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Abstract:

The theory of ideal MHD-equilibria in three-dimensional geometry is revisited with particular emphasis on the issue of rational magnetic surfaces in non-axisymmetric configurations. In stellarators parallel plasma currents tend to diverge on rational magnetic surfaces if the Hamada condition is not satisfied. It is shown that by modifying the force balance of the plasma by friction terms, viscous terms and inertial terms the mathematical structure of the problem changes appreciably leading to elliptic partial differential equations. This feature allows one to solve the equilibrium problem together with boundary conditions. Singularities on rational magnetic surfaces are removed by retaining dissipative effects like friction with a neutral background or viscosity. The paper discusses the friction model of steady state equilibria, the Stokes model and the Navier-Stokes model.

1. Introduction

The commonly used model to describe equilibrium in toroidal configurations is the one-fluid model of ideal magnetohydrodynamics. The gradient of a scalar pressure is balanced by the force $\boldsymbol{j} \times \boldsymbol{B}$

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla p \qquad \text{Eq. 1.1}$$

As it is wellknown, in systems with symmetries like axisymmetry in tokamaks or helical symmetry of linear stellarators, the solution of Eq. 1.1 reduces to a two-dimensional quasilinear elliptic equation, the Schlüter-Grad-Shafranov equation. In a three-dimensional equilibrium, however, no such equation exists. As pointed out by Kruskal and Kulsrud[1], Eq. 1.1 can be derived from the variation of a positive functional

$$U = \int_V \left(\frac{B^2}{2} + \frac{p}{\gamma - 1} \right) d^3\boldsymbol{x} \qquad \text{Eq. 1.2}$$

where the variation is subject to some constraints. Minimization of this functional is the method how 3d-equilibria are computed numerically, however, it should be noted, that the variational formulation of the problem does not imply the existence of a solution, at least in a classical sense, where $\boldsymbol{j}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$ are continuously differentiable functions. Nevertheless, several numerical codes have been developed in the past to solve equation Eq. 1.1 by starting from minimization of the energy (Eq. 1.2). An excellent review of these approaches was published by J. Johnson[2]. A major difference between the axisymmetric equilibrium and the non-axisymmetric equilibrium is the additional condition

$$\oint \frac{dl}{B} = const. \qquad \text{Eq. 1.3}$$

on all rational magnetic surfaces. This condition follows from

$$\nabla \bullet \boldsymbol{j} = 0 \;\; ; \;\; \boldsymbol{j} = \frac{\boldsymbol{B} \times \nabla p}{B^2} + \sigma \boldsymbol{B} \;\; ; \;\; p = p(\psi) \qquad \text{Eq. 1.4}$$

ψ = const. defines the magnetic surfaces. The magnetic differential equation

$$\nabla \bullet \boldsymbol{j}_\perp + \boldsymbol{B} \bullet \nabla \sigma = 0 \qquad \text{Eq. 1.5}$$

leads to the following condition of integrability (Newcomb's condition)

$$\oint \frac{\nabla \bullet \boldsymbol{j}_\perp}{B} dl = 0 \qquad \text{Eq. 1.6}$$

[1] M. Kruskal and R. Kulsrud, Phys. Fluids, (1958)
[2] J. Johnson, Comp. Phys. Reports 4, 39 (1986).

which is equivalent to Eq. 1.3. If these conditions cannot be satisfied singularities of the parallel current density will arise[3]. In this paper A. Boozer argues that the pressure gradient becomes zero on all rational surfaces thus making the parallel plasma current again finite.

As has been pointed out by H. Grad[4] the equation of an ideal equilibrium may lead to a very pathological pressure distribution, which could be continuous in radial direction, but which has no continuous derivatives. This extra global condition (Eq. 1.3) has been strongly criticized by H. Grad as something "completely alien to any concept, which has ever arisen in connection with the theory of partial differential equations". Another feature of the ideal model is the occurrence of islands and stochasticity, which is a generic property of 3-dimensional magnetic fields. Usually it is argued to make the pressure constant in such a region, however, islands and stochasticity depend on the plasma currents and these on the pressure distribution. If the plasma pressure is large, islands can overlap and lead to stochasticity all over the plasma column. This effect already arises, if one tries to compute the stellarator equilibrium following the iterative scheme proposed by L. Spitzer[5].

Starting from equation (1) a sequence $\boldsymbol{B}_n$ of magnetic fields is constructed, which are assumed to converge towards a self-consistent equilibrium. However, as pointed out by A. Boozer[6] magnetic surfaces can be destroyed, even if the process begins with a set of nested surfaces. Numerical codes following this procedure have been developed by A. Reiman and H. Greenside[7] and J. Kisslinger, H. Wobig[8]. At low β this procedure yields results which look acceptable, however, at higher β the occurrence of islands prohibits the convergence.

A further objection against the ideal model is its incapability to satisfy boundary conditions neither in plasma pressure nor in density or electric potentials, which even do not occur in the model. In the ideal model, plasma flow - either diffusion velocity or $\boldsymbol{E} \times \boldsymbol{B}$ - drifts of parallel flows have no feedback on the force balance. The flow velocity $\boldsymbol{v}$ has to be calculated from Ohm's law

$$-\nabla\Phi + \boldsymbol{v}\times\boldsymbol{B} = \eta\boldsymbol{j} \qquad \text{Eq. 1.7}$$

and the equation of continuity

$$\nabla\bullet\rho\boldsymbol{v} = Q \qquad \text{Eq. 1.8}$$

[3] A. Boozer, *Plasma equilibrium with rational magnetic surfaces*, Phys. Fluids 24, (1981) 1999-2003
[4] H. Grad, Phys. Fluids **10** (1967) 137
[5] L. Spitzer, {Phys. Fluids 1, (1958) 253.
[6] A. Boozer, Phys. Fluids **27** (1984) 2110
[7] A. Reiman and H. Greenside, Comp. Phys. Com. **43** (1986) 147
[8] J. Kisslinger, H. Wobig, Europhys. Conf. Abstracts Vol. 9F Part I, 453

Q is the density source term. In the frame of this model electric potential and parallel flow velocity are calculated from a magnetic differential equation. The potential is computed from the magnetic differential equation

$$-\boldsymbol{B} \bullet \nabla\Phi = \eta \boldsymbol{j} \bullet \boldsymbol{B} \qquad \text{Eq. 1.9}$$

In stellarators without toroidal loop voltage the Newcomb condition requires

$$\oint \eta \boldsymbol{j} \bullet \boldsymbol{B} \frac{dl}{\boldsymbol{B}} = 0 \qquad \text{Eq. 1.10}$$

on all closed field lines. Thus, even if an ideal equilibrium exists and condition Eq. 1.3 is satisfied, this new condition Eq. 1.10 may or may not be satisfied. There is no free parameter left for adjustment. If singularities of the parallel current exist, these also occur in the electric potential. The perpendicular plasma velocity is given by

$$v_\perp = -\eta \frac{\nabla p}{B^2} - \frac{\nabla\Phi \times \boldsymbol{B}}{B^2} \qquad \text{Eq. 1.11}$$

The parallel plasma velocity is computed from the equation of continuity

$$\nabla \bullet \rho v_\perp + \nabla \bullet \rho v_{\parallel} = Q \qquad \text{Eq. 1.12}$$

The ansatz $v_{\parallel} = \lambda \mathbf{B}$ yieds the magnetic differential equation

$$\nabla \bullet \rho v_\perp + \boldsymbol{B} \bullet \nabla \rho \lambda = Q \qquad \text{Eq. 1.13}$$

Again, on rational magnetic surfaces a condition of integrability occurs

$$\oint \nabla \bullet \rho v_\perp \frac{dl}{B} = \oint Q \frac{dl}{B} \qquad \text{Eq. 1.14}$$

Eq. 1.13 is another magnetic differential differential equation, which may lead to singular solutions if the integral condition 1.14 cannot be satisfied. The magnetic differential equation 1.5 can be solved by a Fourier ansatz which gives the parallel current density in a Fourier series.

$$j_{\parallel} \propto p'(s) \sum \frac{m\delta_{m,n}}{n - \iota(s) m} \sin(n\varphi - m\theta) \qquad \text{Eq. 1.15}$$

where the coefficients are given by the expansion

$$\frac{1}{B^2} = \frac{1}{B_0^2}\left(1 + \sum_{m,n \neq 0} \delta_{m,n} \cos(n\varphi - m\theta)\right) \qquad \text{Eq. 1.16}$$

If the resonant coefficients δ_{nm} do not vanish the parallel current density is infinity on rational surfaces with $n - \iota m = 0$.

The shape of the current density close to the resonant surface is sketched in the following figure.

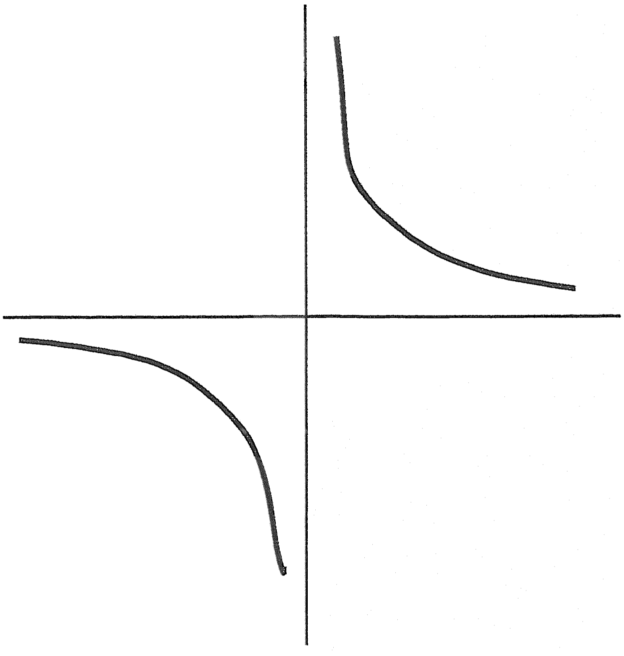

Fig. 1: Parallel current density close to a resonant magnetic surface.

In a paper[9] A. Boozer has argued that the pressure gradient should be zero on rational surfaces, since the particle flux through the magnetic surface is finite. Integrating Ohm´s law over the magnetic surface yields the relation

$$-p'(s)\Gamma = \eta \iint_{V=s} j^2 \frac{df}{|\nabla V|} \qquad \text{Eq. 1.17}$$

which implies that the plasma current density is zero on all rational surfaces. This makes the current density discontinuous and non-differentiable. Methods to solve Maxwell´s equation with such kind of current distributions do not exist in the literature. In summarizing the objections against the ideal model of equilibrium in stellarators we find

- Mathematically the model may be not well-posed and a solution in the classical sense may not exist
- The existence of everywhere dense nested magnetic surfaces is be a too strong condition for non-axisymmetric equilibria.

[9] A. Boozer Phys. Fluids 24, (1981) p. 1999)

- On rational magnetic surfaces several integral conditions exist, which generally cannot be satisfied.
- Boundary conditions on plasma parameters cannot be imposed.
- A rotating plasma equilibrium cannot uniquely be described by the ideal MHD-model

Even if an ideal equilibrium exists the computation of the plasma potentila and the parallel flow velocity can lead to singularities. In perpendicular direction to the magnetic field all other forces like inertial and viscous forces are small and may be neglected. Parallel to the magnetic field, however, even small additional forces can lead to a decoupling of pressure surfaces and magnetic surfaces. Instead of the strong condition

$$\boldsymbol{B} \bullet \nabla p = 0 \qquad \text{Eq. 1.18}$$

which follows from Eq. 1.1, any parallel inertial or viscous force leads to

$$\boldsymbol{B} \bullet \nabla p \approx 0 \qquad \text{Eq. 1.19}$$

Since finite plasma flow velocity decouples magnetic surfaces and pressure surfaces this effect removes the reason of the criticism against the ideal model. Therefore, from a physics point of view, one could argue, that an "approximate " solution of the ideal MHD - model may still be a good description of the plasma behaviour, since corrections to the equilibrium conditions arising from the finite flow velocity are small. How good the approximation is, can only be answered after a well-posed equilibrium has been established and the properties of its solution are understood.

Furthermore, as it is known from hydrodynamics, small additional terms in the force balance can introduce new phenomena like bifurcations. The solutions are no longer unique and several solutions can exists with the same boundary conditions. A well-known example in hydrodynamics are Taylor vortices in a rotating fluid. Therefore it cannot be excluded a priori, that small additional terms in eq. 1.1 - especially those containing higher order derivatives of the velocity - have a strong effect on the solution.

2. The issue of magnetic islands

In non-axisymmetric configurations like stellarators magnetic islands and stochastic regions may occur. Since no lower limit of the island size exists, the issue arises at which limit the ideal MHD model fails to describe the islands properly. Because of the non-ideal effects the pressure surfaces and the magnetic surfaces no longer coincide

Let us assume that the variation of pressure on the magnetic surface is δp and the pressure can be written as p(r) + δp. r is the average radius of the magnetic surface. The distance between magnetic surface and pressure surface is

$$\delta r = -\frac{\delta p}{p'(r)} \quad \text{or} \quad \frac{\delta r}{a} = -\frac{\delta p}{p}\frac{p}{a p'(r)} \qquad \text{Eq. 2.1}$$

a is the plasma radius. The second factor on the right hand side is of order unity. If the magnetic islands are smaller than the distance given by Eq. 2.1 magnetic surfaces and pressure surfaces certainly will not coincide and the ideal MHD-model fails to describe the equilibrium. On the other hand if the islands are larger than the limit given by Eq. 2.1 one may neglect the difference between magnetic surfaces and pressure surfaces and treat the gross structure of the islands in the frame of ideal MHD.

Let us consider the plasma boundary region where interaction with a neutral background provides a friction force

$$\boldsymbol{F} = m_i n f_{io} v \,. \qquad \text{Eq. 2.2}$$

f_{io} is the frequency of the charge exchange process, it is in the order of 10^3 Hz as will be shown later. The pressure gradient along field lines is

$$\nabla_{\|} p = m_i n f_{io} v_{\|} \qquad \text{Eq. 2.3}$$

from which we get the estimate

$$\frac{\delta p}{p} = m_i n f_{io} v_{\|} \frac{L}{p} \approx \frac{f_{io}\, v_{\|} L}{v_{th}^2} \approx \frac{v_{\|}}{v_{th}} f_{io} \tau_{\|} \qquad \text{Eq. 2.4}$$

L is a characteristic length along field lines and $\tau_{\|}$ the parallel transit time. v_{th} is the ion thermal velocity and $v_{\|}$ the parallel flow velocity. The thermal velocity in the boundary region (100 eV) is in the order of 10^5 m/s which yields a transit time of 10^{-4} s (L = 10 m) and the pressure drop becomes

$$\frac{\delta p}{p} \approx \frac{v_{\|}}{v_{th}} 10^{-1} \qquad \text{Eq. 2.5}$$

which is in the order of 10^{-2} - 10^{-1}. Interaction with neutral gas mainly occurs in the edge region. Therefore, especially in the edge region computations of small islands using the ideal MHD model may give the wrong information about the size of these islands and the position of the last closed surface.

In the bulk of the plasma neoclassical effects, entering the macroscopic equations via the viscous forces, lead to a decoupling of pressure surfaces and magnetic surfaces. In the CGL approximation the anisotropic pressure is

$$\pi = \left(p_{\|} - p_{\perp}\right)\left(\frac{\boldsymbol{B}{:}\boldsymbol{B}}{B^2} - \frac{1}{3}I\right) \qquad \text{Eq. 2.6}$$

which in this formulation is valid in all regimes of collisionality. Parallel to the magnetic field the force balance is

$$\boldsymbol{B}\bullet\nabla p = \boldsymbol{B}\bullet\nabla\bullet\pi = \boldsymbol{B}\bullet\nabla\bullet\left(p_{\parallel} - p_{\perp}\right)\left(\frac{\boldsymbol{B}:\boldsymbol{B}}{B^2} - \frac{1}{3}I\right) \qquad \text{Eq. 2.7}$$

In the collisional regime the difference in parallel and perpendicular pressure is

$$p_{\parallel} - p_{\perp} = -3p\tau\sum_{l,m}\left(b_l b_m \frac{\partial V_l}{\partial x_m} - \frac{1}{3}\frac{\partial V_m}{\partial x_m}\right) \qquad \text{Eq. 2.8}$$

τ is the like-particle collision time. The plasma velocity in the magnetic surfaces is the sum of a poloidal rotation and a parallel flow

$$V_0 = -E(\psi)\boldsymbol{e}_p + \Lambda(\psi)\boldsymbol{B} \qquad \text{Eq. 2.9}$$

where $\boldsymbol{e}_p$ is the poloidal Hamada vector, which is defined by

$$\boldsymbol{e}_p \times \boldsymbol{B} = \nabla\psi\ ; \quad \nabla\bullet\boldsymbol{e}_p = 0 \qquad \text{Eq. 2.10}$$

This flow velocity is incompressible and the solution of

$$\boldsymbol{E} + \boldsymbol{V}_0 \times \boldsymbol{B} = 0\ ; \quad \nabla\bullet\boldsymbol{V}_0 = 0 \qquad \text{Eq. 2.11}$$

$\boldsymbol{E}$ is the radial electric field which is written as

$$\boldsymbol{E} = E(\psi)\nabla\psi \qquad \text{Eq. 2.12}$$

Inserting this in eq. 2.8 yields

$$p_{\parallel} - p_{\perp} = -3p\tau\, V_o \frac{\nabla B}{B} \qquad \text{Eq. 2.13}$$

And from this relation we obtain the estimate

$$\frac{p_{\parallel} - p_{\perp}}{p} = -3\frac{\lambda}{R}\frac{|V_o|}{V_{th}} \qquad \text{Eq. 2.14}$$

λ is the mean free path of ion-ion collisions and R the major radius. Here the approximation has been used

$$\frac{\nabla B}{B} \approx \frac{1}{R} \qquad \text{Eq. 2.15}$$

This collisional approximation holds until the mean free path is comparable with the major radius R. In conclusion we find that plasma rotation which is 10^{-2}-10^{-1} of the thermal velocity leads to a pressure anisotropy of the same order. This implies that the ideal MHD-model with isotropic pressure fails to describe the equilibrium. Numerical codes[10,11] which solve 3-D MHD-equilibria with magnetic islands, are limited to an island

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[10] K. Harafuji, T. Hayashi, T. Sato, J. Comp. Phys. **81**, 169 (1989)

size larger than the limits discussed above. This also implies that computations of islands based on the ideal MHD-model with scalar pressure[12] are valid only for large islands. lowest order (neglecting radial diffusive losses).

In the plateau regime or in the long-mean-free-path regime eq. 2.8 is no longer valid and the anisotropy of the pressure must be computed by kinetic theory. The definition of the anisotropic pressure term is

$$p_{\parallel} - p_{\perp} = m\int\left(v_{\parallel}^{2} - \frac{v_{\perp}^{2}}{2}\right) f d^{3}v \qquad \text{Eq. 2.16}$$

and the distribution function follows from

$$Lf = -\frac{m}{kT} F_{0}\left(v_{\parallel}^{2} - \frac{1}{2}v_{\perp}^{2}\right) V_{0} \bullet \frac{\nabla B}{B} \qquad \text{Eq. 2.17}$$

F_0 is the Maxwellian and the operator L is given by

$$L = v_{\parallel}\frac{\boldsymbol{B}}{B} \bullet \nabla + \boldsymbol{v}_{D} \bullet \nabla - C \qquad \text{Eq. 2.18}$$

C is the linearised collision operator and v_D the drift velocity of the guiding center. In the collisional limit L -> C eq. 2.13 coincides with Eq. 2.10. In the collisionless regime the distortion of the distribution function is larger than in the collisional regime resulting in a larger pressure anisotropy than in the collisional regime. Decoupling of magnetic surfaces and pressure surfaces grows with growing mean free path.

Another approach to solve the problem of island formation has been made by Hegna and Bhattacharjee[13], who treated the islands within the framework of resistive interchange modes. In the vicinity of the rational surface plasma resistivity has been retained, which allows for surface breaking and reconnection. This procedure removes the singularity of the plasma current, however, the method is limited to a single island and does apply to the issue of island overlap and stochasticity. Since the force balance in their theory is not modified magnetic surfaces and pressure surfaces coincide on all scales and for this reason it fails in stochastic regions.

Vacuum fields of stellarator configurations very often exhibit islands on low order rational surfaces. These islands are the result of symmetry breaking fields, which are introduced by the toroidal curvature. These "natural" islands are related to the periodicity of the configuration, in 5-period Wendelstein stellarators these islands occur on

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[11] H.S. Greeenside, A.H. Reiman, A. Salas, PPPL-2469
[12] J.R. Cary, M. Kotschenreuther, Phys. Fluids **28**, 1392 (1985)
[13] C.C. Hegna, A. Bhattacharjee, Phys. Fluids B1, (1989) 392-397

rational surfaces with $\iota = 5/n$, $n = 4,5, \ldots$.. A next generation of smaller islands occurs on $\iota = 10/m$, where m is the sum of the denominators in two adjacent islands of the lowest generation. With rising plasma pressure these islands will be distorted by plasma currents, in particular, parallel currents in the close vicinity will contribute to the modification of the island. In this context the neoclassical bootstrap current has to be taken into account since this effect may be larger than the currents driven by an electric field.

3. The friction model

Several models to incorporate the feedback of the plasma flow on the momentum balance have been developed in the past, the most simple ones are those with a friction term $-\alpha \boldsymbol{v}$ in Eq. 1.1- which leads to dissipation of energy. Such a friction term has been used in the Chodura - Schlüter code[14] to accelerate relaxation towards an equilibrium. In equlibrium, however, the flow velocity is set to zero, and all objections raised against Eq. 1.1 remain valid. In the frictional model proposed by H. Wobig[15] a friction term $-\alpha \boldsymbol{v}$ and finite resistivity have been added. A physical mechanism, leading to a dissipative term $-\alpha \boldsymbol{v}$ arises from charge exchange processes between charged ions and neutral gas background. However, this effect is restricted to the boundary region where the density of neutral gas is high. This model allows to calculate plasma flow and plasma currents in term of ∇p and $\nabla \Phi$. Then from $\nabla \cdot \boldsymbol{j} = 0$ and the equation of continuity a quasilinear elliptic system for p and Φ is obtained. The basic equations of the friction model are

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B} - \alpha \boldsymbol{v} \qquad \text{Eq. 3.1}$$

$$\nabla \Phi = \boldsymbol{v} \times \boldsymbol{B} - \eta \boldsymbol{j} \qquad \text{Eq. 3.2}$$

and

$$\nabla \bullet \boldsymbol{j} = 0 \ ; \quad \nabla \bullet n\boldsymbol{v} = Q \qquad \text{Eq. 3.3}$$

together with

$$\nabla \bullet \boldsymbol{B} = 0 \ ; \quad \nabla \times \boldsymbol{B} = \boldsymbol{j} \qquad \text{Eq. 3.4}$$

$$\boldsymbol{j} = -\frac{\alpha}{B^2 + \alpha\eta} \nabla_\perp \Phi - \sigma \nabla_\parallel \Phi - \frac{1}{B^2 + \alpha\eta} \nabla p \times \boldsymbol{B} \qquad \text{Eq. 3.5}$$

[14] Chodura ,A. Schlüter, J. of Comp. Phys. 41, 68 (1981)

and

$$v = -\frac{\eta}{B^2+\alpha\eta}\nabla_{\perp}p - \frac{1}{\alpha}\nabla_{\parallel}p - \frac{1}{B^2+\alpha\eta}\nabla\Phi\times\boldsymbol{B} \qquad \text{Eq. 3.6}$$

Inserting these results into eqs. 3.3 yields

$$-\nabla\bullet\left(\frac{\alpha}{B^2+\alpha\eta}\nabla_{\perp}\Phi + \sigma\nabla_{\parallel}\Phi\right) = \nabla\bullet\left(\frac{1}{B^2+\alpha\eta}\nabla p\times\boldsymbol{B}\right) \qquad \text{Eq. 3.7}$$

and

$$-\nabla\bullet\left(\frac{\eta n}{B^2+\alpha\eta}\nabla_{\perp}p + \frac{n}{\alpha}\nabla_{\parallel}p\right) - \nabla\bullet\left(\frac{n}{B^2+\alpha\eta}\nabla\Phi\times\boldsymbol{B}\right) = Q \qquad \text{Eq. 3.8}$$

The system 3.7 and 3.8 is elliptic and one can impose either Dirichlet or Neumann type boundary conditions. This feature is quite different from the ideal MHD-model, where no boundary conditions can be imposed. The equations can be solved for any given magnetic field, which is continuous and has continuous derivatives. No restriction as in the ideal MHD model arises. It is a basic feature of elliptic systems that the solutions have continuous derivatives up to second order, if the coefficients of the equations are sufficiently regular. Therefore the derivatives of p and Φ remain bounded, which is distinct from the singular behaviour of plasma currents and flow velocity in the ideal model close to rational magnetic surfaces. Furthermore, the model does not require the existence of magnetic surfaces, however, radial losses and plasma pressure will strongly depend on the quality of the magnetic surfaces.

The basic difference to the ideal MHD-model is demonstrated in Eq. 3.7, which is an inhomogeneous elliptic equation for the electric potential. Since the friction term is small and the plasma conductivity σ is large the perpendicular derivatives in Eq. 3.7 are negligible compared to the parallel derivatives, except for the rational surfaces where these derivatives are essential in removing the singularities. Without this friction term Eq. 3.7 becomes a magnetic differential equation again. Eq. 3.8 is an elliptic equation for the plasma pressure. The parallel derivative has a large coefficient, which leads to small pressure variation along field lines. On irrational magnetic surfaces this is a negligible property, however, on rational surfaces this de-coupling of magnetic surfaces and pressure surfaces avoids the singularities.

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[15] H. Wobig, Z. f. Naturforsch. 41a, 1101 (1986)

Since the derivatives of the potential remains bounded the parallel current density stays also bounded and the close to the resonant rational surface is has the shape sketched in the following figure.

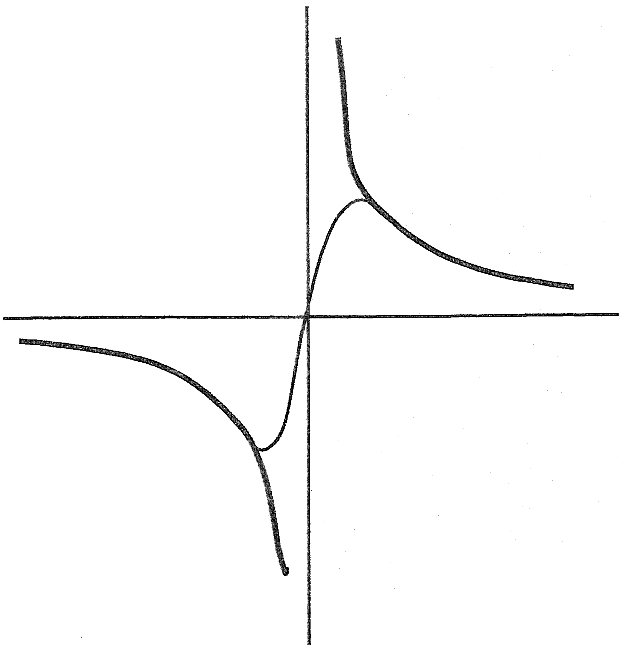

Fig. 2: Parallel current density close to a resonant magnetic surface

In order to find a solution of this system, an iterative scheme has been constructed, where the single steps are well-posed problems. The solution of the "friction - model" of plasma equilibria starts from a given magnetic field $\boldsymbol{B}_0$ and its current $\boldsymbol{j}_0$. Potential and pressure are calculated from Eqs. 3.7 and 3.8. Using Eq. 3.4 we get a new magnetic field $\boldsymbol{B}_1$. Since the current is continuous and has no singularities a solution of Eqs. 3.4 always exists. If no boundary conditions are imposed on the magnetic field, the solution is given by Biot-Savart´s law and added to the field produced by the external coils. In effect, this procedure results in a map T:

$$T\colon \boldsymbol{B}_0 \Rightarrow \boldsymbol{B}_1 \qquad \text{Eq. 3.9}$$

The problem of a self-consistent equilibrium is now reduced to the existence of a fixed point of this map. According to the Schauder fixed point theorem[16], a continuous compact map has a fixed point, if at maps a closed convex subset of a Banach space onto itself. In our particular case the Banach space is the space of all magnetic fields with continuous and bounded derivatives and a norm defined by the maximum of these derivatives. In ref. 14 it has been shown that a fixed point of this map exists, the sequence of magnetic fields obtained by iteration of the map T, $(\boldsymbol{B}_n)$ converges. The condition for convergence is that the source term Q be small enough, or the plasma beta be small enough. Indeed, it can be easily shown, that in case of Q=0 only the trivial solution $\boldsymbol{j}$=0 v=0 exists. One major difference to the iterative scheme proposed by L. Spitzer is, that the existence of magnetic surfaces is not required and the sequence of plasma currents remains bounded. Uniqueness of the solution is not automatically guaranteed, because the basic equations are nonlinear in the magnetic field.

As shown in ref. 14, the frictional model exhibits a strong similarity to the equations describing Bénard convection of a fluid heated from below. The two equations 3.7 and 3.7 have the same structure as the equation for velocity flux function and temperature in the theory of Benard convection. If the source term is small enough and the dissipative term large enough only one solution of these two equations exists. Thus, bifurcation and convective solutions are expected, as it is wellknown from Bénard convection.

4. The Stokes model

In hydrodynamics the Stokes model is applicable to fluids with large viscous forces which slow down the flow velocity far below the sound velocity, which allows one to neglect the inertial forces. In the following the equivalent model of a plasma will be discussed. Although the frictional model removes some of the difficulties of the ideal MHD-model, it is limited to the boundary region, where the plasma-neutral interaction provides a physical basis for the friction forces. In the bulk of the plasma this dissipative process is small and the only term playing the same role is the viscosity. In general the viscosity is only known in a collisional plasma. The equations are given by Braginskii[17], who also shows that viscosity gives rise to a positive entropy production for any finite flow v. Mathematically speaking, the viscous forces define a second order differential operator Vv, which is selfadjoint and positive definite. A similar case exists in hydrodynamics of incompressible fluids where $-\mu\Delta v$ represents the effect of viscosity. As it is shown in textbooks of hydrodynamics[18] the selfadjoint and positive properties of the

[16] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations, Springer 1983, p. 280
[17] Braginskii, Rev. of Plasma Phys. Vol. I
[18] O.A Ladyshenskaja, Funktionalanalytische Untersuchungen der Navier-Stokeschen Gleichungen, Akademie Verlag Berlin 1965 and R. Temam, Navier Stokes Equations, North-Holland, Amsterdam, New York , Oxford 1984

viscosity operator are the basis of existence and uniqueness theorems. Therefore many of these results can be transferred to the plasma case without the need to know the viscous operator in detail.

With viscosity and finite resistivity the stationary equilibrium is described by the following equations:

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B} - V v \qquad \text{Eq. 4.1}$$

$$\nabla \Phi = v \times \boldsymbol{B} - \eta \boldsymbol{j} \qquad \text{Eq. 4.2}$$

$$\nabla \bullet \boldsymbol{j} = 0 \;;\quad \nabla \bullet n v = Q \qquad \text{Eq. 4.3}$$

$$\nabla \bullet \boldsymbol{B} = 0 \;;\quad \nabla \times \boldsymbol{B} = \boldsymbol{j} \qquad \text{Eq. 4.4}$$

Furthermore there is an equation of state, which correlates pressure p and density n. For the very moment we have skipped the inertial forces, their effect will be discussed later. Another problem of the model is the compressibility of the plasma, which can lead to schock formation if the flow velocity is large enough. However, in a slowly diffusing plasma a model where pressure and density are decoupled and the density is homogeneous may be applicable. In particular, if in a stellarator the particle source is provided by cold gas refuelling from the wall, the density will be more or less constant over the plasma radius. In this case the inhomogeneity of the pressure represents the inhomogeneity of the temperature. With this approximation the equation of continuity is replaced by

$$\nabla \bullet \boldsymbol{v} = Q \qquad \text{Eq. 4.5}$$

All these approximations have their limits, the essential feature of this model is the modified momentum balance Eq. 4.1 which is considered as an equation for the flow velocity $\boldsymbol{v}$. The solution of this "Stokes - model" of a plasma equilibrium starts from a given magnetic field $\boldsymbol{B}_0$. Velocity, pressure, current density and electric potential are then calculated using equations 4.1, 4.2 and 4.3. From the current density a new magnetic field is computed and by iterating the procedure one obtains a sequence of magnetic fields (B_n), which converges if certain conditions are satisfied.

In order to obtain an estimate of the electric potential we solve eq. 4.1 for the perpendicular plasma current

$$\boldsymbol{j}_\perp = \frac{\nabla p \times \boldsymbol{B}}{B^2} - \frac{\nabla \bullet \pi(V) \times \boldsymbol{B}}{B^2} \qquad \text{Eq. 4.6}$$

and approximate the velocity in this equation by

$$V_0 = \frac{B \times \nabla\Phi}{B^2} \qquad \text{Eq. 4.7}$$

Ohm´s law yields the parallel component of the electric current and from eq. 8 we get the equation

$$\nabla \bullet \sigma \nabla_{\|}\Phi + \nabla \bullet \frac{\nabla \bullet \pi(V_0) \times B}{B^2} = \nabla \bullet \frac{\nabla p \times B}{B^2} \qquad \text{Eq. 4.8}$$

This equation is an equation for the electric potential in a given magnetic field and a given pressure profile. All derivatives of the potential occur and the parallel derivative has a large weight due to the conductivity σ. Neglecting the viscous term would reduce the equation to a magnetic differential equation for the parallel current, which yields divergent solutions on rational surfaces if the Hamada condition is not satisfied.

$$\oint \frac{dl}{B} = const. \qquad \text{Eq. 4.9}$$

Retaining the viscous term modifies the problem completely. Eq. 4.8 is a partial differential equation for the potential where boundary conditions can be imposed and a condition on rational surfaces does not occur.

5. The Navier-Stokes model

The "Stokes - model" may be applicable, if the viscosity is large enough and the plasma flow velocity is damped to a small level, so that inertial forces do not play a role. In a hot plasma with rare collisions, one has to expect that just the opposite is true and inertial terms are stronger than viscous forces. Taking into account the inertial forces leads us to a "Navier-Stokes-model" of a steady state plasma with the following equations

$$\rho \boldsymbol{v} \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} - V\boldsymbol{v} \qquad \text{Eq. 5.1}$$

$$\rho = const. \quad ; \quad \nabla \bullet \boldsymbol{v} = Q \qquad \text{Eq. 5.2}$$

$$\nabla \Phi = \boldsymbol{v} \times \boldsymbol{B} - \eta \boldsymbol{j} \; ; \quad \nabla \bullet \boldsymbol{j} = 0 \qquad \text{Eq. 5.3}$$

As has been proven in hydrodynamics (see ref. 17), equations 5.1 and 5.2 have a unique solution $\boldsymbol{v}, p$, if the force $\boldsymbol{j} \times \boldsymbol{B}$ is small enough and the viscosity is large enough. In a fluid with small viscosity (large Reynolds number) bifurcations and multiple solutions occur. It is the nonlinearity of the inertial terms, which gives rise to these phenomena and it has to be expected that these bifurcations also occur in a steady state plasma. Viscous damping will limit the maximum flow velocity in such bifurcated equilibria, from the results in hydrodynamics, however, these convective solutions may result in enhanced plasma losses because of the higher viscous entropy production. Another field of interest is the H-mode confinement regime where a poloidal plasma rotation is observed. The Navier-Stokes model is appropriate to describe bifurcation of a non-rotating towards a rotating plasma.

After solutions of the momentum balance and the equation of continuity, the subsequent procedure follows the same line as described above. Existence and uniqueness of solutions have been investigated by Spada and Wobig[19]. Utilising the same procedure as in the friction model the method of proving existence is to contruct a compact map in a Banach space of vector functions $(\boldsymbol{v},\boldsymbol{B})$ and to invoke Schauder's fixed point theorem. As in the friction model it was found that existence of continuous solutions can be proven if the source term Q is small enough and the dissipative effects are large enough. This procedure is completely different from the method to minimize the energy integral as is applied in ideal MHD. If the conditions on sources and dissipative effects are satisfied, the convergence of the iterative scheme is guaranteed.

In a rotating plasma equilibrium magnetic surfaces and pressure surfaces obviously do not coincide. Let us assume that a rotating equilibrium with toroidally nested pressure surfaces exists. It can be shown that inertial forces must provide the driving mechanism to maintain the flow against viscous damping. For this purpose we take the scalar product of eq. 5.1 with the current density and the magnetic field and average over the pressure surface.

$$\langle \boldsymbol{j}\bullet \nabla\bullet \rho v{:}v\rangle - \langle \boldsymbol{j}\bullet V v\rangle = 0$$

$$\langle \boldsymbol{B}\bullet \nabla\bullet \rho v{:}v\rangle - \langle \boldsymbol{B}\bullet V v\rangle = 0 \qquad \text{Eq. 5.4}$$

Since viscous forces and inertial forces in rotating equilibria are small compared to the pressure gradient and the Lorentz forces we have $\boldsymbol{B}\cdot\nabla p \approx 0$, which implies that magnetic surfaces and pressure surface nearly coincide. The compensation of viscous damping forces by inertial forces is the essential feature of the Stringer spin-up mechanism[20].

This simplified Navier-Stokes model may still not represent the real behaviour in a toroidal plasma, when plasma flow parallel to the magnetic field becomes large and even may approach the velocity of sound. Hazeltine, Lee and Rosenbluth[21] have investigated such case. In this case the approximation with constant density is no longer valid and the complete equation of continuity must be taken into account.

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[19] M. Spada, H. Wobig, J. Phys. A: Math. Gen. 25 (1992) 1575 -1591

[20] T. Stringer, Phys Rev. Letters **22**, 770 (1969)

[21] Hazeltine, Lee and Rosenbluth Phys. Fluids, 14,361 (1971).

$$\nabla \bullet \rho v = Q \quad \text{Eq. 5.5}$$

Compressibility of the plasma allows formation of schock fronts as has been discussed by Hellberg, Winsor and Dawson[22]. In this case pressure and density are no longer decoupled and the equation of continuity introduces another non-linearity. The basic iteration scheme to find a self-consistent solution persists, however the hydrodynamic step - the solutions of eqs. 5.1 and 5.2 are more complicated than in the case of an incompressible fluid. The theory of compressible fluids has been studied extensively in the past, however, it is only a few years ago that general existence and uniqueness theorem have been proven by A. Valli, W.M. Zajaczkowski[23] and others. The main condition for existence and uniqueness is that for solving the momentum balance and the equation of continuity the external force $j\mathrm{x}B$ has to be small enough. The plasma current grows with β and this is limited by the source terms (heating and fuelling). Therefore, this condition can only be met, if the plasma pressure controlled by the source term Q is small enough.

6. Conclusions

One of the limitations of the ideal MHD-model of plasma equilibria is the existence of magnetic islands on rational magnetic surfaces. These islands are characteric for three-dimensional configurations and in principle the can occur on any rational surface, thus forming a dense set of islands and leading to a pathological pressure distribution. On the other hand the radial pressure profile in real plasmas is the result of a diffusion process and therefore continuous with continuous derivatives. The mathematical implications of the ideal MHD-model in 3D-geometry have been extensively discussed by H. Grad[24], who argues that in general smooth solutions of the ideal model with nested surfaces do not exist. Unless a convergent procedure has been found any numerical effort to compute ideal 3D-equilibria may lead to erroneous results.

One way to eliminate the deficienccies of the ideal model is to incorporate dissipative effects, which in particular of rational magnetic surfaces truncate the singularities and decouple pressure surfaces and magnetic surfaces on a small spatial scale. This leads to a feedback of plasma motion on the force balance and mathematically it introduces

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[22] Hellberg, Winsor and Dawson, Phys. Rev. Letters, **28**, 1922 (1972).
[23] A. Valli, W.M. Zajaczkowski, Commun. Math. Phys. 103, 259 (1986)
[24] H. Grad, *Theory and Applications of the Non-existence of Simple Toroidal Plasma Equilibria,* Int. J. of Fusion Energy 3, No 2 April 1985, p 33

second order derivatives of the plasma velocity. As described above the problem to find a self-consistent equilibrium in non-axisymmetric configurations is then reduced to a fixed point problem of a non-linear map T in a Banach space of magnetic fields. The properties of this map depend on the plasma model, which determines the pressure, plasma flow velocity and the plasma currents for a given magnetic field. Any plasma model with frictional damping or viscous damping leads to classical elliptic problems, and use can be made of the rich mathematical literature in this field (see for example the paper of Sermange and Temam[25] who investigate the incompressible MHD-flow). These damping terms remove the singularities of the ideal MHD-model, however, inertial forces and compressibility of the plasma can give rise to multiple solutions and enhanced plasma losses. This problem has already been recognized and in the past and many authors (H.P. Zehrfeld, B.J. Green[26] and Hazeltine, Lee and Rosenbluth[27]) have studied the effect of these terms. Bifurcations, multiple solutions and enhanced plasma losses arising from inertial forces and other non-linearities can occur in axisymmetric and non-axisymmetric configurations. The mathematical theory of stellarator equilibria with a compressible plasma is still incomplete, however, since after retaining inertia and viscosity, the major task is to solve hydrodynamic equations, there is a chance that on the basis of far developed theory of hydrodynamic flow the theory of stellarator equilibria can established. The ideas outlined here only illustrate some steps on the way towards such a theory, these steps have to be completed with rigorous proofs.

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[25] M. Sermange and R. Temam, *Some Mathematical Questions Related to the MHD-Equations*, Com. of Pure and Applied Mathematics Vol. XXXVI, 635-664 (1983)
[26] H.P. Zehrfeld, B.J. Green ,Phys. Rev.Letters, {\bf 23}, 961 (1969)
[27] B.J. Green [27]and Hazeltine, Lee and M.N. Rosenbluth, Phys. Fluids, {\bf 14 },361 (1971)