Stability Studies of Ideal Plasma Flow Equilibria

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Introduction. Experimental and theoretical evidence obtained recently that poloidal and toroidal plasma rotation may considerably affect stability, raises the question regarding a theoretically and computationally adequate description of this phenomenon. In this paper we study the linear, in general resistive, stability of toroidally-axisymmetric ideal flow equilibria using single-fluid MHD equations. It is found that for moderate and high Mach numbers the stabilizing character of differential flow strongly depends on the consistent treatment of plasma compressibility and inertia not only in the stability analysis but also in the equilibrium calculation.

MHD Background Theory. Equilibrium and stability are derived from the following time-dependent MHD equations

$$\rho \frac{d\mathbf{v}}{dt} = \nabla\times\mathbf{B}\times\mathbf{B}/\mu_0 - \nabla p \tag{1}$$

$$\frac{d\mathbf{B}}{dt} + \mathbf{B}\nabla\cdot\mathbf{v} = \mathbf{B}\cdot\nabla\mathbf{v} - \nabla\times(\eta\nabla\times\mathbf{B}/\mu_0) \tag{2}$$

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{3}$$

$$\frac{dT}{dt} + (\Gamma - 1)\,T\nabla\cdot\mathbf{v} = 0 \tag{4}$$

i.e. the balance equations for momentum and magnetic flux and conservation of mass and energy, where most quantities have their usual meanings. d/dt denotes the total derivative $\partial/\partial t + \mathbf{v}\cdot\nabla$. η is the electrical resistivity, Γ is the ratio of the specific heats and T the temperature in eV.
Considering (3) as the equation for the mass density ρ and (4) as determining the temperature T, the plasma pressure follows from the equation of state $p = \rho T/m$. This way (1-4) form a closed set of equations for the nonlinear evolution of the velocity field $\mathbf{v}$, magnetic induction $\mathbf{B}$, mass density ρ and temperature T. Note that due to $\nabla\cdot\mathbf{B} = 0$ (2) is satisfied for all times if it is satisfied initially. Naturally, for $\eta = 0$ the ideal MHD model is recovered.

Equilibrium. Time-independent axisymmetric solutions of the ideal MHD equations represent ideal flow equilibria. In stationary equilibrium both magnetic induction $\mathbf{B}$ and momentum density $\rho\mathbf{v}$ are divergence-free vector fields with the flux-representations

$$\mathbf{B} = \frac{1}{2\pi}\Big(\nabla\Psi\times\nabla\varphi + C\nabla\varphi\Big) \quad \text{and} \quad \rho\mathbf{v} = \frac{1}{2\pi}\Big(\nabla\Psi_M\times\nabla\varphi + C_M\nabla\varphi\Big), \tag{5}$$

where φ is the angle around the axis of symmetry in right-handed co-ordinates (R, φ, z). Ψ and Ψ_M are the poloidal fluxes of $\mathbf{B}$ and $\rho\mathbf{v}$. $C = \mu_0 J$ (where J is the poloidal current) and C_M are the corresponding toroidal circulations of these vector fields. Equation (2) implies that $\mathbf{v}\times\mathbf{B}$ is irrotational, so that Φ_M in $\mathbf{v}\times\mathbf{B} = \nabla\Phi_M$ is the electric potential caused by the mass flow. From $\mathbf{B}\cdot\nabla\Phi_M = 0$ and $\mathbf{v}\cdot\nabla\Phi_M = 0$ it can be concluded that Φ_M, as well as Ψ_M, are constants on magnetic surfaces. Evaluation of $\nabla\Psi\cdot\nabla\Phi_M$ implies

$C_M = \mu_0 J\Psi_M' + 4\pi^2 R^2 \rho \Phi_M{}'$, where $' = d/d\Psi$. In contrast to the fast adiabatic (isentropic) perturbations of the equilibrium state on the Alfvén time scale, we assume for the steady-state an isothermal behaviour ($\Gamma = 1$) so that $\mathbf{v}\cdot\nabla T = 0$ and $T = T(\Psi)$. The equilibrium components of the momentum balance (1) parallel to $\mathbf{B}$ and in the toroidal direction turn out to be exactly integrable relations so that the dynamic free energy G_M per mass unit and the poloidal current J_M

$$G_M = \frac{B^2 \Psi_M'^{\,2}}{2\rho^2} - 2\pi^2 R^2 \Phi_M'^{\,2} + G(T, \rho T/m), \qquad J_M = \left(1 - \frac{\mu_0 \Psi_M'^{\,2}}{\rho}\right) J - 4\pi^2 R^2 \Psi_M' \Phi_M{}' \quad (6)$$

also become surface quantities with $G_M = G_M(\Psi)$ and $J_M = J_M(\Psi)$. G is the Gibb's free energy $G(T, p) = (T/m)\ln(p/p_M)$ of an ideal gas, where p_M is an arbitrary constant. The last but most salient equation to be considered is the equilibrium component of equation (1) normal to the magnetic surfaces which represents a quasilinear partial differential equation for the poloidal magnetic flux Ψ

$$\nabla\cdot\left\{\left(1 - \frac{\mu_0 \Psi_M'^{\,2}}{\rho}\right)\frac{\nabla\Psi}{R^2}\right\} + \frac{\mu_0^2 J J_M'}{R^2}$$
$$+4\pi^2\mu_0\left(\rho(G_M{}' - \frac{\partial G}{\partial T} T') + \mathbf{v}\cdot\mathbf{B}\,\Psi_M{}'' + C_M \Phi_M{}''\right) = 0 \quad (7)$$

where

$$\mathbf{v}\cdot\mathbf{B} = \frac{\Psi_M' |\nabla\Psi|^2 + \mu_0 J C_M}{4\pi^2 \rho R^2} \quad (8)$$

The 5 functions $\mathbf{F}_D = (J_M, G_M, T, \Psi_M, \Phi_M)$ must be specified and J and ρ be determined as solutions of the equations (6). Note that in contrast to static equilibria the quantities J, $p = \rho T/m$ and the toroidal rotation frequency $\Omega_t = v_t/R = \mu_0 J \Psi_M'/(2\pi\rho R^2) + 2\pi\Phi_M{}'$ for non-zero poloidal flow are not constant on magnetic surfaces.
We determine the flux functions $\mathbf{F}_D$ from the distributions of poloidal current, mass density, temperature and of the poloidal and toroidal flow components on some line leading from the magnetic axis to the plasma boundary. Once $\mathbf{F}_D$ is calculated, (7) can be solved. Here $\mu_0 \rho v_p^2/B_p^2 < \beta$ must be satisfied in order not to violate the elliptic nature of the partial differential equation (7).
With ρ, T, $\mathbf{v}$ and $\mathbf{B}$ determined we obtain an initial state for the evolution equations (1-4) suitable for a stability analysis.

Stability. The linearization of (1-4) together with the ansatz $\sim \exp(\lambda t)$ for the time-dependency yields

$$\lambda\rho_0\mathbf{v} = -\rho\mathbf{v}_0\cdot\nabla\mathbf{v}_0 - \rho_0(\mathbf{v}_0\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}_0) - \nabla(\rho_0 T + \rho T_0)$$
$$+(\nabla\times\mathbf{B}_0\times\mathbf{B} + \nabla\times\mathbf{B}\times\mathbf{B}_0)/\mu_0 \quad (9)$$

$$\lambda\mathbf{B} = -\mathbf{v}_0\cdot\nabla\mathbf{B} - \mathbf{B}\,\nabla\cdot\mathbf{v}_0 + \mathbf{B}\cdot\nabla\mathbf{v}_0 + \nabla\times(\mathbf{v}\times\mathbf{B}_0 - \eta\nabla\times\mathbf{B}/\mu_0) \quad (10)$$

$$\lambda\rho = -\mathbf{v}_0\cdot\nabla\rho - \rho\,\nabla\cdot\mathbf{v}_0 - \mathbf{v}\cdot\nabla\rho_0 - \rho_0\nabla\cdot\mathbf{v} \quad (11)$$

$$\lambda T = -\mathbf{v}_0\cdot\nabla T - (\Gamma-1)T\,\nabla\cdot\mathbf{v}_0 - \mathbf{v}\cdot\nabla T_0 - (\Gamma-1)T_0\nabla\cdot\mathbf{v} \quad (12)$$

where the subscript 0 denotes zeroth-order equilibrium quantities and ρ, T, $\mathbf{v}$ and $\mathbf{B}$ now (and in what follows) denote the first-order stability quantities.

Computational Realization. The stationary equilibria described by solutions of the equations (6-8) were calculated using the DIVA program [1] for axisymmetric free-boundary, separatrix- or limiter-defined equilibria in the magnetic field of an external conductor system. For the purpose of being able to study their linear stability a significantly extended version of the CASTOR static-equilibrium spectral stability code [2] was developed by allowing for *compressible* as well as for *inertial* equilibrium flow (represented by the coloured terms in the stability equations (9-12)). For the transfer of equilibrium data and flux-surface geometry from DIVA to CASTOR an interface has been developed between both the codes which arranges for an efficient coupling of equilibrium and stability calculations. The stability equations are solved in CASTOR by conversion into a large non-hermitian eigenvalue problem of the form $(\mathbf{A}-\lambda\mathbf{B})\cdot\mathbf{u}=0$ and determining the complex eigenvalues λ and eigenvectors $\mathbf{u}$ by inverse vector iteration.

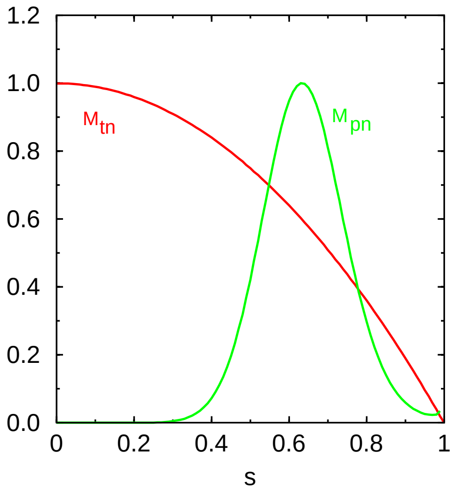

Figure 1: Normalized poloidal and toroidal Mach profiles M_{pn} and M_{tn}.

Results. For a case study of a particular ASDEX Upgrade discharge (the flat-top phase of shot 12224 at 0.72 s with neutral beam injection and reversed shear profile) we distinguish between data which have been held fixed and parametrized data.

To the first group belong the plasma current I_p = 0.786 MA, $\beta_p = 0.644$, the temperature on axis T_a = 9.6 keV, the external magnetic field and the forms ρ_n, T_n and J_n of the reference-line profile functions entering the equilibrium fields ρ_0, T_0 and $\mathbf{B}_0$. In an equilibrium state these profiles are completely determined by the relations

$$T_0(s) = T_a T_n(s), \qquad \rho_0(s) = m\, p_a(I_p,\beta_p)\rho_n(s)/T_a;$$

$$J_0(s) = J_b\Big(1 - C_s(I_p,\beta_p)(1-J_n(s)^2)\Big)^{1/2}$$

where s is the square root of the normalized poloidal flux. Note that the poloidal current at the plasma boundary, J_b, is determined by the toroidal vacuum magnetic field and the pressure on axis p_a as well as the diamagnetism parameter C_s by the plasma current and beta-poloidal. As parametrized data we have chosen the poloidal and toroidal rotation Mach number distributions on the reference line:

$$M_p = v_p/c_s = M_{pm}M_{pn}(s), \qquad M_t = v_t/c_s = M_{ta}M_{tn}(s)$$

where $c_s^2 = T/m$. As shown in Fig.1 the normalized distributions M_{pn} and M_{tn} have been chosen such that they have a value of 1 at the maximum of M_p and M_t, respectively. After having made a choice for the distributions M_{pn} and M_{tn} the only free parameters in the problem are M_{pm} and M_{ta}. We have carried through stability studies for $M_{ta} \in (0, 0.6)$ and $M_{pm} \in (0, 0.01)$, where at $M_{ta} = 0.32$ the toroidal Mach number on axis passes the experimentally observed value. As a starting point we refer to the static equilibrium $M_{pm} = M_{ta} = 0$. Due to the presence of reversed shear there are two $(2,-1)$ rational magnetic surfaces in the variability range of $q \in (-5.53, -1.87)$. They lead to the formation of a double tearing mode (where we have assumed a Lundquist-Reynolds number $S \simeq 2.7 \times 10^6$) which also persists for non-zero flow. Fig.2 (left and middle) illustrates that up to a certain value a small amount of differential rotation already produces considerable damping of this mode. Beneath this value, the growth rates start to

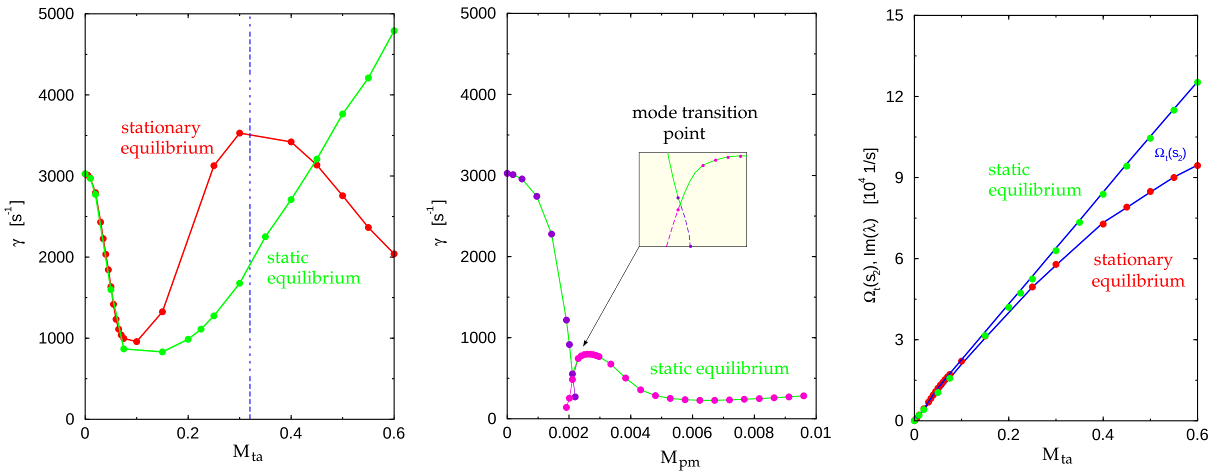

Figure 2: Left - Growth rates due to toroidal flow ($M_p = 0$). Middle - Growth rates due to poloidal flow ($M_t = 0$). Right - Ω_t on the outer resonant surface (solid) and $\omega = \mathrm{Im}(\lambda)$ (points).

increase and then, for stationary equilibria, drop again. With toroidal flow, γ reaches its minimum near $M_{ta} = 0.10$ with a difference between the rotation frequencies on the rational surfaces of $\Delta\Omega_t = 11.0$ kHz, for poloidal flow this minimum is at $M_{pm} \simeq 0.006$ ($\Delta\Omega_p = 8.2$ kHz). Fig.2 (left and middle, green curves) shows a growth rate calculation on the basis of the static equilibrium with $\mathbf{v}_0 \neq 0$ only in the stability equations. For small M_t there is good agreement with the growth rates of the stationary equilibria (left, red curve), but for moderate and high toroidal Mach numbers where the signatures of the stationary equilibrium change noticeably, quantitative discrepancies are obtained. A similiar ascertainment holds for the frequency $\omega = \mathrm{Im}(\lambda)$ of the solution of the stability equations (9-12) (Fig.2, right, points). Due to q-profile variations with increasing M_{ta} the outer rational surface moves radially outward into regions where Ω_t is smaller so that there is no linear relation between M_{ta} and Ω_t as for static equilibria with fixed q-profile (blue curves). Note that there is remarkably good numerical agreement between Ω_t on the outer resonant surface and ω.

Conclusions. We have investigated the damping of a double-tearing mode due to poloidal and toroidal plasma rotation. For purely toroidal flow we studied the behaviour of a double-tearing mode for a whole family of stationary equilibria parametrized by the toroidal Mach number trough the calculation of growth rates and oscillation frequencies. It was found that for moderate and high Mach numbers correct results can be only obtained if plasma compressibility and inertia are taken into account both for equilibrium and stability calculations. The results indicate that the stabilization of the plasma is more sensitive to poloidal than to toroidal flow, and suggest an investigation of the combined effects of both. It should be noted that the accurate Fourier representation of the poloidal variation of mass density ρ and poloidal current J as described by the nonlinear equations (6) in the CASTOR stability code is highly complex and presently only has been done completely for the mass density. The reason is the inherent constancy on magnetic surfaces of ρ and J for static equilibria which is removed by plasma flow.

[1] H. P. Zehrfeld, Resistive Equilibrium States of Axisymmetric Plasmas with Compressible Viscous Fluid Flow, in *26th EPS Conference on Controlled Fusion, Contributed Papers*, Maastricht, Netherlands, 1999.

[2] W. Kerner et al., CASTOR: Normal-Mode Analysis of Resistive MHD Plasmas, J. Comput. Phys. **142**, 271–303 (1998).