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Abstract

The work investigates the Alfvén instabilities that can be driven by circulating energetic ions in optimized stellarators of Wendelstein line. A general expression for the growth rate of the instabilities is obtained and analyzed. It is shown that the absence of the axial symmetry makes it possible that various types of Alfvén eigenmodes will be destabilized; both the kind of destabilized Alfvén eigenmodes and the type of the resonances driving the instability may differ from those in tokamaks. In particular, an important role of the helicilty-induced resonance is predicted. The upper limits of the local energy losses of circulating α-particles caused by various Alfvén instabilities in a four-period Helias reactor are evaluated. It is found that certain destabilized Alfvén eigenmodes will affect only alphas with the energy well below 3.5 MeV, which seems to open a possibility to remove the helium ash by exciting the corresponding Alfvén eigenmodes by either energetic particles or an antenna system.

I. INTRODUCTION

Recent studies of the Alfvén continuum and Alfvén eigenmodes in optimized stellarators of Wendelstein line have shown that there are many gaps in the continuum, where discrete eigenmodes can reside [1–4]. In particular, the existence of the MAE modes (mirror-induced Alfvén eigenmodes) was predicted; it was found that the elongation of the plasma cross section, which rotates poloidally along the large azimuth of the torus, results in a very wide gap with the HAE_{21} modes (helicity-induced eigenmodes, where subscripts denote the poloidal and toroidal coupling numbers) [3,4]. A generalized kinetic energy integral was constructed [5]. Experiments on Wendelstein 7-AS [6] have demonstrated that neutral beam injection ions can lead to Alfvén instabilities (AI) and concomitant losses of energetic ions in Wendelstein-line stellarators [7,8]. Thus, one can expect that AIs driven by alpha particles may arise in a Helias reactor [9–11]. On the other hand, analyses of Alfvén instabilities observed experimentally on stellarators often rely to a considerable extent on the theory developed for tokamaks. However, it is clear that because of the mentioned variety of discovered Alfvén modes such approach may be not always justified to interpret the experimental data and cannot be used for a reliable prediction of the role of Alfvén instabilities in Wendelstein 7-X [6,9] and a Helias reactor. Moreover, as we will show in this work, the resonant wave-particle interaction in stellarators has a number of peculiarities; therefore, description of instabilities in these systems, in general, requires a new theory.

In the present work we consider the destabilization of various Alfvén eigenmodes by circulating energetic ions in optimized stellarators. Using the obtained family of the resonances responsible for the interaction of Alfvén waves with energetic ions, we evaluate possible influence of Alfvén instabilities on the losses of alpha particles in a Helias reactor, following the approach of Refs. [12].

The structure of the work is as follows. A general expression for the instability growth rate is obtained in Sec. II. The response of circulating energetic ions and a corresponding expression for the growth rate are found in Sec. III. The obtained expressions are applied to the optimized stellarators and analyzed in Sec. IV. The effect of Alfvén instabilities on confinement of alpha particles is considered in Sec. V. Section VI summarizes the obtained results and contains the conclusions drawn.

II. BASIC EQUATIONS

When the energetic ions destabilize the waves and the bulk plasma effect is stabilizing, the instability growth rate can written as follows:

$$\gamma = \gamma_\alpha + \gamma_d, \tag{1}$$

where $\gamma_\alpha > 0$ and $\gamma_d < 0$ describe the wave destabilization and damping, respectively; γ_α is associated with the energetic ions, whereas γ_d is due to the bulk plasma. Below we will be interested only in γ_α, which we will refer to as the growth rate.

The energy exchange between a particle and a wave is described by the following equation:

$$\frac{d\mathcal{E}}{dt} = e(v_\parallel \tilde{E}_\parallel + \mathbf{v}_D \cdot \tilde{\mathbf{E}}_\perp) + \mu_p \frac{\partial \tilde{B}}{\partial t}, \tag{2}$$

where $\mathcal{E}$ is the particle energy, μ_p is the particle magnetic moment, $\mathbf{v}_D$ is the drift velocity caused by the curvature and inhomogeneity of the magnetic field, $\tilde{\mathbf{E}}$ is the wave electric field, $\tilde{\mathbf{B}}$ is the wave magnetic field, the subscripts "$\perp$" and "$\|$" label the vector components across and along the magnetic field, respectively. Let us evaluate the terms in this equation. To do it, we use the equation $\tilde{E}_\parallel/\tilde{E}_\perp \approx \beta_e \omega^2 k^2 / \left(2\omega_B^2 k_\perp k_\parallel\right)$ [13], where ω_B is gyrofrequency, $\beta_e = 8\pi n_e T_e / B^2$ is the ratio of the electron pressure to the magnetic field pressure, $\mathbf{k}$ is the wave vector. Assuming $\omega = k_\parallel v_A$ with v_A the Alfvén velocity, $k_\parallel \ll k_\perp$, we find:

$$\frac{v_\parallel \tilde{E}_\parallel}{v_D \tilde{E}_\perp} \leq \frac{\beta_e v_A^2 k_\perp k_\parallel L_D}{2 v_\alpha \omega_B}, \tag{3}$$

where L_D is the characteristic drift length, in particular, $L_D = R_0$ (R_0 is the large radius of the torus) for the toroidicity-induced drift. This ratio is small for the typical plasma parameters and the waves with $k_\parallel R_0 \sim 1$. The third term in Eq. (2) may considerably contribute to the transverse motion but it tends to zero for the particles with $\mu_p \to 0$. Thus, although $v_\parallel \gg v_D$ for most particles, the first term in Eq. (2) is negligible for the energetic ions interacting with shear Alfvén waves. Therefore, below we will take into account only the transverse part of the wave-induced current of the energetic ions, $\tilde{\mathbf{j}}_\perp^\alpha$.

In order to describe Alfvén eigenmodes in a plasma with energetic ions, we use the following equation for the wave-induced current:

$$\nabla \cdot \left(\tilde{\mathbf{j}}^{MHD} + \tilde{\mathbf{j}}_\perp^\alpha \right) = 0, \tag{4}$$

where $\tilde{\mathbf{j}}^{MHD}$ is the ideal-MHD current, tilde labels perturbed parts of quantities. Using the fact that for the ideal-MHD Alfvén waves the longitudinal components of the electric field ($\tilde{E}_\parallel$) and the magnetic field ($\tilde{B}_\parallel$) vanish, we take $\tilde{\mathbf{A}}_\perp = 0$, where $\mathbf{A}$ is the vector potential of the electromagnetic field. Then $\tilde{\mathbf{E}} = -\nabla_\perp \tilde{\Phi}$ with $\tilde{\Phi}$ the scalar electric potential, which we assume to depend on time as $\exp(-i\omega t)$. All other perturbed quantities can also be expressed in terms of $\tilde{\Phi}$. Taking this into account, we derive a general expression for the growth rate as follows. We multiply Eq. (4) by $\tilde{\Phi}$ and integrate the product over the plasma volume, which yields:

$$\int d^3x \, \tilde{\mathbf{j}}^{MHD} \cdot \nabla \tilde{\Phi} + \int d^3x \, \tilde{\mathbf{j}}_\perp^\alpha \cdot \nabla_\perp \tilde{\Phi} = 0. \tag{5}$$

This equation was derived assuming that $\int d\mathbf{s} \cdot \tilde{\mathbf{j}} \tilde{\Phi} = 0$, where the integral is taken over the plasma boundary. We consider weakly damped Alfvén waves that can be destabilized by a small population of the energetic particles. This makes it possible to apply the perturbation approach to find a solution of Eq. (5). We present the wave frequency as $\omega = \omega_0 + \delta\omega$, where ω_0 is the eigenmode frequency in ideal MHD, $\delta\omega \ll \omega_0$. Then the zero-order approximation yields the equation

$$\int d^3x \, \tilde{\mathbf{j}}^{MHD} \cdot \nabla \tilde{\Phi} = 0, \tag{6}$$

which determines the eigenfrequency ω_0, whereas the first-order terms determine $\delta\omega$. For $\gamma \equiv \mathrm{Im}\omega$ we obtain:

$$\gamma_\alpha = -\mathrm{Im} \frac{\int d^3x \, \tilde{\mathbf{j}}_\perp^\alpha \cdot \nabla_\perp \tilde{\Phi}}{\int d^3x \, (\partial \tilde{\mathbf{j}}^{MHD} / \partial \omega_0) \cdot \nabla \tilde{\Phi}}. \tag{7}$$

For further analysis we expand perturbed quantities ($\tilde{X}$) in Fourier series:

$$\tilde{X} = \sum_{m,n} X_{m,n}(r)\exp(im\vartheta - in\varphi - i\omega t). \tag{8}$$

Here we use the flux coordinates $(x^1, x^2, x^3) = (r, \vartheta\varphi)$ [14,15], $r = r(\psi)$, ϑ, and φ being the radial, poloidal, and toroidal coordinates; ψ is the toroidal magnetic flux; r is defined by $\psi = \bar{B}r^2/2$; $\bar{B}$ is defined by the expression for the equilibrium magnetic field given by

$$B_0 = \bar{B}\left[1 + \sum_{\mu,\nu} \epsilon_B^{(\mu\nu)} \cos\left(\mu\vartheta - \nu N\varphi\right)\right]; \tag{9}$$

N is the number of the field periods along the large azimuth of the torus; the large radius of the torus is defined by $B_3 = \bar{B}R_0$; B_3 is a covariant component of $\mathbf{B}_0$. The normalized Fourier harmonics, $\epsilon_B^{(\mu\nu)}$, are small. In particular, in a Helias reactor [10,11] the dominant harmonic is the mirror one, $\epsilon_m \equiv \epsilon^{(01)} \approx 0.1 > 0$; the other important harmonics are $\epsilon_h \equiv \epsilon^{(11)} < 0$, $\epsilon_t \equiv \epsilon^{(10)} < 0$, and $\epsilon_0 \equiv \epsilon^{(00)} > 0$ (see Fig. 1).

In order to express γ_α in terms of Φ_{nm}, we use the following Alfvén eigenmode equation for the optimized stellarators derived in Ref. [3,4], where we have added a term with $\tilde{\mathbf{j}}_\perp^\alpha$:

$$\sum_{m,n}\left\{\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\omega^2}{v_A^2} - k_{mn}^2\right)\frac{\partial\Phi_{m,n}}{\partial r} - \left(\frac{\omega^2}{v_A^2} - k_{mn}^2\right)\frac{m^2}{r^2}\Phi_{m,n} - \frac{4\pi i\omega}{c^2}\nabla\cdot\tilde{\mathbf{j}}_{\perp m,n}^\alpha \right.$$
$$\left. + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\omega^2}{v_A^2}\sum_{\mu,\nu}\left[O\left(\epsilon_B^{(\mu\nu)}\right) + O\left(\epsilon_g^{(\mu\nu)}\right)\right]\frac{\partial\Phi_{m,n}}{\partial r}\right\}e^{-i\omega t + im\vartheta - in\varphi} = 0, \tag{10}$$

where $v_A = \bar{B}/\sqrt{4\pi n_i M_i}$, $k_{mn} \equiv k_\parallel(m,n) = (m\iota - n)/R_0$ is the longitudinal wavenumber, ι is the rotational transform, $O\left(\epsilon_B^{(\mu\nu)}\right)$ and $O\left(\epsilon_g^{(\mu\nu)}\right)$ represent small terms coupling waves with various mode numbers, $\epsilon_g^{(\mu\nu)}$ are normalized Fourier harmonics of a metric tensor component,

$$g^{11} = 2\psi\delta_0\bar{B}\left[1 + \sum_{\mu,\nu}\epsilon_g^{(\mu\nu)}\cos\left(\mu\vartheta - \nu N\varphi\right)\right]. \tag{11}$$

In Eq. (10) we have neglected some terms containing $k_\parallel$ because $k_\parallel \ll m/r$ and the magnetic shear in the optimized stellarators is small. Because of the presence of parameters $\epsilon_B^{(\mu\nu)}$, $\epsilon_g^{(\mu\nu)}$, Eq. (10) is essentially a set of coupled equations for perturbations with various mode numbers. Therefore, Eq. (10) can be written in a matrix form:

$$\left(\hat{\mathcal{P}}_{ij}^{(0)} + \hat{\mathcal{P}}_{ij}^{(1)}\right) X_j = 0, \tag{12}$$

where X_j is a column containing the elements $\Phi_{m+\mu,n+N\nu}$ arranged in a certain order, μ and ν are integers determined by Eqs. (9), (11); $\hat{\mathcal{P}}_{ij}^{(0)}$ and $\hat{\mathcal{P}}_{ij}^{(1)}$ are the matrix operators describing the ideal-MHD part and the kinetic part of the equation, respectively. Note that the $\hat{\mathcal{P}}_{ij}^{(1)}$ matrix is diagonal, whereas the $\hat{\mathcal{P}}_{ij}^{(0)}$ matrix is non-diagonal, but its non-diagonal elements are small (of the order of ϵ). Multiplying Eq. (12) by X_i^{*T} and applying again the perturbation approach, we arrive at Eq. (7) in the form:

$$\gamma_\alpha = \frac{2\pi}{c^2} \frac{Re \sum_{m,n} \int d^3x \, \mathbf{j}^{\alpha}_{\perp m,n} \cdot \nabla_\perp \Phi^*_{m,n}}{\sum_{m,n} \int d^3x \, v_A^{-2} \left[\left| \Phi'_{m,n} \right|^2 + (m^2/r^2) \left| \Phi_{m,n} \right|^2 \right]}, \tag{13}$$

where

$$\nabla \Phi_{m,n} \equiv \Phi'_{m,n} \mathbf{e}^1 + im\Phi_{m,n} \mathbf{e}^2 - in\Phi_{m,n} \mathbf{e}^3, \tag{14}$$

$\mathbf{e}^1 = \nabla r$, $\mathbf{e}^2 = \nabla \vartheta$, $\mathbf{e}^3 = \nabla \varphi$ are the basis vectors, $\Phi'_{m,n} \equiv \partial \Phi_{m,n}/\partial r$.

III. DERIVATION OF γ_α ASSOCIATED WITH THE CIRCULATING ENERGETIC IONS

We proceed from a collisionless linearized kinetic equation for the distribution function of the energetic ions, f. Its solution can be obtained by the method of characteristics and written as follows:

$$\tilde{f} = -\frac{e}{M} \int_{-\infty}^{t} d\tau \left(\tilde{\mathbf{E}} + \frac{1}{c} \mathbf{v} \times \tilde{\mathbf{B}} \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \tag{15}$$

where f_0 is the equilibrium distribution function, the integral is taken over time, the space coordinates $\mathbf{r} = \mathbf{r}(t)$ and the velocity $\mathbf{v} = \mathbf{v}(t)$ in the integrand are determined by the equation of the particle orbital motion. We are interested in well-circulating particles for which $v_\parallel \approx$ const. In addition, we neglect effects of the particle Larmor rotation, which is justified when $k_\perp \rho_L \ll 1$ (ρ_L is the Larmor radius) and the width of the region where the Alfvén mode is localized exceeds the Larmor radius of the energetic ions. Then the constants of motion approximately coincide with those in axisymmetric magnetic configurations. They are the particle energy ($\mathcal{E}$), the magnetic moment (μ_p), and the canonical angular momentum (J), which we take in the form $J = \psi_p - v_\| B_3/\omega_B$, where ψ_p is the poloidal magnetic flux. Taking this into account, we write Eq. (15) as

$$\tilde{f} = -\frac{e}{M}\hat{\Pi} f_0 \int_{-\infty}^{t} d\tau\,(\mathbf{v}\cdot\tilde{\mathbf{E}}) + \frac{c}{i\omega}\frac{\partial f_0}{\partial J}\tilde{E}_3, \tag{16}$$

where

$$\frac{1}{M}\hat{\Pi} = \frac{\partial}{\partial\mathcal{E}} + \frac{cn}{e\omega}\frac{\partial}{\partial J}, \tag{17}$$

M is the particle mass. In many cases of practical importance the pitch-angle variable $\lambda = \mu_p\bar{B}/\mathcal{E}$ is more convenient than μ. Indeed, when the fast ion energy is sufficiently high, $\mathcal{E} \gg (M_i/M_e)^{1/3}T_e$, the dominant effect of Coulomb collisions on fast ions is the slowing down conserving the particle pitch-angle. In addition, particle orbit width is typically small, which justifies the use of r instead of J. Therefore, below we assume that $f_0 = f_0(\mathcal{E},\lambda,r)$. Then we can present the operator $\hat{\Pi}$ as follows:

$$\hat{\Pi} = M\frac{\partial}{\partial\mathcal{E}} - \frac{M\lambda}{\mathcal{E}}\frac{\partial}{\partial\lambda} + \frac{n}{\omega\omega_B\iota}\frac{1}{r}\frac{\partial}{\partial r}. \tag{18}$$

We are interested only in the resonance part of $\tilde{f}$. For this reason, we omit the second term in Eq. (16). Then, allowing for the fact that $\mathbf{E} = -\nabla_\perp\Phi$ and writing $\mathbf{v} = \mathbf{v}_\| + \mathbf{v}_D$, we obtain:

$$\tilde{f}(t) = \frac{e}{M}\hat{\Pi} f_0 \int_0^\infty dt'\,\mathbf{v}_D(\tau)\cdot\nabla\tilde{\Phi}(\tau)|_{t-t'}. \tag{19}$$

Now we expand the perturbed quantities into the Fourier series according to Eq. (8). In addition, we use the Fourier expansion for the drift velocity:

$$\mathbf{v}_D = \sum_{p,s}\mathbf{u}_{ps}(r)e^{ip\vartheta+is\varphi}, \tag{20}$$

where p and s are integers. Taking $r(t) = \text{const}$ in Eqs. (8), (20), we approximate the time dependence of the particle angular coordinates as follows:

$$\vartheta(\tau) \approx \vartheta(t) + \omega_\vartheta(\tau - t), \quad \varphi(\tau) \approx \vartheta(t) + \omega_\varphi(\tau - t). \tag{21}$$

Here ω_ϑ and ω_φ are the frequencies of the poloidal and toroidal particle rotation, respectively, which are approximately conserved. Due to this fact, we can easily calculate the integral in Eq. (19). As a result, we obtain the following expression for a Fourier harmonic of the distribution function:

$$f_{m,n} = -\frac{ie}{M}\hat{\Pi} f_0 \sum_{p,s} \frac{(\mathbf{u}_{ps} \cdot \nabla\Phi_{m,n})}{\Omega^{ps}_{mn}} e^{ip\vartheta + is\varphi}, \tag{22}$$

where

$$\Omega^{ps}_{mn} = \omega - (p+m)\omega_\vartheta - (s-n)\omega_\varphi. \tag{23}$$

Noting that the energetic particle current is

$$\mathbf{j}^\alpha_{m,n} = e\int d^3v\,(\mathbf{v}_\| + \mathbf{v}_D) f_{m,n}, \tag{24}$$

where $d^3v = (2\pi/M^2) d\mathcal{E} d\mu_p B/v_\|$, we can calculate the numerator of Eq. (13). First of all, we find the angle-averaged product $\mathbf{j}^\alpha_{\perp m,n} \cdot \nabla_\perp \Phi^*_{m,n}$. Using Eqs. (22) and (24), we obtain:

$$\langle \mathbf{j}^\alpha_{\perp m,n} \cdot \nabla_\perp \Phi^*_{m,n} \rangle = -\frac{ie^2}{M}\sum_{p,s}\int d^3v\,\hat{\Pi} f_0 |\mathbf{u}_{p,s} \cdot \nabla\Phi_{m,n}|^2 \frac{1}{\Omega^{ps}_{mn}}, \tag{25}$$

where $\langle(...)\rangle \equiv \int d\vartheta\, d\varphi\,(...)/(2\pi)$. When deriving Eq. (25), we used the fact that $u_{-p,-s} = u^*_{p,s}$. Note that $1/\Omega^{ps}_{mn}$ in the integrand in this equation has both real and imaginary parts:

$$\frac{1}{\Omega^{ps}_{mn}} = \frac{P}{\Omega^{ps}_{mn}} - i\pi\delta(\Omega^{ps}_{mn}), \tag{26}$$

where P means the principal value. The imaginary part in Eq. (26) determines the resonance condition for the wave-particle interaction, which can be written for the well-circulating particles as

$$\omega - k^{ps}_{mn} v_\| = 0, \tag{27}$$

where

$$k^{ps}_{mn} = [(m+p)\iota - n + s]R_0^{-1} \tag{28}$$

is the effective longitudinal wavenumber. Substituting Eq. (25) to Eq. (13), we obtain:

$$\gamma_\alpha = \frac{2\pi^2 e^2}{c^2 M} \frac{\sum_{m,n} \int dr\, r \int d^3v\, \hat{\Pi} f_0 \sum_{p,s} \left|\mathbf{u}_{ps}\cdot\nabla\Phi_{m,n}\right|^2 \delta(\Omega^{ps}_{mn})}{\sum_{m,n} \int dr\, r v_A^{-2} \left[\left|\Phi'_{m,n}\right|^2 + (m^2/r^2)\left|\Phi_{m,n}\right|^2\right]}. \qquad (29)$$

Here the velocity integral should be taken over the region corresponding to the circulating particles.

This expression can be considerably simplified for well-localized modes, for which the $\Phi'_{m,n}$ terms in the numerator and denominator dominate. Assuming that there are only two essential terms in the sum over m,n in Eq. (29)) [i.e., only two waves are coupled in Eq. (10)], we find:

$$\gamma_\alpha = \frac{2\pi^2 e^2 v_A^2(r_l)}{Mc^2} \int d^3v\, \hat{\Pi} f_0 \sum_{p,s} \left|u^1_{ps}\right|^2 \delta(v_\parallel - v^r_\parallel)/|k^{ps}_{mn}|. \qquad (30)$$

where $v^r_\parallel$ is the resonance longitudinal velocity, $v^r_\parallel = \omega/k^{ps}_{mn}$, r_l is the radius around which the mode is localized. This equation is valid provided that the resonant particles with both $v^r_\parallel > 0$ and $v^r_\parallel < 0$ equally contribute, which is the case when $f_0(v_\parallel)$ is an even function. The growth rate associated with non-balanced injection is less approximately by a factor of 2. Note that Eq. (30) does not contain the wave amplitudes and can be also obtained in the local approach.

One can see that Eqs. (29), (30) are valid also for the description of the instability of the eigenmodes with the frequency just below the cylindrical Alfvén continuum with given mode numbers m and n, so-called "global Alfvén eigenmodes" (GAE). In this case, $\sum_{m,n}$ in Eqs. (29), (30) contains the only term.

IV. ALFVÉN INSTABILITIES THAT CAN BE DRIVEN BY ENERGETIC IONS IN OPTIMIZED STELLARATORS

In order to analyze the derived expression for γ_α, we have to find the dominant Fourier harmonics of the drift velocity. We proceed from the expression $\mathbf{v}_D = \omega_B^{-1}\mathbf{b}\times\left(v_\parallel^2\mathcal{K} + \mu_p\nabla B_0/M\right)$, where $\mathbf{b} = \mathbf{B}_0/B_0$, $\mathcal{K} = B_0^{-2}\nabla_\perp\left(B_0^2/2 + 4\pi p_0\right)$ is the magnetic field line curvature, p_0 is the plasma pressure. Taking into account that the plasma current in the optimized stellarators vanishes, $|B|^2\sqrt{g} = B_3$ (g is the determinant of the metric tensor), and neglecting the terms of the order of ϵ^2, we obtain:

$$
\begin{aligned}
v_D^1 &= \frac{\left(v^2+v_\parallel^2\right)\bar{B}}{2\omega_B}\sum_{\mu,\nu}\epsilon^{(\mu\nu)}\mu\sin(\mu\vartheta-\nu N\varphi),\\
v_D^2 &= \frac{\left(v^2+v_\parallel^2\right)\bar{B}}{2\omega_B}\sum_{\mu,\nu}\epsilon^{(\mu\nu)\prime}\cos(\mu\vartheta-\nu N\varphi)+\frac{4\pi v_\parallel^2}{\omega_B B}\frac{\partial p_0}{\partial\psi},\\
v_D^3 &\approx 0.
\end{aligned}
\tag{31}
$$

The term with p_0 in v_D^2 can be excluded due to the fact that the plasma pressure weakly affects the field line curvature:

$$
v_D^2 = \frac{\left(v^2+v_\parallel^2\right)\bar{B}}{2\omega_B}\sum_{\mu\cdot\nu\neq 0}\epsilon^{(\mu\nu)\prime}\cos(\mu\vartheta-\nu N\varphi)+\frac{v_\perp^2\bar{B}}{2\omega_B}\frac{\partial\epsilon_0}{\partial\psi},
\tag{32}
$$

where ϵ_0 is the diamagnetic Fourier harmonic of the magnetic field strength. Comparing Eqs. (31) and (32) with Eq. (8) and proceeding from ψ to the radial coordinate r, we find:

$$
\begin{aligned}
u_{ps}^1 &= -\frac{i(v^2+v_\parallel^2)}{4r\omega_B}p\sum_{\mu,\nu}\epsilon_B^{(\mu\nu)}(\delta_{p,\mu}\delta_{s,-\nu N}+\delta_{p,-\mu}\delta_{s,\nu N}),\\
u_{ps}^2 &= \frac{(v^2+v_\parallel^2)}{4r\omega_B}\sum_{\mu\cdot\nu\neq 0}\epsilon_B^{(\mu\nu)\prime}(\delta_{p,\mu}\delta_{s,-\nu N}+\delta_{p,-\mu}\delta_{s,\nu N})+\frac{v_\perp^2\epsilon_0'}{2\omega_B r}\delta_{p,0}\delta_{s,0},\\
u_{ps}^3 &= 0,
\end{aligned}
\tag{33}
$$

where $\epsilon' = \partial\epsilon/\partial r$, δ_{ij} is the Kronecker symbol. Using Eq. (33), we can calculate $|\mathbf{u}_{ps}\cdot\nabla\Phi_{m,n}|^2$ in the numerator of Eq. (29):

$$
\begin{aligned}
|\mathbf{u}_{ps}\cdot\nabla\Phi_{m,n}|^2 = &\sum_{\mu\cdot\nu\neq 0}\frac{\left(|v|^2+v_\parallel^2\right)^2}{16r^2\omega_B^2}\left|p\epsilon^{(\mu\nu)}\Phi_{m,n}'-m\epsilon^{(\mu\nu)\prime}\Phi_{m,n}\right|^2\\
&\times(\delta_{p,\mu}\delta_{s,-\nu N}+\delta_{-p,\mu}\delta_{-s,-\nu N})\\
&+\frac{m^2v_\perp^4\epsilon_0^{\prime 2}}{4r^2\omega_B^2}|\Phi_{m,n}|^2\delta_{p,0}\delta_{s,0}.
\end{aligned}
\tag{34}
$$

It follows from Eqs. (34), (29) that in the optimized stellarators the energetic particles drive the instability through the following resonance:

$$
\omega-(m+j\mu)\omega_\vartheta+(n+j\nu N)\omega_\varphi=0,
\tag{35}
$$

where $j=0,\pm 1$. This resonance condition can be written in a form more convenient for analysis as

$$
\omega-(k_{mn}+2jk^{\mu\nu})v_\parallel=0,
\tag{36}
$$

where $k^{\mu\nu} = (\mu\iota - \nu N)/(2R_0)$. The magnitude of $k^{\mu\nu}$ at a certain point r_* is essentially the longitudinal wavenumber for which two cylindrical wave branches, $\omega_1(r) = |k_{m,n}|v_A$ and $\omega_2(r) = |k_{m+\mu,n+\nu N}|v_A$ intersect provided that these branches intersect at the point r_*. But we should note that the numbers μ, ν in Eqs. (33)-(36) come from the dependence of $\mathbf{v}_D$ on B_0 and, thus, they have nothing to do with the branch crossing. The latter determines approximate magnitudes of the frequency of various Alfvén eigenmodes, namely, $\omega = |k_*^{\mu_0,\nu_0}|v_{A*}$ with μ_0 and ν_0 the coupling numbers that determine the kind of AEs, i.e., TAEs (toroidicity-induced Alfvén eigenmodes), HAEs, MAEs etc. If we approximate ω in this way and take $k_{mn} \approx k_*^{\mu_0,\nu_0}$, $\iota \approx \iota_*$, we find that the resonance longitudinal velocity, $|v_\parallel^r|$, coincides with that in tokamaks for a special case of $\mu_0 = \mu$, $\nu_0 = \nu$, being equal to v_{A*} and $v_{A*}/3$. Note that the small magnetic shear in optimized stellarators justifies the used approximation even for modes which a rather wide region of localization; on the other hand, relatively strong multi-mode coupling, first of all, large magnitude of $\epsilon_g^{(21)}$ [3,4], deteriorates the applicability of the used approximation.

Equation (29) and Eqs. (34)–(18) determine γ_α in the optimized stellarators with taking into account the finite width of the mode location. We can draw certain conclusions from these equations. First, there exist resonances which are absent in tokamaks (note that one usually uses the only sideband resonance, $\omega = [1 \pm \iota/(k_\parallel R_0)]k_\parallel v_\parallel$, for the description of the interaction of circulating ions and the waves in both tokamaks and stellarators). Further, several Fourier harmonics of the magnetic field strength, rather than only ϵ_t as in tokamaks, contribute to the growth rate (but $\epsilon_g^{(\mu\nu)}$ does not contribute). Moreover, because ϵ_t is relatively small in optimized stellarators, its role can even be negligible. However, the magnitude of $\epsilon^{(\mu\nu)}$ is not the only factor determining the growth rate. Another important factor is the number of the resonant particles, which is different for different μ, ν and determined by the velocity distribution function of the energetic ions. The latter is typically characterized by sharp decrease at a certain velocity. In particular, the distribution function of alpha particles almost vanishes at $v > v_\alpha$, where $v_\alpha = \sqrt{2\mathcal{E}_\alpha/M_\alpha}$, $\mathcal{E}_\alpha = 3.5MeV$. Therefore, alphas interact with waves only when $v_\alpha > v_r \approx v_\parallel^r/\sqrt{1-\lambda}$. The resonance curves $v_r(\lambda)$ for a Helias reactor are presented in Fig. 2.

Let us consider the case of strongly localized waves. Using Eqs. (30), (34) and taking $\omega = |k_{mn}|v_{A*}$, we find:

$$\gamma_\alpha = \frac{\pi^2 M_\alpha v_{A*}^2}{8\omega \bar{B}^2 r^2} \sum_{\mu,\nu,j} \mu^2 |\epsilon^{(\mu\nu)}|^2 |v_\parallel^r| \int d^3v\, \delta(v_\parallel - v_\parallel^r)(|v|^2 + v_\parallel^2)^2 \hat{\Pi} f_0, \tag{37}$$

where $j = \pm 1$, $\mu = 1$, $\nu = 0, 1$, and the resonance velocity is given by Eq. (36).

The latter yields for the "gap" modes:

$$v_\parallel^r = v_{A*} \left(1 + 2j \frac{\iota_* - \nu N}{\mu_0 \iota_* - \nu_0 N}\right)^{-1} \mathrm{Sgn} k_{mn}. \tag{38}$$

When deriving Eq. (38), we approximated the mode location radius by $\iota_* = (2n + \nu_0 N)/(2m + \mu_0)$. We observe that two Fourier harmonics of the magnetic field, ϵ_t and ϵ_h, contribute to γ_α when the modes are localized. The resonance magnitudes of $v_\parallel$ may coincide for the $\nu = 1$ resonance and the usual tokamak resonance with $\mu = 1$, $\nu = 0$. One can see that it will be the case when $\mu_0 = 0$, $\nu_0 = 1$ (MAE modes) and $\mu_0 = 2$, $\nu_0 = 1$ (HAE_{21} modes). On the other hand, non-coinciding $v_\parallel^r$ are less than the coinciding ones, except for one of them for HAE_{21}, which, however, is too high to interact with $3.5 MeV$ alpha particles in a Helias reactor. Therefore, one can expect that the $\mu = \nu = 1$ resonance will be responsible for the well-localized MAE and HAE_{21} instabilities.

For the localized GAE modes (when the name "global" is hardly justified, which is the case for $m > 1$, $n > 1$), we obtain from Eq. (36):

$$v_\parallel^r = v_{Al} \left(1 \pm \frac{(\mu \iota_l - N\nu) r_l}{2 m s_l L_{\rho_l} \iota_l}\right)^{-1} \mathrm{Sgn} k_{mn}, \tag{39}$$

where s is the magnetic shear, $L_\rho = -\partial\rho/\partial r$ with ρ the plasma mass density. Equation (39) implies that the mode frequency is close to a minimum of $|k_\parallel| v_A(r)$ and the modes have the radial distribution peaked near the mentioned minimum.

In the case of isotropic velocity distribution of the energetic ions, Eq. (37) can be written as follows:

$$\gamma_\alpha = \frac{3\pi\beta_\alpha}{64 k_{mn}^2 r^2} \sum_{\nu=0,1;j=\pm 1} |\epsilon^{(1\nu)}|^2 \frac{w \int_w^{w/\sqrt{\epsilon_{eff}}} du\, u(u^2 + w^2)^2 (\omega\partial/\partial u^2 + \omega_d) f_0}{\int_0^\infty du\, u^4 f_0}, \tag{40}$$

where the dimensionless quantities $w = |v_\parallel^r|/v_0$ and $u = v/v_0$ have been introduced; v_0 is the characteristic velocity of the energetic ions; ϵ_{eff} is an effective magnitude of Fourier harmonics of B_0, such that a particle is circulating when $|v_\parallel|/v > \sqrt{\epsilon_{eff}}$ ($\epsilon_{eff} \approx \epsilon_m$ for the core localized modes); $\omega_d = -cn\mathcal{E}_0/(e_\alpha B\iota r L_\alpha)$ is the diamagnetic drift frequency of the energetic ions; $\mathcal{E}_0 = M_\alpha v_0^2/2$; L_α is the characteristic length of the inhomogeneity of the energetic ion density (typically $L_\alpha^{-1} \approx -d\ln n_\alpha/dr$, where n_α is the density of the energetic ions); $\beta_\alpha = 8\pi p_\alpha/\bar{B}^2$; p_α is the pressure of the energetic ions defined as follows:

$$p_\alpha = \int d^3v \frac{Mv^2}{3} f_0. \tag{41}$$

It follows from Eq. (40) that the instability can arise when the term with ω_d dominates, destabilized modes being characterized by $n < 0$ for p_α decreasing with radius.

In order to find the magnitude of the growth rate for various kinds of destabilized AEs, we have to specify the distribution function of the energetic particles. Let us take the Maxwellian distribution, $f_0 \propto exp(-u^2)$, for which γ_α can be easily found analytically. We obtain:

$$\gamma_\alpha = \frac{\sqrt{\pi}\beta_\alpha}{8k_{mn}^2 r^2}(\omega_d - \omega) \sum_{j=\pm 1} \left[\epsilon_t^2 F(w_t^{(j)}) + \epsilon_h^2 F(w_h^{(j)})\right], \tag{42}$$

where

$$F(w) = w[1 + 2w^2 + 2w^4]e^{-w^2} - w\left[1 + w^2\left(1 + \epsilon_{eff}^{-1}\right) + 0.5w^4\left(1 + \epsilon_{eff}^{-1}\right)^2\right] e^{-w^2\epsilon_{eff}^{-1}}, \tag{43}$$

$$w_t^{(j)} = \frac{v_{A*}}{v_0|1 + j\iota_*/(m\iota_* - n)|}, \quad w_h^{(j)} = \frac{v_{A*}}{v_0|1 + j(\iota_* - N)/(m\iota_* - n)|}. \tag{44}$$

Note that when the particle density profile is flat and the magnetic shear is small (which is the case in the optimized stellarators), the resonance velocities, w_t and w_h weakly depend on the mode numbers and are determined mainly by the kind of AEs. To see it, we write $|m\iota_* - n| = |\nu_0 N - \mu_0 \iota_*|/2$, see also Eq. (38).

The Maxwellian distribution function of the energetic ions was used to calculate the growth rate of the TAE instability in tokamaks in Ref. [16] and is often used to analyze destabilized AE modes in stellarators. However, this is not always justified for two main reasons. First, the velocity distribution can be strongly anisotropic (e.g., during neutral beam injection); therefore, AEs can be destabilized even when the ω_d term is small [17].

Second, as was noted above, typically the energy distribution sharply drops at a certain energy $\mathcal{E}_\alpha$, which is a stabilizing factor. One can see that the latter prevents the ω_d-driven instabilities when $v_\parallel^r$ is close to v_A. Indeed, for $w/\sqrt{\epsilon_{eff}} > 1$, where we have taken $v_0 = v_\alpha \equiv (2\mathcal{E}/m)^{1/2}$, the velocity derivative in Eq. (40) results in a term with $\delta(v - v_\alpha)$, which dominates when $w \longrightarrow 1$ (i.e., when $v_\parallel^r \to v_\alpha$). Below we will consider two examples in which the energetic ions are characterized by $f_0 = 0$ for $\mathcal{E} > \mathcal{E}_\alpha$.

Let us analyze the destabilization of Alfvén eigenmodes by energetic alpha particles with the distribution function formed by Coulomb collisions,

$$f_0 \propto \frac{\eta(v_\alpha - v)}{v^3 + v_c^3}, \tag{45}$$

where $\eta(x) = \int_{-\infty}^{x} dx\, \delta(x)$, $v_c = \sqrt{2\mathcal{E}_c/M_i}$ with $\mathcal{E}_c \sim (M_i/M_e)^{1/3}$. One can see from Eq. (40) that in this case a necessary condition of the instability is $\omega_d > 1.5\omega$ [we use the equation $\partial \ln f_0/\partial u^2 = -1.5u/(u^3 + u_c^3)$ and take $v_0 = v_\alpha$]. This condition is satisfied, first of all, for low-frequency AEs, e.g., TAE modes. However, both w_t and w_h are rather low for TAEs. Therefore, the actual threshold value of ω_d/ω required for γ_α being positive for these modes may be considerably higher because $v_\parallel^r \le v_A \sim v_c$ in a reactor.

The growth rate for f_0 given by Eq. (45) can be calculated analytically, but the expression obtained this way is not convenient for analysis, especially when $v_\parallel^r$ is about v_c. Therefore, we apply a numerical approach. We consider a Helias reactor with $n_i(r) = 2.6 \times 10^{14}/[1 + (x/x_n)^{10}]\,\mathrm{cm}^{-3}$, $T_i(r) = 15/[1 + (x/x_T)^4]\,\mathrm{keV}$, and take $n_\alpha(r) \sim n_i^2\langle\sigma v\rangle$, where $\langle\sigma v\rangle \propto T_i^2$, $x = r/a$, $x_n = 0.7$, $x_T = 0.5$, which yields the alpha profile and L_α shown in Fig. 3. Then we calculate the ratio ω_d/ω for various AEs localized in the plasma core, using the fact that ι weakly depends on r, see Fig. 4. Finally, we calculate the instability growth rate and the threshold magnitudes of $(\omega_d/\omega)_{cr}$ for which $\gamma_\alpha = 0$. To clarify the role of the resonance associated with the presence of the helical harmonic in the magnetic field, ϵ_h, we also make calculations, omitting the terms with this harmonic, i.e., keeping only the "standard" terms associated with the toroidicity; the growth rate calculated in the approximation of conventional resonance (i.e., without the term proportional to ϵ_h^2) is labeled by the superscript "0". The results of the calculations for various gap modes are given in Tables I and II. Table I contains the results obtained for $\omega_d/\omega = 10$. This magnitude exceeds the threshold magnitudes, which are also presented in Table I [$3.7 < (\omega_d/\omega)_{cr} < 5.7$]. The chosen magnitude of ω_d/ω corresponds to, e.g., TAE-modes with $n \sim 1$ and other AEs with higher n localized at $r/a \sim 0.3$ in a Helias reactor, see Fig. 4. Data in Table II are relevant to various AEs with given mode numbers localized at $r/a \sim 0.3$. The mode numbers were taken, using the $\iota(r)$ profile of the Helias reactor given in Fig. 1. It follows from Tables I and II that the "non-conventional" term associated with the helicity strongly affects the growth rate of all AEs except for TAE and HAE_{22}, being dominant in the case of the MAE and HAE_{21} instabilities. On the other hand, it only weakly changes $(\omega_d/\omega)_{cr}$. The MAE and TAE instabilities have the largest growth rates, whereas the EAE and HAE_{21} instabilities are characterized by the smallest ones. Note that the threshold magnitude of $(\omega_d/\omega)_{cr}$ is minimum for the MAE instability. The results of calculations relevant to the GAE instability are shown in Table III. We observe that for certain modes the growth rate of GAE is very large and significantly exceeds that of the gap modes. Non-conventional resonances have strong influence on the GAE instability except for the mode with $n/m = -4/-4$. The considered GAE modes are localized in the region around r_l/a shown in Fig. 5.

In order to demonstrate effects of anisotropy, we assume that the distribution function of energetic ions is strongly anisotropic and take it in the form:

$$f_0 = \frac{\eta(v_\alpha - v)}{v^3}\delta(\chi - \chi_\alpha), \tag{46}$$

where $\chi = v_\parallel/v$ is the pitch angle, $\chi^2 \approx 1 - \lambda$, and $|\chi_\alpha| > \sqrt{\epsilon_{eff}}$. Equation (46) is justified when $|v_\parallel^r \chi_\alpha|^3 \gg v_c^3$ (then the energy of the ions interacting with the waves well exceeds $\mathcal{E}_c$) or at the initial phase of the NBI (neutral beam injection) heating, i.e., for $\Delta t \ll \tau_s$, where Δt is the time interval since the start of NBI, τ_s is the slowing down time of the energetic ions. Using Eqs. (37), (41), and (46), we obtain for the case of $|\chi_\alpha| > w$ ($v_\parallel^r/\chi_\alpha < v_\alpha$):

$$\gamma_\alpha = \frac{3\pi}{32}\frac{\beta_\alpha}{k_\parallel^2 r^2} \sum_{\nu=0,1;j=\pm1} |\epsilon^{(1\nu)}|^2 \left[\frac{3}{\chi_\alpha^4} - \frac{2}{\chi_\alpha^2} - 5 + \frac{2\omega_d}{\omega}\left(1 + \frac{1}{\chi_\alpha^2}\right) w^2\right], \tag{47}$$

where w is given by Eq. (44) for the gap modes and Eq. (39) for GAEs. When $|\chi_\alpha| < w$, $\gamma_\alpha = 0$ because there are no energetic ions interacting with the waves; when $|\chi_\alpha| = w$, the waves are strongly damped (formally, $\gamma_\alpha \to -\infty$). The first term in Eq. (47) drives the instability caused by the velocity anisotropy. It overrides the second and third ones when $\chi_\alpha^2 < 0.6$, c.f. Ref. [17]. This implies that $\gamma_\alpha > 0$ for $w < |\chi_\alpha| < 0.77$ and arbitrary $\omega_d \geq 0$. As an example, we take $\chi_\alpha^2 = 0.5$, $w = v_A/v_\alpha = 1/3$. Then we find that the ratio of the last term to the first one in Eq. (47) equals to $\omega_d/(18\omega)$, from which it follows that the anisotropy can essentially contribute to the instability growth rate and even be the main factor destabilizing AE-modes.

V. EFFECT OF THE DESTABILIZED ALFVÉN EIGENMODES ON ALPHA PARTICLES IN A HELIAS REACTOR

It is of interest to predict possible influence of Alfvén instabilities (AI) on α-particles in a Helias reactor. This can be done by studying various destabilizing and stabilizing mechanisms, determining the spatial structure and the wave numbers of destabilized waves, and developing a nonlinear theory of the instabilities. There is, however, another way which is simple but sufficient to evaluate the *maximum possible* energy loss of alphas caused by AIs. This way is based on the assumption that the amplitudes of the destabilized waves are so large that all the energetic ions which enter the resonance region in the velocity space are lost from the AI localization region. Based on the mentioned assumption, alpha losses in tokamaks were evaluated in Refs. [12,18]. Here we will apply this approach to evaluate the alpha loss and the energy of the particles affected by the destabilized AEs in a four-period Helias reactor [10,11]. We restrict ourselves to studying the influence of AIs on the circulating particles and a certain group of transitioning particles ("quasi-circulating" particles), which constitute the majority of the energetic alpha population. In this case we can use the results obtained above. Let us consider the resonance curves shown in Fig. 2. Taking into account that most important resonances are associated with the helicity and toroidicity (see previous section) we consider only the curves which correspond to these resonances. We observe that for most pitch angles each toroidicity-induced resonance curve lies either higher or coincides with the corresponding helicity-induced one. Therefore, we evaluate alpha losses in an approximation which describes only losses caused by the toroidicity-induced resonance (although, as was shown above, helicity can be the main factor driving the instability). This approximation considerably simplifies the analysis and, on the other hand, will give a reasonable estimates of the energy loss. Thus, we assume that the destabilized waves affect alpha particles through the following resonance:

$$\omega \equiv |k_{\parallel}|v_A = \left(k_{\parallel}R\iota^{-1} \pm 1\right)\omega_\theta \ , \tag{48}$$

where

$$k_{\parallel} \approx (N\nu_0 - \mu_0\iota_*)\frac{v_{A*}}{2R_0}, \quad \iota_* = \frac{2n+\nu_0 N}{2m+\mu_0}. \tag{49}$$

The frequency of the poloidal motion of the particles equals to $\iota v_{\parallel}/R_0$ for the well-circulating particles (which was used in previous sections) and tends to zero for the marginally circulating ones. Therefore, the resonance particle velocity, v_r, determined by Eq. (48) is a monotonically growing function of the pitch-angle variable λ in the region $(0, \lambda_{max})$, where λ_{max} is the maximum possible magnitude of λ for the particles interacting with the waves through the considered resonance. Taking into account that λ is approximately conserved during the particle slowing down, we can write the fraction of the energy lost from a given flux surface as [12]:

$$\nu_\epsilon = \frac{W_-(r)}{W_+(r)} = \frac{\int_{\Delta v,\Delta\lambda} \overline{d^3v}\, S_\alpha v_r^2}{\int \overline{d^3v}\, S_\alpha v^2} \ , \tag{50}$$

where $W_+(r)$ and $W_-(r)$ are the produced and lost power at the flux surface with the radius r, correspondingly; $S_\alpha \equiv S_\alpha(r, \mathbf{v})$ represents the source of the energetic particles [in particular, $S_\alpha = n_i^2\langle\sigma v\rangle\delta(v - v_\alpha)/4$ for α-particles]; $n_i^2\langle\sigma v\rangle/4$ characterizes the rate of alpha production; n_i is the bulk ion density; the bar means the flux surface averaging. The integral in the numerator of Eq. (50) is taken over the region in the (v, λ) space from which the energetic ions can reach the resonance curve $v_r(\lambda)$ during the collisional slowing down. If we take into account only the ϵ_m and ϵ_0 harmonics, we can easily find:

$$|\omega_\theta| = \frac{2\pi}{\tau_\theta}, \quad \tau_\theta = \frac{4R\mathrm{K}(\kappa^{-1})}{\iota v\kappa\sqrt{2\lambda_l\tilde{\epsilon}_m}}, \tag{51}$$

where $\mathrm{K}(\kappa^{-1})$ is the complete elliptic integral of the first kind, κ is the trapping parameter ($\kappa > 1$) given by

$$\kappa^2 = \frac{\lambda^{-1} - \epsilon_0 + \epsilon_m - 1}{2\epsilon_m} = \frac{\lambda_l^{-1}(r) + \tilde{\epsilon}_m - 1}{2\tilde{\epsilon}_m}, \tag{52}$$

$\lambda_l(r) = \mu\hat{B}(r)/\mathcal{E}$ is the local pitch-angle parameter, $\hat{B}(r) = \bar{B}[1+\epsilon_0(r)]$ is the average magnetic field at the flux surface with the radius r, $\tilde{\epsilon}_m = \epsilon_m/(1+\epsilon_0)$. Equation (51) is valid for $\lambda_l \leq (1+\tilde{\epsilon}_m)^{-1}$. It follows from Eq. (52) that the plasma diamagnetism leads to decrease of κ, and, because $d\epsilon_0(r)/dr > 0$, it can result in the transformation of a circulating particle moving outwards into a trapped one.

Taking into account more Fourier harmonics complicates the picture. Indeed, the maximum magnetic field strength is $B_{max} = B(\theta = \pi, \phi = 0) = \hat{B}(1 + \tilde{\epsilon}_m + |\tilde{\epsilon}_h| + |\tilde{\epsilon}_t|)|$. This implies that the circulating particles are characterized by $\lambda_l \leq (1+\tilde{\epsilon}_m+|\tilde{\epsilon}_h|+|\tilde{\epsilon}_t|)^{-1}$ ($\tilde{\epsilon}_{h,t} = \epsilon_{h,t}/(1+\epsilon_0)$), i.e., adding the helical and toroidal Fourier harmonics reduces the region of circulating particles. But, on the other hand, a particle moving along a field line can pass through the point $(\theta = \pi, \phi = 0)$ only when a part of the field line passed by the particle for the considered time goes through that point. Otherwise, the particles with $\lambda_l = (1+\tilde{\epsilon}_m+|\tilde{\epsilon}_h|+|\tilde{\epsilon}_t|)^{-1}$ will not be reflected. Furthermore, a particle moving through $(\theta = 0, \phi = 0)$ will not be reflected even when $\lambda_l \leq (1+\tilde{\epsilon}_m - |\tilde{\epsilon}_h| - |\tilde{\epsilon}_t|)^{-1}$. Therefore, the resonance (48) can be applicable also to particles with $1 + \tilde{\epsilon}_m + |\tilde{\epsilon}_h| + |\tilde{\epsilon}_t| \leq \lambda_l \leq (1+\tilde{\epsilon}_m - |\tilde{\epsilon}_h| - |\tilde{\epsilon}_t|)^{-1}$. These are transitioning particles. Eventually they are reflected either due to the motion along the field line or due to precession. After the reflection they can become locally trapped particles or remain locally passing ones. Equation (48) is applicable not to all locally passing particles but only to those ones which have $\tau_\theta \ll \tau_b$, where τ_b is the bounce period. The latter can be referred to as quasi-circulating particles.

We calculate the energy losses of (i) circulating particles; (ii) circulating plus quasi-circulating particles. Correspondingly, we take $0 \leq \lambda_l \leq (1+\tilde{\epsilon}_m+|\tilde{\epsilon}_h|+|\tilde{\epsilon}_t|)^{-1}$ and $0 \leq \lambda_l \leq (1+\tilde{\epsilon}_m - |\tilde{\epsilon}_h| - |\tilde{\epsilon}_t|)^{-1}$. In the latter case we overestimate the influence of AIs on alphas through the resonance given by Eq. (48) because this resonance is valid not for all the transitioning particles. To calculate ω_θ, we use Eqs. (51) and (52) with $\tilde{\epsilon}_m$ replaced by the effective magnitude $\epsilon_{eff} = \tilde{\epsilon}_m + |\tilde{\epsilon}_h| + |\tilde{\epsilon}_t|$ or $\epsilon_{eff} = \tilde{\epsilon}_m - |\tilde{\epsilon}_h| - |\tilde{\epsilon}_t|$, respectively.

At first, let us consider the case when a resonance curve lies close to v_α. Then marginally circulating particles are not affected by the waves; therefore [12],

$$\nu_\epsilon = \frac{v_{r0}^2}{2v_\alpha^2} \int_0^{\lambda_{max}} \frac{d\lambda_l}{(1-\lambda_l)^{3/2}} = \frac{v_{r0}}{v_\alpha}\left(1 - \frac{v_{r0}}{v_\alpha}\right), \tag{53}$$

where $v_{r0} \equiv v_r(\lambda_l = 0)$, $\lambda_{max} = 1 - v_{r0}^2/v_\alpha^2$. The loss fraction given by Eq. (53) cannot exceed 25% and tends to 0 when $v_{r0} \to v_\alpha$.

Now we calculate ν_ϵ numerically. The results are presented in Figs. 6, 7. It follows from these figures that the destabilization of MAE modes would be the most dangerous, as it could result in the loss of about 35% of the whole α-particle energy. Uncertainties associated with the behavior of transitioning particles become essential for $r/a \geq 0.4$, where the differences between the losses shown in Figs. 6 and 7 constitute about 5%. The smallest of the shown losses is associated with the destabilization of TAE modes; however, this loss is still considerable (more than 20%). On the other hand, the HAE_{21} modes, which are located in the largest frequency gap of the optimized stellarators [3] can result in the loss much less than those shown in Figs. 6 and 7, affecting only particles with the energy $\mathcal{E} \ll 3.5\,\mathrm{MeV}$. This conclusion can be drawn from Fig. 2, from which it follows that α-particles can reach only the lower branch of the HAE_{21} resonance curve located well below v_α.

The obtained results are relevant to the local losses, i.e., to the losses from a flux surface. A necessary condition for AEs to transport the particles over a distance Δr is that the resonant wave-particle interaction persists during the transportation. But the increase of v_r with the radius tends to destroy the interaction of the particles moving outwards. Therefore, the finite width of the resonances must be taken into account to answer the question of whether the wave-particle interaction can take place in the whole region Δr. The magnitude of Δr is the largest for a given resonance width when the plasma density is flat, which is expected to be the case in the plasma core of a Helias reactor.

A specific feature of weak-shear systems is that their AEs are localized in the plasma core (at the periphery, where the plasma density is strongly decreasing, the waves are damped because of the continuum damping). Therefore, the most probable consequence of the excitation of AIs in a Helias reactor is the expulsion of α-particles from the plasma core to the periphery where they can be either thermalized or lost. The losses in the periphery can be induced by various mechanisms; for instance, the transitioning particles can be lost because of the stochastic diffusion predicted in Ref. [19]). In the latter case an AI will be a trigger of the alpha losses. But even when the AIs do not result in alpha loss they will deteriorate the plasma energy balance by moving fast alphas to the periphery.

On the other hand, the transport of fast alphas to the plasma periphery is not necessarily harmful. The destabilized waves that resonate with alphas of relatively low energy (e.g, HAE_{21} instability) will influence the particles only after they have already transferred most of their energy to the plasma. Such instabilities could play a positive role by removing the helium ash from the plasma core. If the corresponding eigenmodes are stable, it would be worth to excite them by an external antenna. Note that a possibility to destabilize TAE modes by using an antenna system was demonstrated in experiments on the JET tokamak [20].

VI. SUMMARY AND CONCLUSIONS

In the present work we have developed a perturbative theory of destabilization of Alfvén eigenmodes by energetic circulating ions in stellarators, the main attention being paid to optimized stellarators of Wendelstein line, in particular, to a Helias reactor. Destabilization of various Alfvén gap modes, such as MAEs, several kinds of HAEs (HAE_{21}, HAE_{11}, and HAE_{22}-modes), TAEs, and EAEs, as well as GAE-modes, is considered. The results relevant to a Helias reactor are summarized in Tables I–III. In addition, we evaluated possible energy losses of alpha particles induced by Alfvén instabilities in a Helias reactor.

We have shown for the first time that because of the presence of several main Fourier harmonics of the magnetic field strength, several sideband resonances rather than the only one associated with toroidicity (known from a theory relevant to tokamaks) may essentially contribute to the instability growth rate. The number of important resonances decreases when the modes are well-localized. In particular, it is shown that in this case the resonance $\omega = [k_\| \pm (\iota - N)/R_0]v_\|$ associated with the helicity-induced drift motion plays a dominant role in the destabilization of MAE and HAE$_{21}$ modes by α-particles in a Helias reactor. This is explained by the fact that for the mentioned modes the upper resonance curves associated with helicity and toroidicity coincide [see Figs. 2(b) and 2(d)], whereas $\epsilon_h^2 \gg \epsilon_t^2$. The fact that these curves lie closer to v_α for MAEs than for HAE$_{21}$ modes explains why the MAE instability is characterized by a higher growth rate. On the other hand, new resonances have a minor influence on the well-localized TAEs because the corresponding resonance curves are situated well below the curve associated with the toroidicity, see Fig. 2(f). One can expect, however, that TAEs of global character may be considerably affected by the diamagnetic-drift-induced resonance because $r\epsilon_0' > \epsilon_t$ for $r/a > 0.3$, $\beta \sim 5\%$, and the corresponding resonance curve coincides with the one associated with the toroidicity. Note that in the optimized stellarators the dominant Fourier harmonic of the magnetic field, ϵ_m, weakly affects the stability because only its derivative, which is small, contributes to γ_α, see Eq. (34). Non-conventional resonances play an important role for both gap modes and GAE modes, which is seen from Tables I–III.

An important practical consequence of the found new resonances is that the tangential neutral beam injection (NBI) can destabilize TAE modes not only when $v_b \geq v_A$ or $v_b \geq v_A/3$, where v_b is the velocity of injected particles, but also for lower beam velocities (energies), e.g., for $v_b \geq v_A|2N/\iota - 1)|^{-1}$ or $v_b \geq v_A|2N/\iota - 3)|^{-1}$ due to the helicity-induced resonance. On the other hand, other instabilities can be caused by particles with $v_b > v_A$ or $v_b < v_A/3$. For instance, the MAE instalility can arise when $v_b \geq |v_\|^r|$ with $|v_\|^r|$ equal to $v_A|3 - 2\iota/N|^{-1}$ and/or $v_A|1 - 2\iota/N|^{-1}$, $v_A|1 + 2\iota/N|^{-1}$.

It follows from our calculations that even when the radial profile of the plasma density is rather flat, the spatial inhomogeneity of the fusion-produced alpha particles in a Helias reactor is sufficiently large to result in the drive overcoming the wave damping caused by the decreasing energy distribution of alphas, at least, for the waves localized at $r/a > 0.2$. An additional factor driving the instability is the velocity anisotropy of energetic ions produced due to, e.g., NBI. Using $f_0 \sim v^{-3}\delta(\chi - \chi_\alpha)$, we have shown that the velocity anisotropy can be a dominant factor driving the instability. The drive overrides the damping when $\max\{\sqrt{\epsilon_m}, |v_r|/v_\alpha\} < \chi_\alpha < 0.77$ (like in tokamaks [17]), the contribution of helicity to the growth rate being, in general, important.

The wave damping by the bulk plasma is not considered in this work. It is planned to study it in the future in order to predict the conditions providing the best stability of plasmas with energetic ions and reveal the most important instabilities.

We have evaluated the maximum possible energy losses that can happen when α-particles interact with waves through the resonance given by Eq. (48). Another result of the work is that certain AIs influence only α-particles with $\mathcal{E} \ll 3.5\,\mathrm{MeV}$. This seems to open a way to remove the helium ash from the plasma core in a Helias reactor by means of the destabilization of the corresponding AEs. The desirable destabilization can be implemented with using NBI or an RF heating system. There is also some probability (which seems to be small) that the instabilities required for the ash removal will be excited by α-particles.

In this work, we have disregarded the effect of bounce resonances (known from tokamak studies) and some other resonances which one may expect to appear in stellarators. Further investigation taking into account the mentioned resonances, the finite width of the resonances, and the influence of AIs on all groups of alphas (circulating, transitioning, and trapped particles) is of importance. Then it will be possible to predict the complete picture of the AE-induced alpha losses and assess the possibility to use destabilized AEs for the ash removal.

In conclusion, we note that although our analysis is relevant mainly to optimized stellarators, the drawn conclusion on the role of non-conventional resonances is valid also for other systems, e.g., for the partly optimized stellarator Wendelstein 7-AS (W7-AS).

Furthermore, one can expect that in W7-AS the role of the non-conventional resonances is even larger because the Fourier spectrum of the magnetic field strength contains more harmonics. It would be of interest to carry out experiments on W7-AS and W7-X aimed at destabilizing various AEs by an external antenna and observing the influence of these AEs on the energetic ions (the latter can be produced either by NBI and/or ICRF heating).

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REFERENCES

[1] C. Nührenberg, Phys. Plasmas **6**, 137 (1999).

[2] C. Nührenberg, Plasma Phys. Control. Fusion **41**, 1055 (1999).

[3] Ya. I. Kolesnichenko, V. V. Lutsenko, H. Wobig, Yu. V. Yakovenko, and O. P. Fesenyuk, *Alfvén Eigenmodes in Helias Configurations (Part I), IPP Report III/261* (Max-Planck-Institut für Plasmaphysik, Garching bei München, Mai 2000); Phys. Plasmas **8**, 491 (2001).

[4] Ya. I. Kolesnichenko, V. V. Lutsenko, H. Wobig, Yu. V. Yakovenko, and O. P. Fesenyuk, in *Controlled Fusion and Plasma Physics, Abstracts of Invited and Contributed Papers, 27th European Physical Society Conference, Budapest, 2000* (The European Physical Society, Petit-Lancy, 2000), p. 88.

[5] A. Könies, Phys. Plasmas **7**, 1139 (2000).

[6] F. Wagner, Transactions of Fusion Technology **33**, 67 (1998).

[7] A. Weller, D. A. Spong, R. Jaenicke, A. Lazaros, F. P. Penningsfeld, S. Sattler, W7-AS Team, and NBI group, Phys. Rev. Lett. **72**, 1220 (1994).

[8] A. Weller, M. Anton, R. Brakel *et al.*, in *Fusion Energy 1998, 17th International Atomic Energy Agency Conference Proceedings, Yokohama, 1998*, Vol. 1 (International Atomic Energy Agency, Vienna, 1999), p. 295.

[9] F. Herrnegger, F. Rau, and H. Wobig (editors), *Contributions to Wendelstein 7-X and Helias Reactor 1991-1998, Report IPP 2/343* (Max-Planck-Institut für Plasmaphysik, Garching bei München, 1999).

[10] J. Kißlinger, C. D. Beidler, and E. Strumberger, in *Controlled Fusion and Plasma Physics, Abstracts of Invited and Contributed Papers, 27th European Physical Society Conference, Budapest, 2000* (The European Physical Society, Petit-Lancy, 2000), p. 345.

[11] C. D. Beidler, E. Harmeyer, F. Herrnegger *et al.*, in *Fusion Energy 2000, 17th International Atomic Energy Agency Conference Proceedings, Sorrento, 2000* (International Atomic Energy Agency, Vienna), Report IAEA-CN-77/FT/4, to be published.

[12] V. S. Belikov, Ya. I. Kolesnichenko, and V. V. Lutsenko, Nucl. Fusion **35**, 207 (1995).

[13] Ya. I. Kolesnichenko, V. V. Parail, and G. V. Pereverzev, in *Reviews of Plasma Physics*, Vol. 17 (Consultants Bureau, New York – London, 1992), p. 1.

[14] A. H. Boozer, Phys. Fluids **24**, 1999 (1981).

[15] R. B. White and M. S. Chance, Phys. Fluids **27**, 2455 (1984).

[16] G. Y. Fu and J. W. Van Dam, Phys. Fluids B **1**, 1949 (1989).

[17] V. S. Belikov, Ya. I. Kolesnichenko, and O. A. Silivra, Nucl. Fusion **32**, 1399 (1992).

[18] G. Kamelander, Ya. I. Kolesnichenko, and V. V. Lutsenko, Nucl. Fusion **35**, 1679 (1995).

[19] Ya. I. Kolesnichenko, C. D. Beidler, V. V. Lutsenko, V. S. Marchenko, I. N. Sidorenko, and H. Wobig, in *Controlled Fusion and Plasma Physics, Abstracts of Invited and Contributed Papers, 27th European Physical Society Conference, Budapest, 2000* (The European Physical Society, Petit-Lancy, 2000), p. 90.

[20] R. F. Heeter, A. Fasoli, D. Borba, D. Start, S. Ali-Arshad, and N. Fisch, in *Alpha Particles in Fusion Research, 5th International Atomic Energy Agency Technical Committee Meeting, Abingdon, 1997* (International Atomic Energy Agency, Vienna, 1997), p. 9.

TABLES

TABLE I. Characteristics of α-induced Alfvén instabilities for $\omega_* = 10\omega$ in a Helias reactor

$Mode$	ω	$\gamma_\alpha R_0/(v_{A*}\beta_\alpha)$	$\gamma_\alpha/\gamma_\alpha^0$	$\gamma_\alpha/(\omega\beta_\alpha)$	$(\omega_*/\omega)_{cr}$	$(\omega_*/\omega)_{cr}^0$
HAE_{22}	$v_{A*}(N-\iota_*)/R_0$	0.020	1.068	0.006	3.926	3.812
MAE	$Nv_{A*}/(2R_0)$	0.166	4.371	0.083	3.756	3.751
HAE_{11}	$v_{A*}(N-\iota_*)/(2R_0)$	0.119	2.072	0.076	4.044	3.940
HAE_{21}	$v_{A*}(N-2\iota_*)/(2R_0)$	0.009	4.727	0.008	4.781	4.762
EAE	$\iota_* v_{A*}/R_0$	0.002	1.988	0.003	5.636	5.203
TAE	$\iota_* v_{A*}/(2R_0)$	0.058	1.000	0.131	4.138	4.138

TABLE II. Characteristics of α-induced Alfvén instabilities for given n in a Helias reactor

$Mode$	$\omega/(2\pi), kHz$	n	$\gamma_\alpha R_0/(v_{A*}\beta_\alpha)$	$\gamma_\alpha/\gamma_\alpha^0$	$\gamma_\alpha/(\omega\beta_\alpha)$	$\gamma_\alpha/\beta_\alpha, kHz$
HAE_{22}	112	-11/-16	0.050/0.078	1.074/1.073	0.016/0.025	1.8/2.8
MAE	72	-9/-15	0.546/0.976	4.368/4.370	0.273/0.489	19.7/35.2
HAE_{11}	56	-6/-13	0.334/0.819	2.089/2.100	0.215/0.527	12.0/29.5
HAE_{21}	40	-9/-14	0.064/0.103	4.833/4.721	0.058/0.093	2.3/3.7
EAE	32	-7/-13	0.020/0.040	2.150/2.180	0.023/0.046	0.7/1.5
TAE	16	-4/-10	0.442/1.166	1.000/1.000	0.993/2.621	15.9/41.9

TABLE III. Characteristics of the GAE instability in a Helias reactor

r_l/a	$\omega/(2\pi), kHz$	n/m	$\gamma_\alpha/\gamma_\alpha^0$	$\gamma_\alpha/(\omega\beta_\alpha)$	$\gamma_\alpha/\beta_\alpha, kHz$
0.6	12	-4/-4	1.001	10.10	121
0.6	31	-10/-10	7.277	1.34	42
0.6	46	-15/-15	3.083	4.78	220
0.44	198	-10/-5	5.585	0.105	21

FIGURES

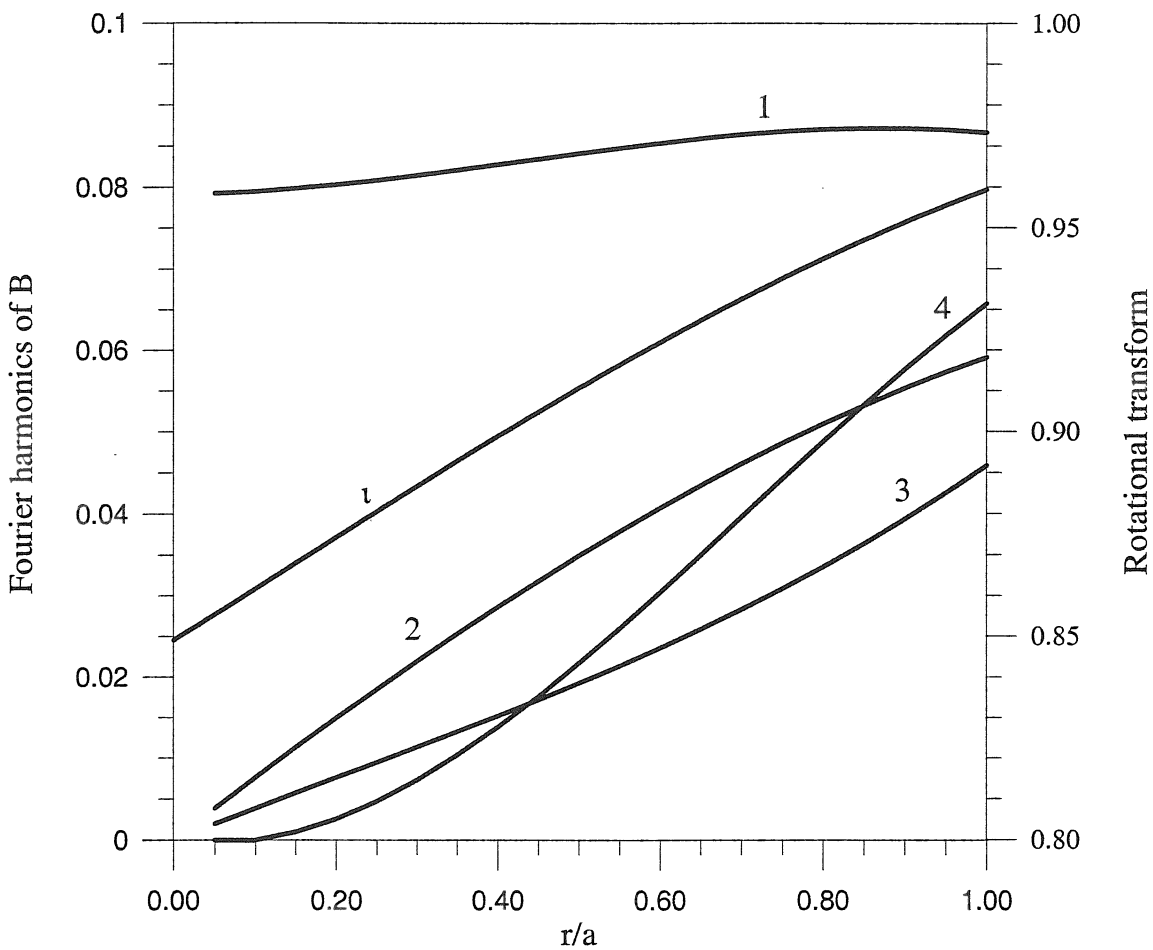


FIG. 1. Iota and main Fourier harmonics of the magnetic field strength in a Helias reactor [11]. 1, the mirror harmonic ($\epsilon_B^{(01)}$); 2, the helical harmonic ($\epsilon_B^{(11)}$); 3, the toroidal harmonic ($\epsilon_B^{(10)}$); 4, the diamagnetic harmonic ($\epsilon_B^{(00)}$).

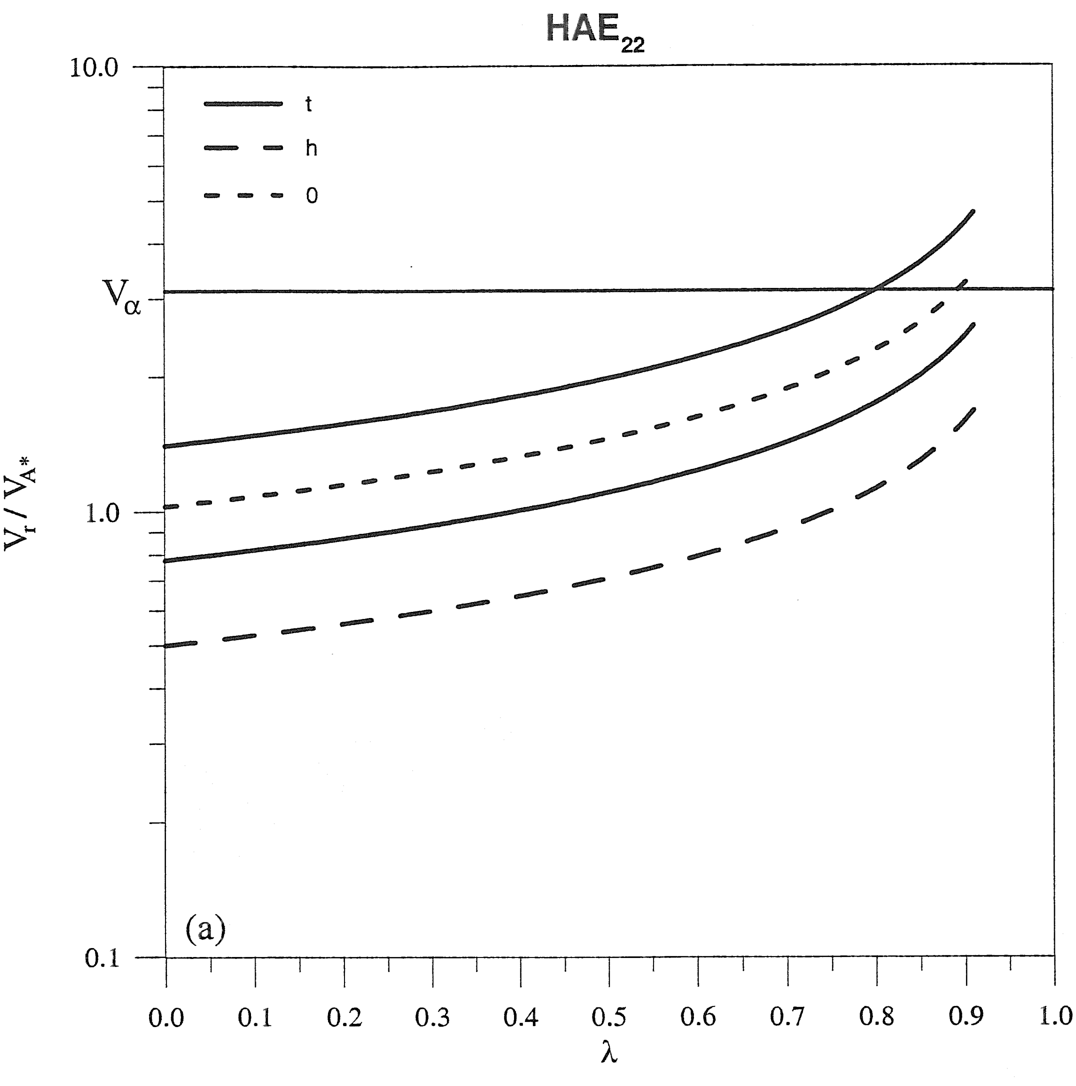
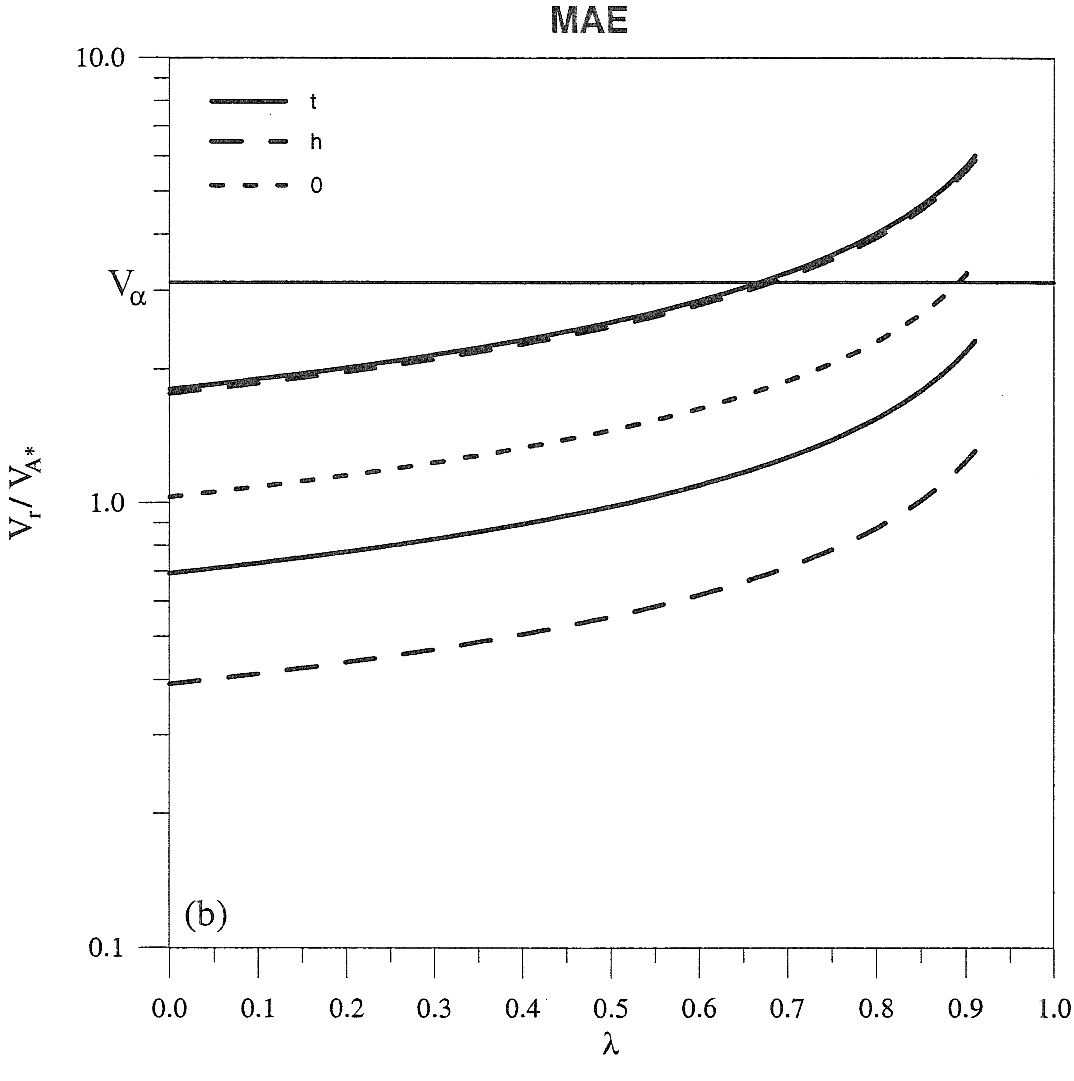


FIG. 2. Resonance velocity $v_r = |v_{\|}^r|/(1-\lambda)^{1/2}$, versus the pitch-angle parameter λ in a 4-period Helias reactor. (a), HAE_{22}; (b), MAE; (c), HAE_{11}; (d), HAE_{21}; (e), EAE; (f), TAE. The conventional resonance curves with $|v_{\|}^r| = v_A$ and $|v_{\|}^r| = v_A/3$ take place in the special case when $\mu_0 = \mu$, $\nu_0 = \nu$ (dashed curves for HAE_{11} modes, solid curves for TAE modes); the curve with $\mu = \nu = 0$ is always a conventional curve with $|v_{\|}^r| = v_A$.



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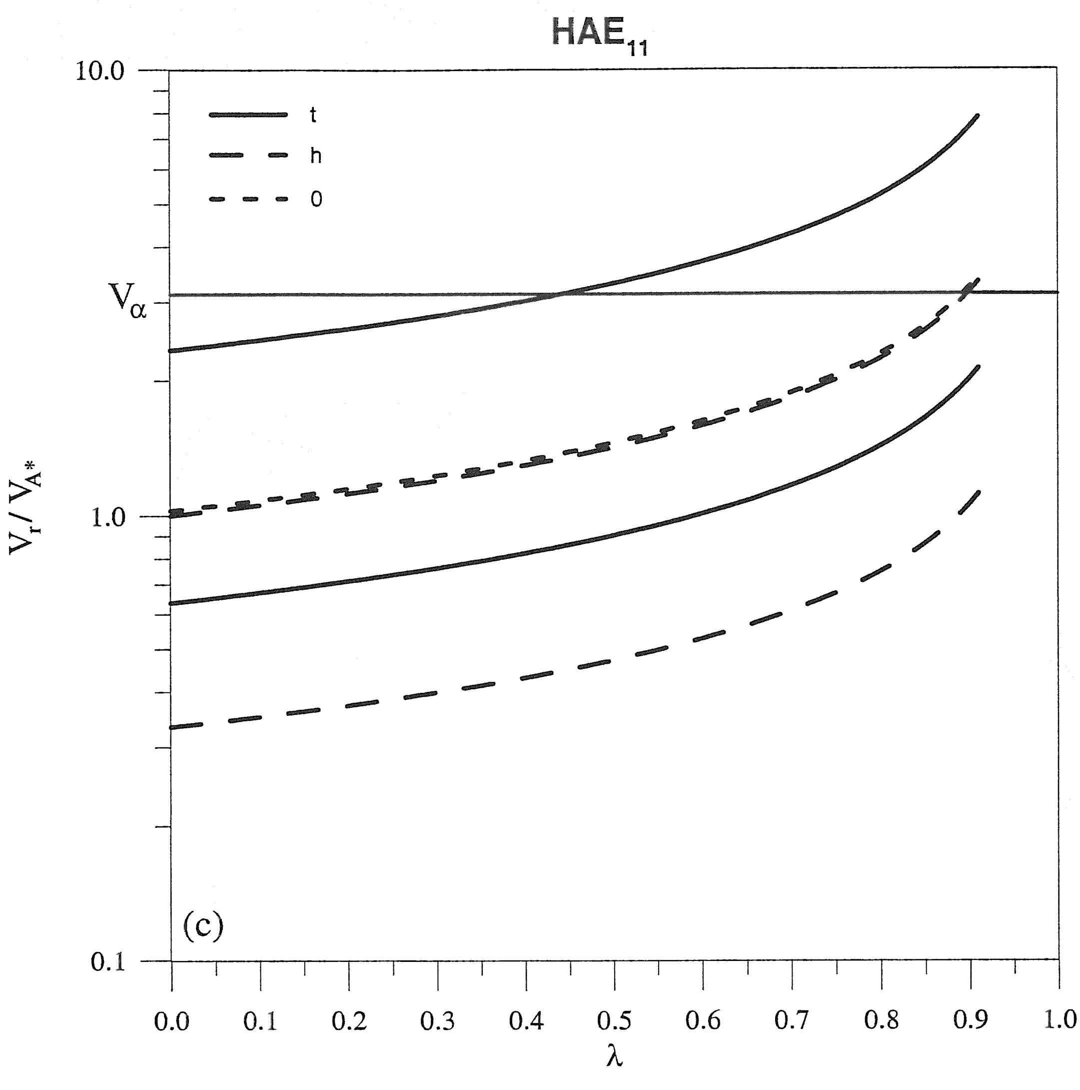

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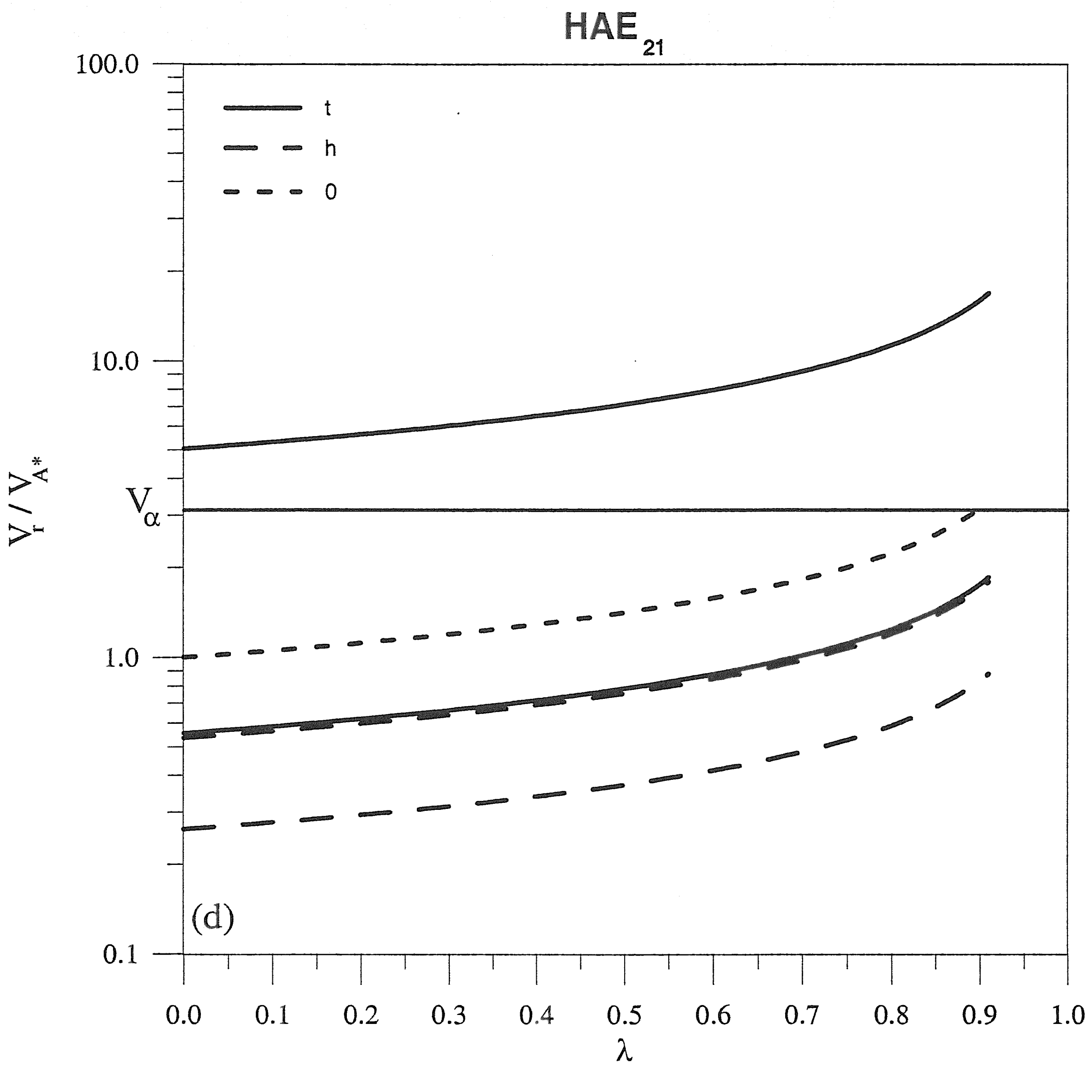

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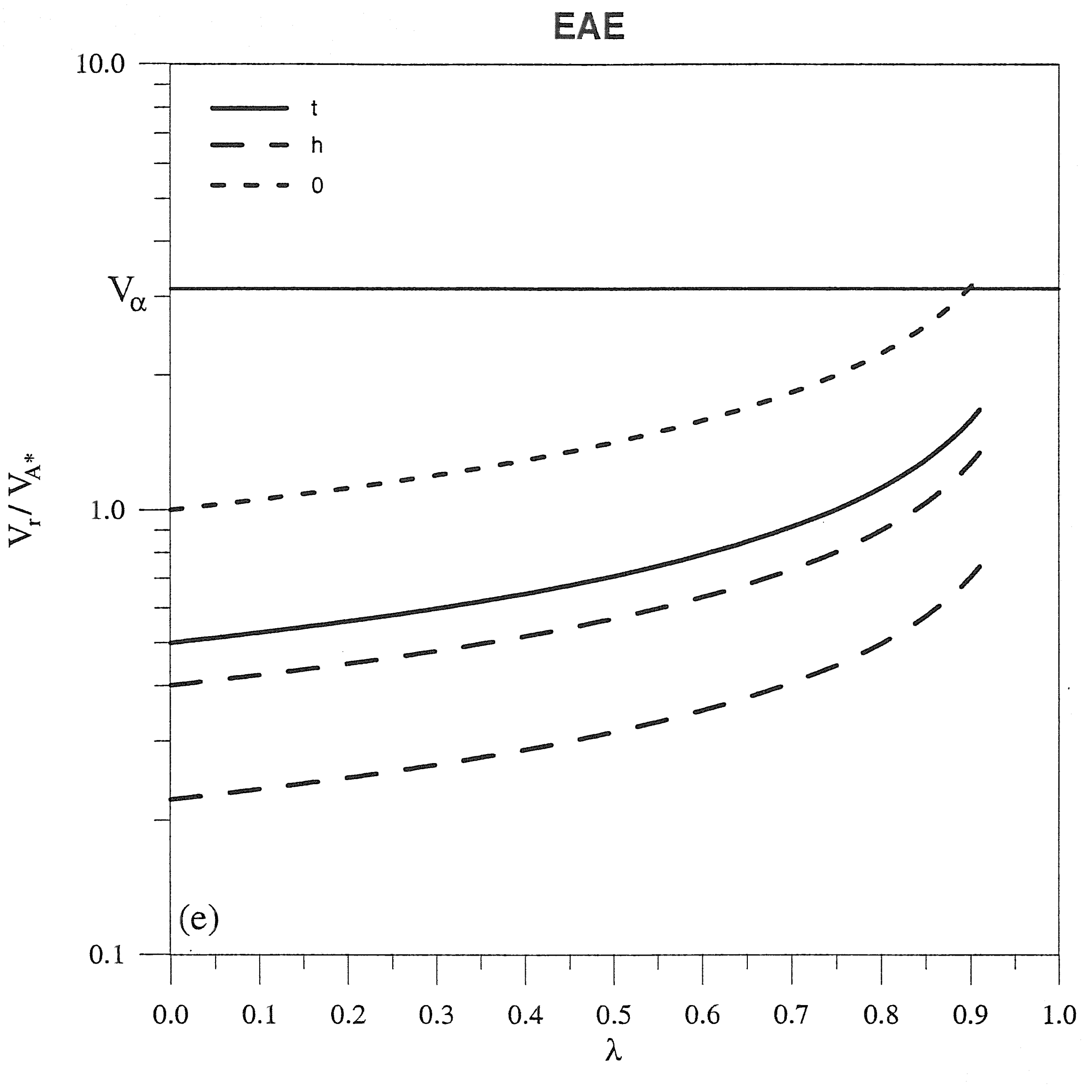

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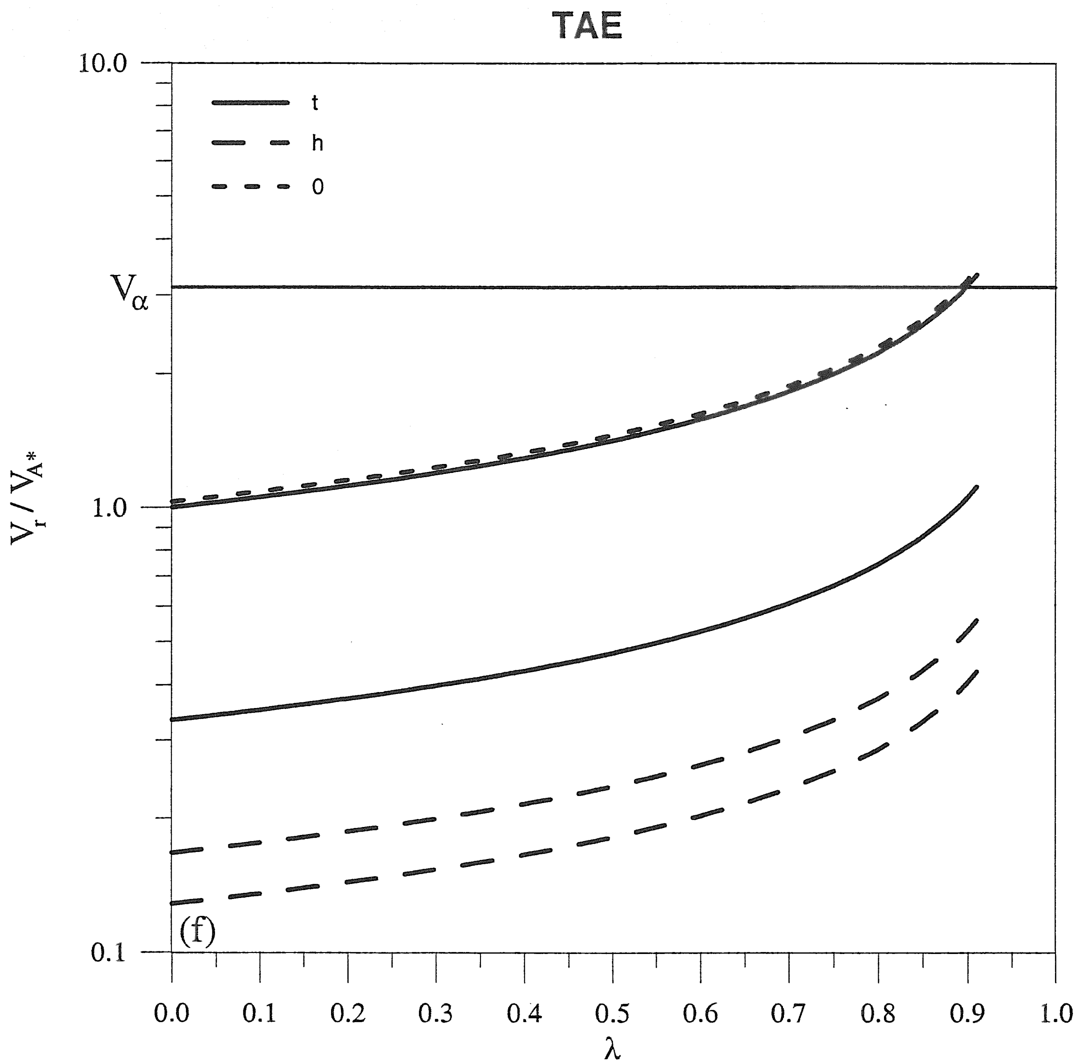

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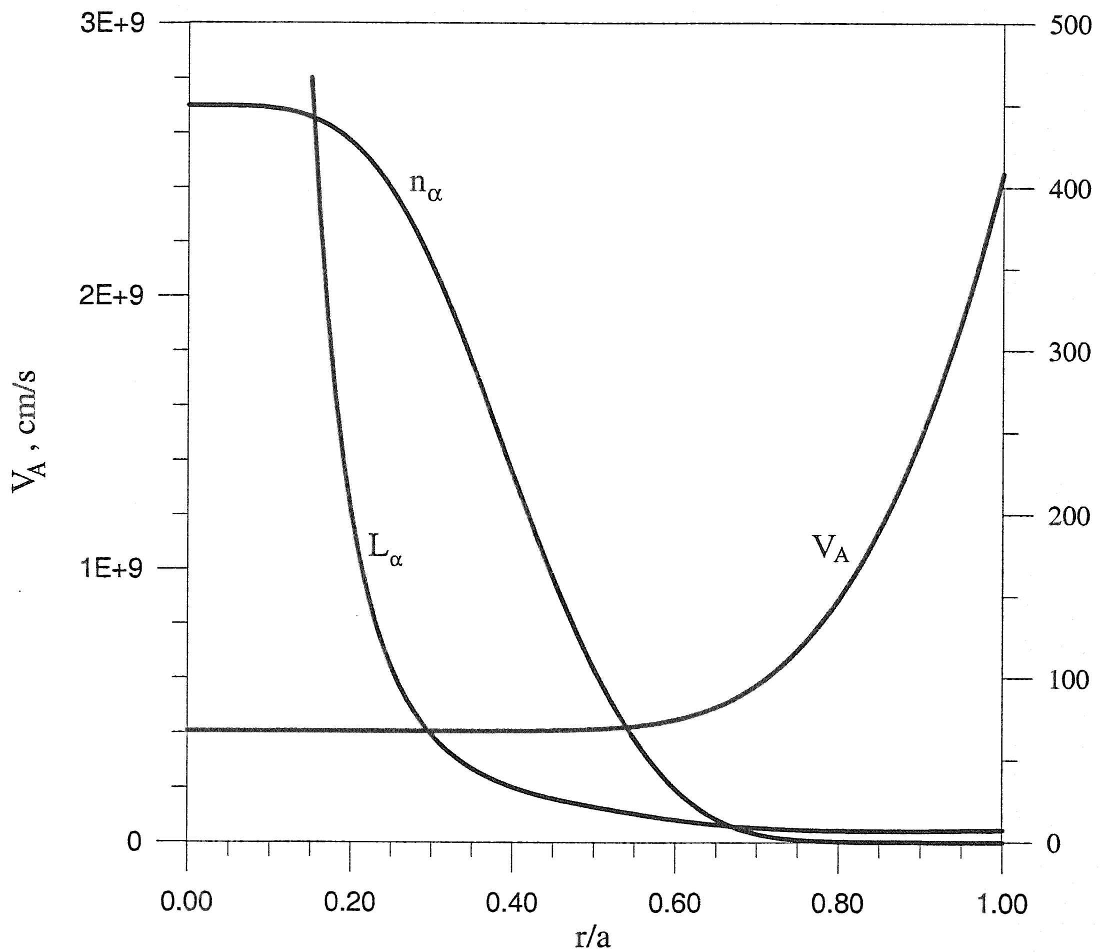

FIG. 3. Alfvén velocity, radial distribution of α-particles, and $L_\alpha(r)$ in a Helias reactor.

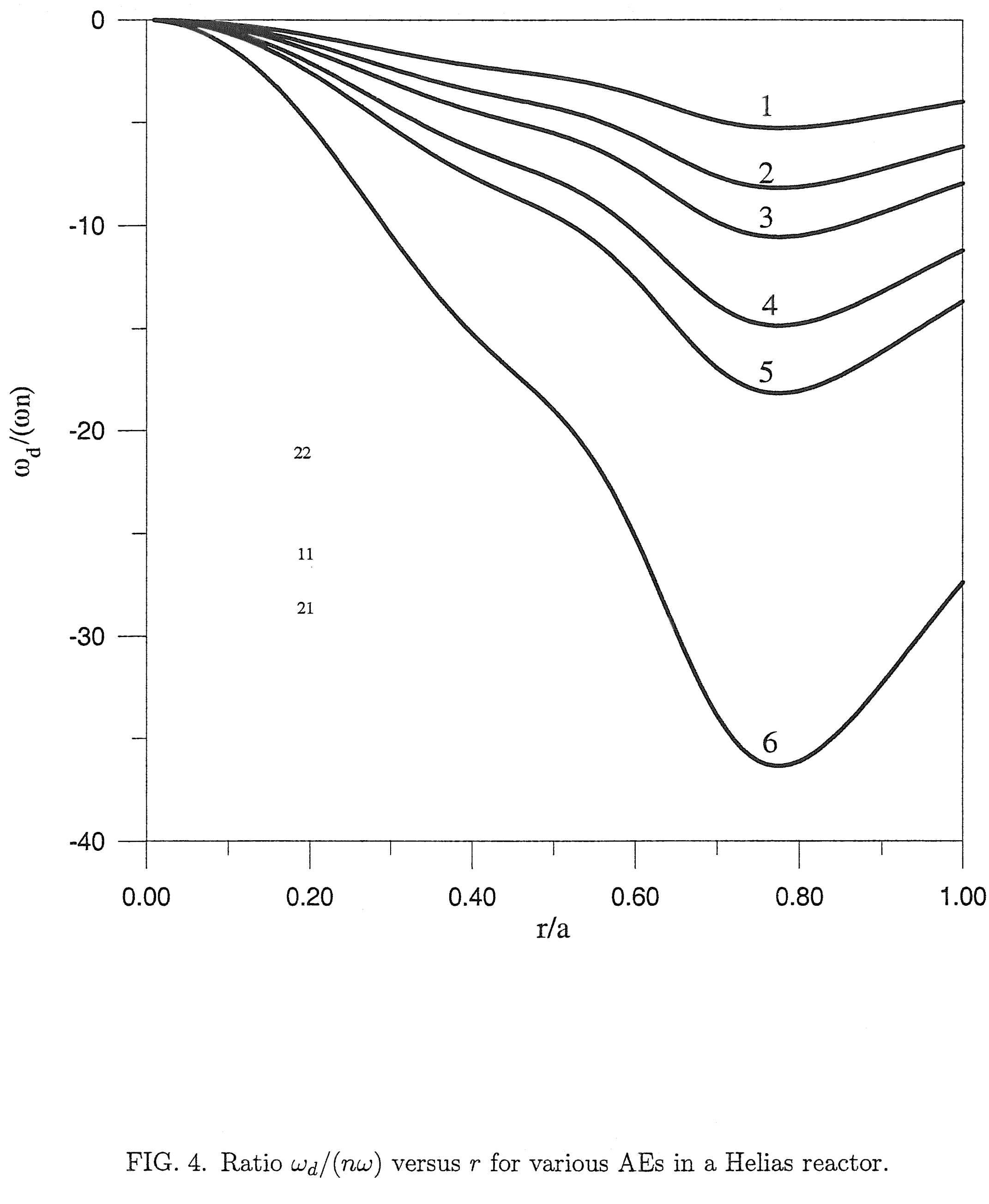

FIG. 4. Ratio $\omega_d/(n\omega)$ versus r for various AEs in a Helias reactor.

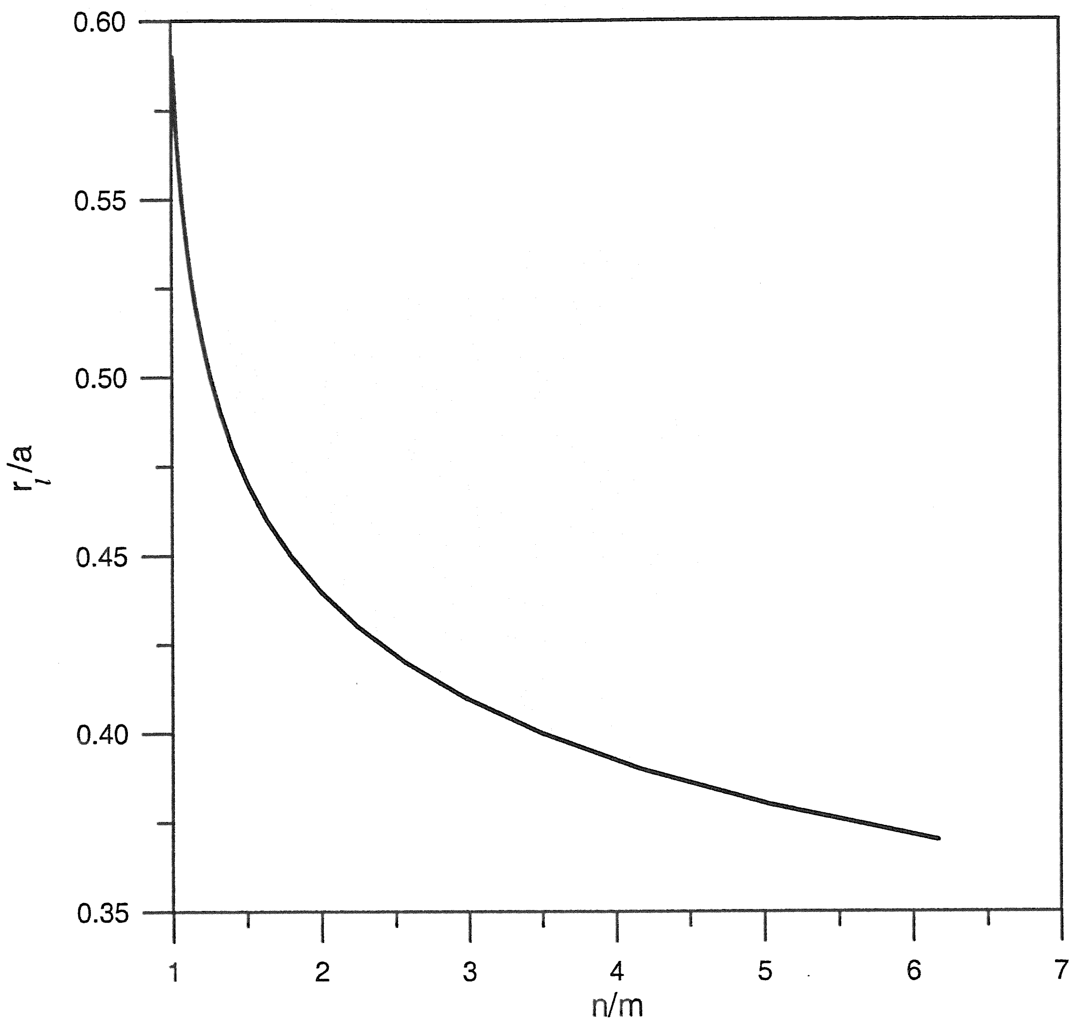

FIG. 5. Location of minimum of the local Alfvén frequency versus n/m in a Helias reactor with the inhomogeneity parameter $x_n = 0.7$.

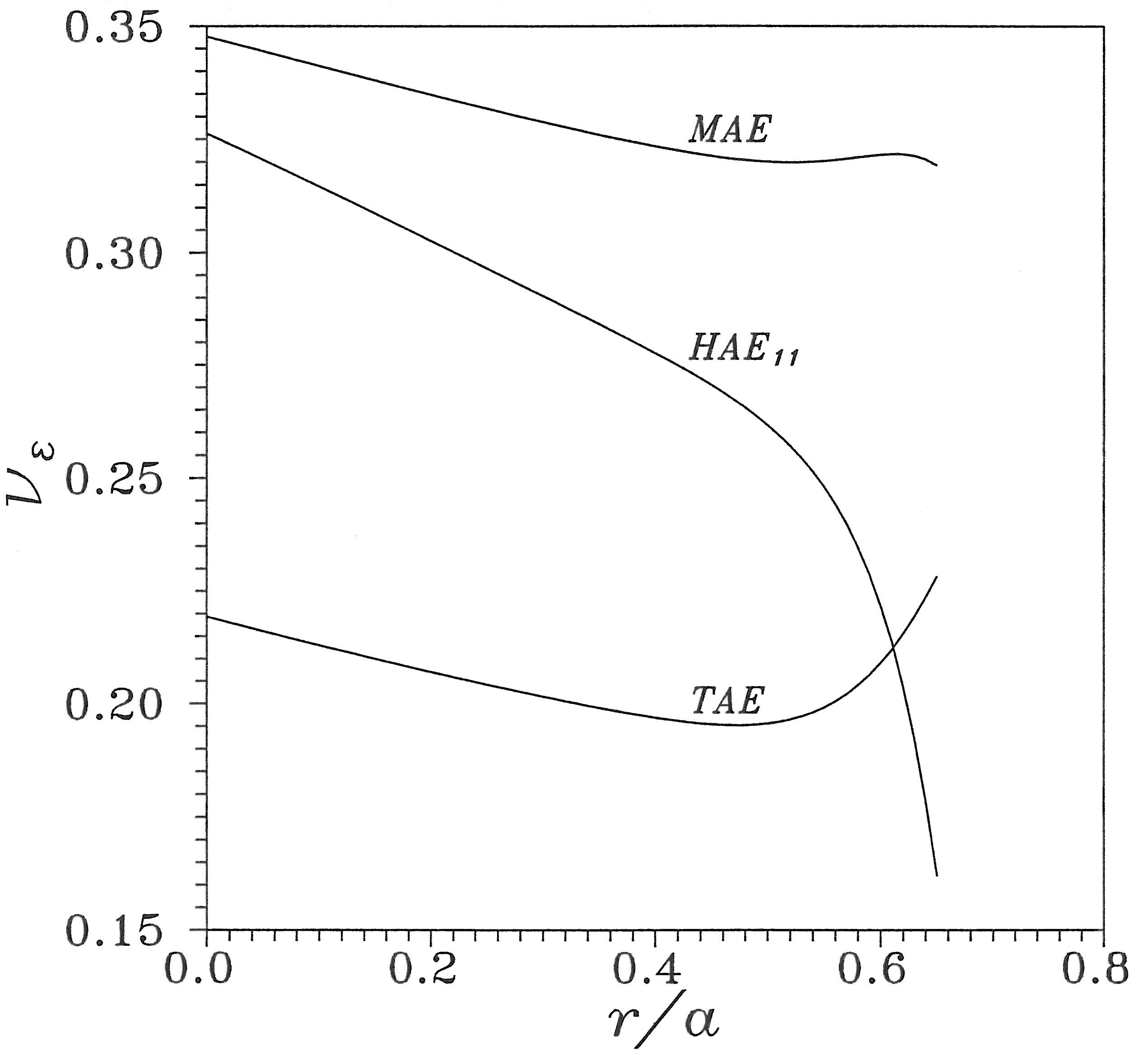

FIG. 6. Local energy losses versus r/a for $v_\alpha = 3.2 v_A(r=0)$ and $\epsilon_{eff} = \epsilon_m + \epsilon_h + \epsilon_t$.

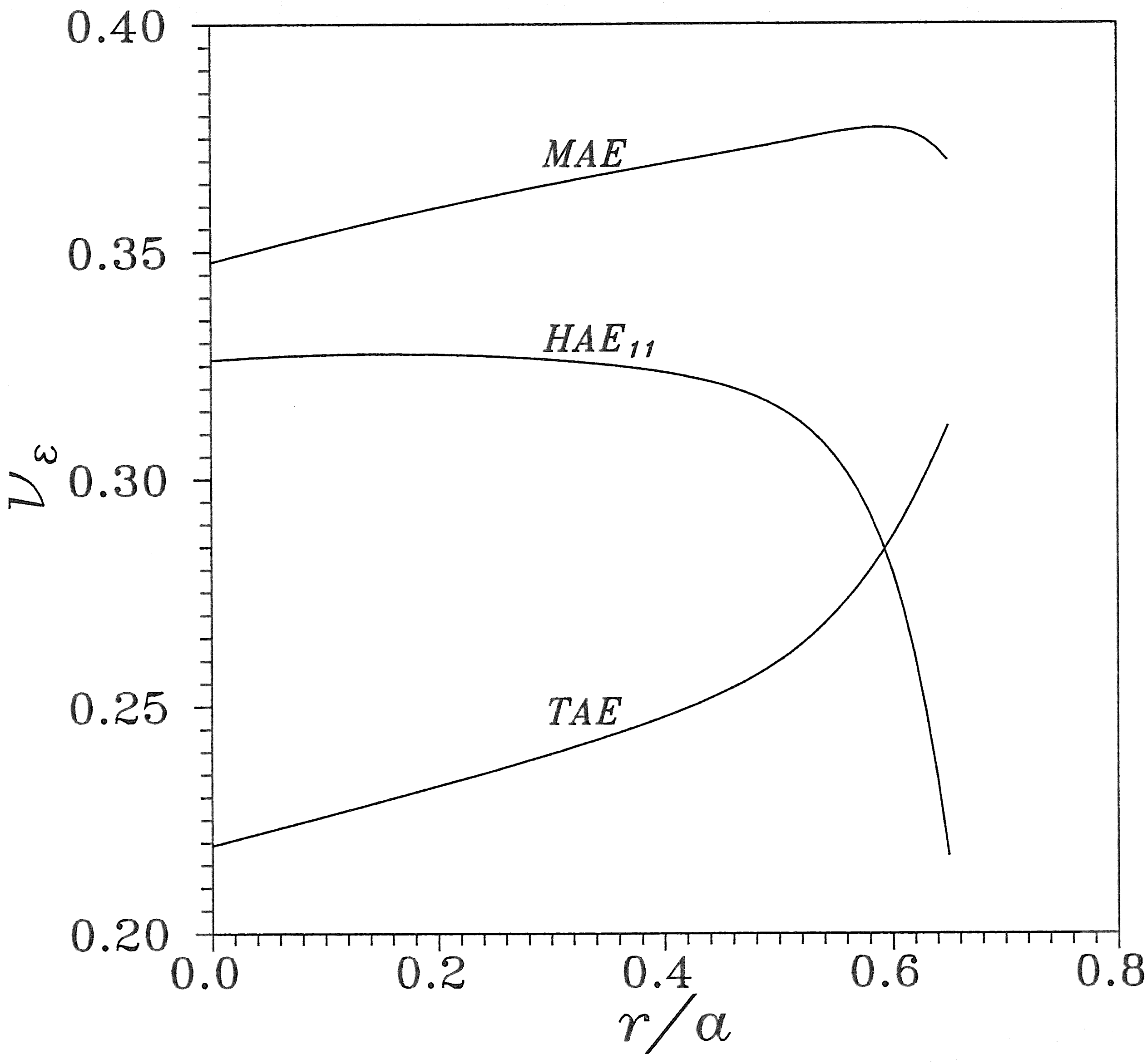

FIG. 7. The same as in Fig. 6 but for $\epsilon_{eff} = \epsilon_m - \epsilon_h - \epsilon_t$.