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3D Equilibrium Averaged Description and Consistency Check

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1 Background Modelling of 3D equilibrium plasma configurations is a challenging task. The existence of magnetic islands and stochastic magnetic field regions makes a direct [1, 2] modelling time consuming, not very robust and flexible, and hardly useful for systematic equilibrium optimization and stability analysis. More tractable conventional model for 3D MHD equilibrium and stability studies is based on the nested magnetic surface approximation as in standard 3D equilibrium code VMEC [3]. However the code convergence is sensitive to the choice of harmonic set for flux surface representation and significantly deteriorates with increasing resolution.

For equilibrium and stability studies based on the averaging methods a lot of numerical codes have been developed and good agreement for both equilibrium and stability in planar-axis stellarators was obtained [4].

The approach to 2D description of MHD equilibrium and stability proposed in [5] is more general. The key idea is to introduce Riemannian space $\mathbb{R}^3$, in which reference 3D equilibrium is symmetric. The first step in such interpretation was carried out in [6, 7], where it was shown that for arbitrary 3D equilibrium (with nested magnetic surfaces at least) one can construct some formal 2D metric tensor and obtain 2D Grad-Shafranov type equation. The equation was obtained by averaging exact 3D equation. In fact, it is the exact zero 2D moment of equilibrium equation, like Kruskal-Kulsrud equation is the exact zero 1D moment.

2 Scalar equations for 3D MHD equilibria description By assuming the magnetic surfaces $a(r) = const$ exist the ideal MHD equilibrium problem

$$\mathbf{j} \times \mathbf{B} - \nabla p = 0, \quad \mathbf{j} = \nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0 \qquad (1)$$

can be reduced to the field equations

$$\nabla \cdot \mathbf{B} = 0, \quad \mathbf{B} \cdot \nabla a = 0, \quad \nabla \cdot (\mathbf{B} \times \nabla a) = 0 \qquad (2)$$

and to the force balance equation

$$|\nabla a|^{-2} (\mathbf{B} \times \nabla a) \cdot \nabla \times \mathbf{B} = dp(a)/da. \qquad (3)$$

The following statement is valid ([6, 8] for example): for any a priori given family of nested toroidal surfaces $a(r) = const$ the full set of solutions of (2) can be represented by the linear combinations

$$\mathbf{B} = \Phi' \nabla a \times \nabla \theta_\psi + \Psi' \nabla a \times \nabla \zeta_\psi, \quad \mathbf{B} = J \nabla_a \theta_F + F \nabla_a \zeta_F \qquad (4)$$

while each summand satisfies (2).

Here

$$(\cdot)' = d(\cdot)/da, \quad \nabla_a(\cdot) = \mathbf{n} \times \nabla(\cdot) \times \mathbf{n}, \quad \mathbf{n} = \nabla a / |\nabla a|,$$

and the pairs of the coefficients $\Phi(a), \Psi(a)$ and $J(a), F(a)$ are arbitrary and refer to toroidal and external poloidal (helicoidal) fluxes or currents. The pairs of the basis vectors $\nabla a \times \nabla \theta_\psi, \nabla a \times \nabla \zeta_\psi$ and $\nabla_a \theta_F, \nabla_a \zeta_F$ are particular solutions of (2). They are generated by the cyclic functions $\theta_{\psi,F}, \zeta_{\psi,F} \in [0, 1) \times [0, 1)$, which can be interpreted as poloidal (helicoidal) and toroidal angles, satisfying equations

$$L_\psi \theta_\psi = L_\psi \zeta_\psi = 0, \quad L_\psi = \nabla \cdot |\nabla a|^2 \nabla_a(\cdot), \quad L_F \theta_F = L_F \zeta_F = 0, \quad L_F = \nabla \cdot \nabla_a(\cdot). \qquad (5)$$

The relations between the fluxes and currents and basis vectors in (4) can be written as

$$\begin{array}{ll} J = -\alpha_{22}\Psi' + \alpha_{23}\Phi', & \alpha_{22}\nabla_a\theta_F + \alpha_{23}\nabla_a\zeta_F = -\nabla a \times \nabla_a\zeta_\psi, \\ F = -\alpha_{23}\Psi' + \alpha_{33}\Phi', & \alpha_{23}\nabla_a\theta_F + \alpha_{33}\nabla_a\zeta_F = \nabla a \times \nabla_a\theta_\psi. \end{array} \tag{6}$$

with matrix elements $\alpha_{ik}(a) = \mathbf{e}_i^F \cdot \mathbf{e}_k^\psi/\sqrt{g_\psi}$ defined by covariant vectors for coordinate variables (5) depending on the shape of magnetic surface only.

Using other combinations of particular solutions of (2) a mixed representation (in terms of poloidal flux and current) of the magnetic field can be obtained:

$$\mathbf{B} = \nabla\Psi \times \mathbf{b}_3^F + F\mathbf{b}_3^\psi, \tag{7}$$

where, as before, each component satisfies equations (2). The vectors $\mathbf{b}_3^{\psi,F}$ can be written as linear combinations of $(\nabla_a\theta_{\psi,F}, \nabla_a\zeta_{\psi,F})$.

Hence for constructing the specific field $\mathbf{B}$, which provides the plasma MHD equilibrium configuration, it is sufficient to know the shape of magnetic surfaces and the distribution of any pair from fluxes and currents over these surfaces. Substitution of the magnetic field (in any form) into the force-balance equation (3) yields the equation for function $a(\mathbf{r})$ and together with boundary conditions complements (4)-(6) to a close system of equations.

Such a method with $\mathbf{B}$ in the form (7) generates three-dimensional analog of the two-dimensional Grad-Shafranov equilibrium equation (see [7]) having one dimensional zero moment – the Kruskal-Kulsrud equation

$$p'V' = J'\Psi' - F'\Phi'.$$

3 2D Grad-Shafranov type equation for 3D plasma equilibrium as the exact zero two-dimensional moment for magnetohydrostatics The following statement was formulated in [6, 7]: for any 3D plasma equilibrium (with nested flux surfaces at least) there exist coordinate system (x^1, x^2, ζ) and corresponding Riemannian space $\mathbb{R}^3$ in which the following conditions are satisfied: i. metric tensor $\hat{g}_{ik}(\mathbf{r})$ is two-dimensional: $\frac{\partial}{\partial\zeta}\hat{g}_{ik} = 0$; ii. poloidal (helicoidal) flux function Ψ is two-dimensional $\Psi = \Psi(x^1, x^2)$ and it is the solution of 2D Grad-Shafranov type equation

$$\hat{\nabla} \cdot \left(\frac{\hat{\nabla}\Psi}{\hat{g}_{33}}\right) + \frac{F}{\hat{g}_{33}}\frac{dF}{d\Psi} - F\hat{\nabla} \cdot \left(\frac{\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}^3}{\hat{g}_{33}}\right) = -\alpha\frac{dp}{d\Psi}; \tag{8}$$

iii. magnetic field takes the form $\mathbf{B} = (\nabla\Psi \times \mathbf{e}_3 + B_3\mathbf{e}_3)/g_{33}$, with $\mathbf{e}_3 = \partial\mathbf{r}/\partial\zeta$, $< B_3 >_\zeta = F(\Psi)$.

Here $\mathbb{R}^3$ is generated by the metric tensor

$$\hat{g}_{ik} = \sqrt{\hat{g}} < \frac{g_{ik}}{\sqrt{g}} >_\zeta, \quad \hat{g} = det\,\hat{g}_{ik} = det^{-2} < \frac{g_{ik}}{\sqrt{g}} >_\zeta, \quad \hat{g}^{ik} = \hat{G}_{ik}/\hat{g}, \tag{9}$$

and

$$\alpha = \alpha(x^1, x^2) = < \sqrt{g} >_\zeta / \sqrt{\hat{g}},$$

$\hat{\nabla}$ is an ∇-operator in $\mathbb{R}^3$. In these formulas $< f >_\zeta = \frac{1}{\zeta_{max}}\int_0^{\zeta_{max}} f(x^1, x^1, \zeta)d\zeta$.

Equation (8) can be obtained by averaging the exact 3D equilibrium equation in any reference coordinate system connected to the natural coordinates (5) by a two-dimensional transformation $x^{1,2} = x^{1,2}(a, \theta_\psi)$, $\quad \zeta = \zeta_\psi - \lambda(a, \theta_\psi)$.

The numerical codes were developed for generation of the Riemannian space metric based on VMEC 3D equilibrium format and for solution of the 2D equation on grid adaptive to magnetic surfaces [9].

4 Consistency check for equilibria with nested flux surfaces The solution of the 2D elliptic equation in the Riemannian space (8) should reproduce the original 3D equilibrium with nested flux surfaces. In particular, it means that in the coordinate system $x^1 = a\cos\theta_\psi, \quad x^2 = a\sin\theta_\psi$ the level lines of the solution $\Psi(x^1, x^2)$ are concentric circles. A deviation from that solution can be estimated by comparing the rotational transform $\iota = -\Psi'/\Phi'$ profile for the 3D configuration and the solution of the averaged equation. Such a consistency check provides more extensive test than the 1D Kruskal-Kulsrud equation balancing the shape of magnetic surfaces against the plasma profiles.

The method was applied to several series of equilibria in conventional stellarators and in the systems with spatial magnetic axes.

The first series was related to the reference LHD configuration [4]. The pressure profile was prescribed in terms of normalized toroidal flux Φ and zero toroidal net current was assumed: $p = p_0(1-\Phi^2)^2, \quad J = 0$. In Fig.1 the rotational transform profiles are compared for a series of equilibria with increasing β. The profiles from the averaged equation match the original profiles with accuracy better than 0.01 for $\beta \lesssim 3\%$ for VMEC resolution with 97 radial nodes equidistributed in toroidal flux and 179 harmonics.

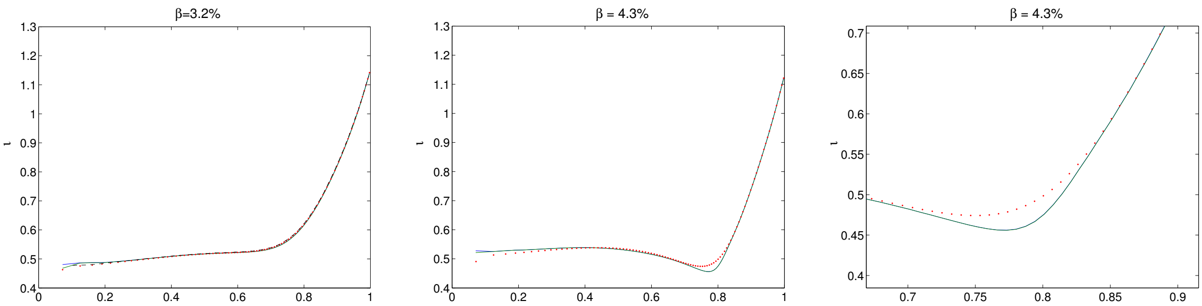

Fig.1 Rotational transform from VMEC solution (dots) and corresponding averaged equation (solid lines) versus square root of normalized toroidal flux. The region of maximal discrepancy for $\beta = 4.3\%$ is zoomed.

However for the equilibrium with $\beta = 4.3\%$ a significant discrepancy (~ 0.05) appeared near the rational surface $\iota = 0.5$ approaching location of the pressure gradient maximum. The discrepancy stays with increased resolution both in VMEC, Riemannian metric generation module to solve elliptic equations (5) and the adaptive grid code to solve the 2D equation.

To check whether it is an indication of magnetic island appearance the pressure profile with another location of the gradient maximum was chosen: $p = p_0(1-\Phi^2)^{10}$. The same effect was reproduced for lower value of $\beta = 1\%$ (Fig.2).

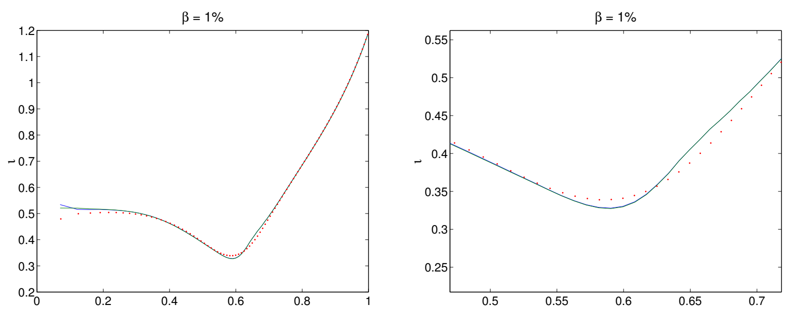

Fig.2 Rotational transform comparison for $\beta = 1\%$ more peaked pressure profile.

Another series is related to the analytic force-free equilibria with $\mathbf{j} = \lambda\mathbf{B}, \quad \lambda = \text{const}$ [10]. Starting from the axisymmetric configuration and applying 3D perturbation generates the series of equilibria including cases with nested flux surfaces, spatial magnetic axis, magnetic islands and stochastic fields. The shape of boundary flux surface was obtained by magnetic field line tracing. The plasma profiles are described by $p = 0, \quad J = \lambda\Phi$. The problem with accuracy in VMEC was encountered: the discrepancy in rotational transform of several percent order was discovered even with radial resolution ~ 200 (Fig.3).

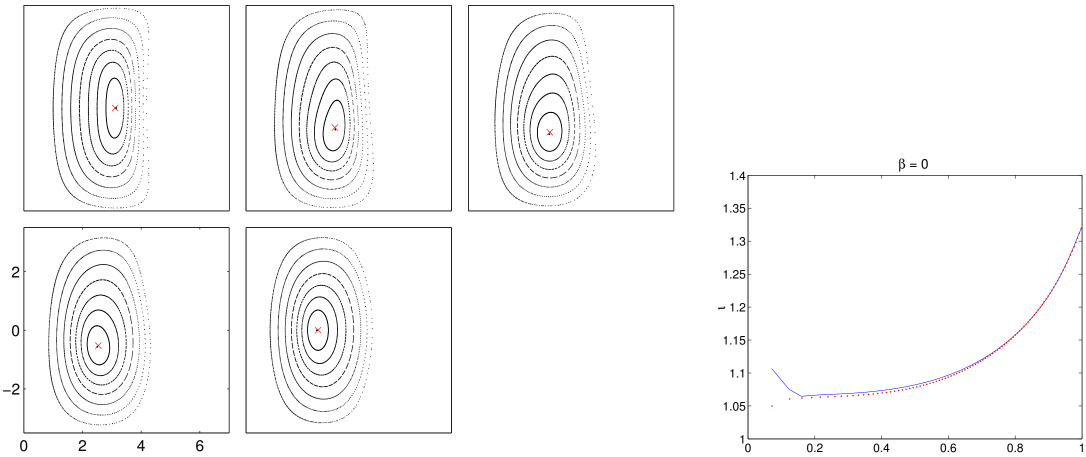

Fig.3 Analytic force-free equilibrium surfaces in different toroidal cross sections (VMEC magnetic axis shown by cross) and rotational transform comparison.

A possible reason for that is a problem with magnetic axis determination. Note that for a configuration with islands the VMEC code also converges to some nested flux surface solution which is not an equilibrium because of uniqueness of the considered force-free configuration. However the same kind of discrepancy in rotational transform profiles was encountered.

5 Discussion A possible use of the averaged description of 3D equilibria is robust and fast computation of approximate equilibria with different β and profiles starting from background 3D vacuum configuration, for example. However the question on the approach applicability range needs to be answered.

An immediate use of the averaged equilibrium description is the consistency check for 3D nested flux surface configurations giving information on numerical equilibrium accuracy and possibly existence of the regular equilibrium problem solution.

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