[\frac{|\langle T_{12,3|4,5}\rangle|^2}{s_{12}}+\frac{|\langle T_{12,4|3,5}\rangle|^2}{s_{12}}+\frac{|\langle T_{12,5|3,4}\rangle|^2}{s_{12}}\Big] \tag{A.22}$$

where we used (A.5) and [23]

$$\sum_x \langle V_1V_2V_x\rangle\langle T_{x,3|4,5}\rangle = \langle T_{12,3|4,5}\rangle+\mathcal{O}(s_{12}) \tag{A.23}$$

Therefore the constraint (A.7) ($\alpha'^2\kappa^2=\pi e^{2\lambda}$) implies the agreement of (A.22) with (A.19) and establishes the correct factorization of the five-point two-loop amplitude.

Appendix B. One-loop kinematic factors: NMPS versus MPS representation

From equations (3.7) and (4.7) of [46] and using the definitions (2.46) it follows that

$$T_{12|3,4,5} = (\lambda\overline{\lambda})V_{12}[36T_{3,4,5}+4(\lambda\gamma^m W_4)(\lambda\gamma^n W_5)F^3_{mn}]+s_{12}J_{12|3,4,5} \tag{B.1}$$
$$T_{1|23,4,5} = (\lambda\overline{\lambda})V_1\left[36T_{23,4,5}+4(\lambda\gamma^m W_{23})(\lambda\gamma^n W_4)F^5_{mn}\right]+s_{23}(J_{13|2,4,5}-J_{12|3,4,5})$$
$$J_{12|3,4,5} \equiv (\lambda\gamma^m W_4)(\lambda\gamma^n W_5)\big[V_1V_2(\overline{\lambda}\gamma_{mn}W_3)+2(\lambda\overline{\lambda})V_2(A_1\gamma_{mn}W_3)-(1\leftrightarrow 2)\big]\ .$$

Since (4.11) is totally symmetric in (345) it is possible to rewrite (B.1) more conveniently by averaging it over its permutations and using the Theorem 1 of [9] to factor out $(\lambda\overline{\lambda})$,

$$T_{12|3,4,5} = 40(\lambda\overline{\lambda})\big[V_{12}T_{3,4,5}-\frac{1}{11}s_{12}A_{12|3,4,5}\big] \tag{B.2}$$
$$T_{1|23,4,5} = 40(\lambda\overline{\lambda})\big[V_1T_{23,4,5}-\frac{1}{11}s_{23}(A_{13|2,4,5}-A_{12|3,4,5})\big]\,,$$

where

$$A_{12|3,4,5} \equiv \frac{1}{6}[V^1(A^2\gamma^{mn}W^3)-V^2(A^1\gamma^{mn}W^3)](\lambda\gamma_m W^4)(\lambda\gamma_n W^5)+(3\leftrightarrow 4,5). \tag{B.3}$$

Note that the admixtures of (B.3) drop out from BRST invariant combinations of (B.2) such as $\langle\frac{T_{1|23,4,5}}{s_{23}}+\frac{T_{12|3,4,5}}{s_{12}}-\frac{T_{13|2,4,5}}{s_{13}}\rangle$, that is why the amplitudes obtained from the minimal [26] and the non-minimal pure spinor formalism [46] agree.

Appendix C. Symmetries of two-loop kinematic factors

This appendix collects the superspace manipulations responsible for some of the symmetry properties of the two-loop scalar and vectorial kinematic factors.

C.1. The Jacobi-like identity of scalar kinematic factors

The kinematic factor (5.8)

$$T_{A,B|C,D}(\lambda,\overline{\lambda}) = \frac{2}{(\lambda\overline{\lambda})^6}(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{def}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\lambda\gamma^{m_1defm_2}\lambda)$$
$$\times\left[(\lambda\gamma^{n_1}W_A)(\lambda\gamma^{p_1}W_B)(\lambda\gamma^{n_2}W_C)(\lambda\gamma^{p_2}W_D)\right],$$

is now demonstrated to satisfy the identity

$$T_{A,B|C,D}(\lambda,\overline{\lambda}) + T_{A,D|B,C}(\lambda,\overline{\lambda}) + T_{A,C|D,B}(\lambda,\overline{\lambda}) = 0. \tag{C.1}$$

To see this one uses the gamma matrix identity

$$(\overline{\lambda}\gamma_{def}r)(\lambda\gamma^{m_1defm_2}\lambda) = 48(\lambda\overline{\lambda})(\lambda\gamma^{m_1}\gamma^{m_2}r) - 48(\lambda\gamma^{m_1}\gamma^{m_2}\overline{\lambda})(\lambda r) \tag{C.2}$$

together with $(\overline{\lambda}\gamma_{m_2n_2p_2}r) = (\overline{\lambda}\gamma^{m_2}\gamma^{n_2}\gamma^{p_2}r)$ and $(\overline{\lambda}\gamma^{m_2})_\alpha(\overline{\lambda}\gamma_{m_2})_\beta = 0$ to obtain

$$\begin{aligned}(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\overline{\lambda}\gamma_{def}r)(\lambda\gamma^{m_1defm_2}\lambda) &= 48(\lambda\overline{\lambda})(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\lambda\gamma^{m_1}\gamma^{m_2}r) \\ &= 48(\lambda\overline{\lambda})(\overline{\lambda}\gamma^{m_2}\gamma^{m_1}\lambda)(r\gamma^{m_2n_2p_2}r)\,,\end{aligned} \tag{C.3}$$

where the cyclic identity $\gamma^{m_2}_{\alpha(\beta}\gamma^{m_2}_{\gamma\delta)} = 0$ and the constraint $(\overline{\lambda}\gamma^m r) = 0$ were used to arrive at the second line. Therefore,

$$(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\overline{\lambda}\gamma_{def}r)(\lambda\gamma^{m_1defm_2}\lambda) = 96(\lambda\overline{\lambda})^2(\overline{\lambda}\gamma^{an_1p_1}r)(r\gamma^{an_2p_2}r)\ . \tag{C.4}$$

After using (C.4), the identity (C.1) follows by noting that it is equivalent to

$$(\overline{\lambda}\gamma^{an_1p_1}r)(r\gamma^{an_2p_2}r) + (\overline{\lambda}\gamma^{ap_2p_1}r)(r\gamma^{an_1n_2}r) + (\overline{\lambda}\gamma^{an_2p_1}r)(r\gamma^{ap_2n_1}r) = 0\ , \tag{C.5}$$

and (C.5) can be shown using $\gamma^a_{\alpha(\beta}\gamma^a_{\gamma\delta)} = 0$.

C.2. Relating vector kinematic factors

In order to prove the symmetry (5.16) of the vectorial kinematic factor in (5.10) and (5.11), the pure spinor constraint can be invoked to decompose the gamma matrices in the factor

$$(W_A\gamma^{a_1a_2a_3}W_B)(\lambda\gamma^{a_2m_1n_1p_1m_2}\lambda) = -(W_A\gamma^{a_2}\gamma^{a_1}\gamma^{a_3}W_B)(\lambda\gamma^{a_2}\gamma^{m_1}\gamma^{n_1}\gamma^{p_1}\gamma^{m_2}\lambda)$$

contained in (5.10). The identity

$$\begin{aligned}(W_A\gamma^{a_2}\gamma^{a_1}\gamma^{a_3}W_B)(\lambda\gamma^{a_2}\gamma^{m_1}\gamma^{n_1}\gamma^{p_1}\gamma^{m_2}\lambda) &= -(W_A\gamma^{a_2}\gamma^{m_1}\gamma^{n_1}\gamma^{p_1}\gamma^{m_2}\lambda)(\lambda\gamma^{a_2}\gamma^{a_1}\gamma^{a_3}W_B)\\ &\quad -(W_A\gamma^{a_2}\lambda)(\lambda\gamma^{m_2}\gamma^{p_1}\gamma^{n_1}\gamma^{m_1}\gamma^{a_2}\gamma^{a_1}\gamma^{a_3}W_B)\end{aligned}$$

then allows applications of the pure spinor constraint in the form of $(\lambda\gamma_r)_\alpha(\lambda\gamma^r)_\beta = 0$; ultimately leading to

$$S^{(1)\,m}_{A,B,C|D,E}(\lambda,\overline{\lambda}) = S^{(2)\,m}_{A,B,C|D,E}(\lambda,\overline{\lambda})+S^{(2)\,m}_{A,D,E|B,C}(\lambda,\overline{\lambda})+S^{(2)\,m}_{B,D,E|A,C}(\lambda,\overline{\lambda})+S^{(2)\,m}_{C,D,E|A,B}(\lambda,\overline{\lambda})$$

which implies (5.16).

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