Shear Alfvén Mode Resonances in Nonaxisymmetric Toroidal Low-Pressure Plasmas

II. Singular Modes in the Shear Alfvén Continuum

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The spatial dependence of modes in the shear Alfvén continuum of nonaxisymmetric toroidal magnetohydrodynamic plasma equilibria is investigated. As in configurations with plane or axial symmetry it is found that these modes may exhibit a singularity extended across whole "singular" magnetic surfaces. Nonaxisymmetry, however, changes the type of singularity from (in general) oscillating to (in general) logarithmic. If the nonaxisymmetry is too strong, the surface-covering property of the singular modes may be lost. Zero plasma pressure is assumed throughout.

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1. Introduction

In a previous paper [1], which is Part I of a two-part study of the shear Alfvén continuum in general toroidal geometry, we derived a system of three linearized equations that govern the modes of a pressure-less magnetohydrodynamic (MHD) plasma of arbitrary geometry. The system is compact, and it is convenient to use when the shear Alfvén and ballooning MHD continua are studied. In the present paper, Part II, of our study, we use this system to explore the shear Alfvén continuum and its associated singularities for general, pressureless, toroidal MHD equilibria. A basic assumption here is that such equilibria exist, possibly in some approximate sense.

Continuum modes are tied to singular solutions of the linearized equations. As regards the shear Alfvén continuum, the nature of these singularities has been the subject of many past investigations for a variety of MHD plasma configurations. In configurations with spatial symmetry, such as the planar sheet pinch, the cylindrical screw pinch and the axisymmetric torus, the shear Alfvén continuum singularities occur about resonant magnetic surfaces $\psi(\mathbf{r}) = \text{const}$ that depend on the frequency ω. A hyperbolic partial differential equation that contains only derivatives along the equilibrium magnetic field lines is solved over each magnetic surface, and the frequency is chosen so that solutions of this equation are bounded and smooth. The allowed frequencies of the singular modes thereby become functions of ψ, i.e. $\omega = \omega(\psi)$. This expression can be inverted to locate the singular magnetic surface as a function of ω, $\psi = \psi_0(\omega)$. Because individual field lines on magnetic surfaces with spatial symmetry are equivalent, the hyperbolic equation can be solved along any field line to obtain the eigenmodes over the entire singular magnetic surface $\psi(\omega)$. The mathematical form of the singularity is derived by solving the complete system of linearized MHD equations with respect to ψ in the vicinity of $\psi(\omega)$. This can be done with power series representions of the modes. Generally, singularities of the type $\ln(\psi - \psi_0)$, $(\psi - \psi_0)^{-1}$ or $(\psi - \psi_0)^{i\tau}$, $(\psi - \psi_0)^{i\tau - 1}$, where τ is a real parameter, are found.

In simple, one-dimensional geometry, such as the cylindrical pinch with coordinates (r, θ, z), Fourier modes are typically assumed with respect to the ignorable coordinates, θ and z. For this case, the magnetic surfaces are surfaces of constant r, and the surface eigenvalue $\omega^2(r)$ satisfies an algebraic equation. The radial dependence of the Fourier modes, out of the magnetic surfaces $r = \text{const}$, can be described by a single second-order equation e.g. for the radial displacement ξ_r [2]. A singularity of the equation occurs if the coefficient of the highest derivative vanishes. This happens where the mode frequency ω coincides with the Alfvén frequency ω_A or the cusp (or slow magnetosonic) frequency ω_C [2], quantities which depend continuously on r. Here, $\omega_A^2 = k_\parallel^2 v_A^2$ where $k_\parallel$ is the wave number parallel to the equilibrium magnetic field $\mathbf{B}$. The Alfvén speed squared v_A^2 is defined by $v_A^2 = \mu_0 \rho / |\mathbf{B}|^2$, where μ_0 is the permeability of free space and ρ is the plasma mass density. In addition, ω_C^2 depends on the plasma pressure P and goes to zero for vanishing P. For fixed frequency ω the displacement ξ_r develops a logarithmic singularity at $r = r_0$, where r_0 is defined by $\omega^2 = \omega_A^2(r_0)$ or $\omega^2 = \omega_C^2(r_0)$.

In axisymmetric toroidal configurations, the surface eigenvalue problem along the field lines reduces to an ordinary differential equation with periodic coefficients in the poloidal angle after a Fourier decomposition with respect to the toroidal angle [3, 4]. The eigenvalue $\omega(\psi)$ must be chosen so that periodic boundary conditions are satisfied. In this case the shear Alfvén and the cusp continua couple together [3, 4] if the pressure is finite. In addition, Floquet theory can be used to show that gaps develop in the continuous spectrum. If the effect of toroidicity is treated as a perturbation of a straight cylindrical geometry the gaps are seen to form at radial positions where two eigenfrequencies of the cylinder coincide [5, 6], so that "level crossing" is avoided. It was claimed originally that the structure of the radial singularity is logarithmic [4] as in the sheet and screw pinches. However, it was recently shown that this is not so in general and that the generic behavior for the normal component of the displacement out of the resonant magnetic surface at $\psi = \psi_0$ is proportional to $(\psi - \psi_0)^{i\tau}$, where τ is a real ψ_0-dependent quantity [5, 7]. The logarithmic law, however, remains valid under special circumstances, e.g. if the equilibrium is up/down symmetric.

Less is known about the continuum singularities of fully three dimensional configurations with no spatial symmetry, i.e. nonaxisymmetric tori, asymmetric configurations with finite-length magnetic field lines, or open-ended configurations without symmetry. Previous investigations of these general cases have focused mainly on the *existence* of continua in the absence of spatial symmetry. It has been established that the spectra of modes, both in the shear Alfvén continuum and in the ballooning continuum, are again determined by eigenvalue problems for ω^2 which contain only derivatives along magnetic field lines [8, 9, 10, 11]. However, unlike in the spatially symmetric configurations, individual field lines in configurations without symmetry are not equivalent. The continuum physics therefore is primarily connected with individual field *lines*. The existence of continua on asymmetric magnetic *surfaces* as a whole, therefore, is a problem which is related [1, 8, 12, 13, 14] but not identical to the continuum problem on field lines. As regards the shear Alfvén continuum, two general classes of solutions can be envisioned: modes that are periodic and smooth over magnetic surfaces, and modes localized to individual field lines with no smooth extension over magnetic surfaces. For applications to plasma heating and current drive, the smooth and periodic eigenmodes could be the more important of the two classes. However, this is an issue that requires further investigation. Regarding the spatial dependence of shear Alfvén continuum modes in the vicinity of a singular magnetic surface in configurations without symmetry, it is suggested in [9] and [15] that the logarithmic singularity persists. As another possibility a power law was mentioned in Ref. [16]. In Refs. [9, 15, 16] either no final conclusions were reached or no explicit arguments for the claims were presented.

The present investigation is devoted to an analysis of the singular eigenfunctions of toroidal pressureless plasmas without spatial symmetry. The focus of the investigation is the class of shear Alfvén continuum modes that are smooth over magnetic surfaces. It will be demonstrated that the logarithmic law $\ln(\psi - \psi)$ is the generic singularity. However, in exceptional cases a power law $(\psi - \psi)^{i\tau}$, where τ is a ψ dependent constant, might still occur. The nature of the singularity, apart from its relevance as a fundamental phenomenon, is relevant for the coupling of external sources of energy into the plasma via resonant absorption, as in the Alfvén wave heating scheme or, e.g., in the interaction of the solar wind with the ionosphere.

Part II of this investigation is organized as follows. In Section 2 the linearized mode equations in the form developed in Part I are presented again. The relevance of an ordinary differential equation along field lines not only for the spectrum of the modes in the shear Alfvén continuum but also for their spatial dependence is pointed out again. Section 3 is devoted to the discussion of this equation, Eq. (16). Different types of boundary conditions are considered. Emphasis is placed on modes which smoothly cover the singular surface. Of particular relevance for the subsequent sections is the number of linearly independent solutions which exist for one eigenvalue. An expansion scheme for the radial dependence is introduced in Section 4. In the generic nonaxisymmetric case it is found that the normal displacement out of the magnetic surface is governed by a logarithm in $\psi - \psi_0$. Section 5 contains a discussion and the conclusions. Appendix A and B are devoted to technicalities. In Appendix C, the case where the continuum equation has two linearly independent eigensolutions per eigenvalue is explored. In these cases the radial dependence does not necessarily follow a logarithmic law. The axisymmetric torus is used as an example. The axisymmetric case is also pursued in Appendix D from the standpoint of the theory of a general 3-D plasma.

2. Linearized MHD Equations of Pressureless Plasma

Our analysis is based on the equations developed in Ref. [1]. The equilibrium configuration is pressureless with magnetic field $\mathbf{B}$ and current density $\mathbf{J}$. Magnetic field line coordinates (r^1, r^2, r^3) that satisfy

$$\mathbf{B} \cdot \nabla r^1 = 0 \quad \text{and} \quad \mathbf{B} \cdot \nabla r^2 = 0 \tag{1}$$

are assumed. Force lines are labeled by the coordinates r^1 and r^2, while points along the force lines are labeled by r^3. In field line coordinates the magnetic field can always be expressed locally in the Clebsch representation [17],

$$\mathbf{B} = f(r^1, r^2)\,[\nabla r^1 \times \nabla r^2], \tag{2}$$

where f is an arbitrary function of its arguments, and $\operatorname{div}\mathbf{B} = 0$ implies the independence of f on r^3. By choosing the variables r^1 and r^2 appropriately, the function f can always be transformed to unity [17]. As in Ref. [1], we shall adopt this convention henceforth. The contravariant and covariant components of an arbitrary vector $\mathbf{a}$ are denoted, respectively, by a^i and a_i, and the metric tensors g^{ik} and g_{ik} are defined by $g^{ik} = \nabla r^i \cdot \nabla r^k$ and $g_{ik} = \mathrm{e}_i \cdot \mathrm{e}_k$, where e_i is the basis vector $\partial \mathbf{r}/\partial r^i$. The determinant of the matrix $\{g_{ik}\}$ is denoted by g. Since $1/\sqrt{g} = [\nabla r^1 \times \nabla r^2] \cdot \nabla r^3$, Eq. (2), with $f = 1$, implies $\sqrt{g}B^3 = 1$.

In Ref. [1] the linearized MHD equations that govern the MHD dynamics of a pressureless plasma in general three-dimensional (3-D) geometry are reduced to a system of three coupled partial differential equations for the two covariant components of the wave electric field, E_1 and E_2, and one covariant component of the wave magnetic field, b_3,

$$i\omega b_3 = B_3\left(\frac{\partial E_1}{\partial r^2} - \frac{\partial E_2}{\partial r^1}\right) + \frac{B_1}{B^3}DE_2 - \frac{B_2}{B^3}DE_1, \tag{3}$$

$$B^3\,\frac{\partial\, i\omega b_3}{\partial r^1} - D\left(\frac{B_1}{B_3}\,i\omega b_3\right) = +\left(\mathcal{L}^{21}E_1 + \mathcal{L}^{22}E_2\right) + \mu_0\frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2}DE_1, \tag{4}$$

$$B^3\,\frac{\partial\, i\omega b_3}{\partial r^2} - D\left(\frac{B_2}{B_3}\,i\omega b_3\right) = -\left(\mathcal{L}^{11}E_1 + \mathcal{L}^{12}E_2\right) + \mu_0\frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2}DE_2, \tag{5}$$

where D and $\mathcal{L}^{ik} = \mathcal{L}^{ki}$, for $i, k = 1$ and 2, are differential operators that contain only derivatives with respect to r^3,

$$D = B^3\frac{\partial}{\partial r^3} \quad \text{and} \quad \mathcal{L}^{ik} \equiv D\frac{g^{ik}}{|\mathbf{B}|^2}D + \mu_0\rho\omega^2\frac{g^{ik}}{|\mathbf{B}|^2}. \tag{6}$$

From Ohm's law, $\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$, and the linearized magnetic pressure $p^* = \mathbf{B}\cdot\mathbf{b}$, it readily follows that $E_1 = -v^2$, $E_2 = v^1$ and $p^* = B^3 b_3$, where v^1 and v^2 are contravariant components of the fluctuating plasma velocity $\mathbf{v}$. Therefore, Eqs. (3) – (5) can also be transformed into an equivalent system of coupled equations that govern v^1, v^2 and p^*. These equations are given in Ref. [1]. Our analysis of the continuum singularities will, however, be based on the coupled equations for (E_1, E_2, b_3).

We now depart from general magnetic configurations and focus our investigation on toroidal configurations with closed nested magnetic surfaces $\psi(\mathbf{r}) = \text{const}$. The configurations need not be symmetric in any way. Since $\mathbf{B}\cdot\nabla\psi = 0$ by definition, and in view of Eq. (1), we can identify one of the coordinates r^1 or r^2 with ψ. We take $r^1 \equiv \psi$. The second coordinate r^2 then labels magnetic field lines on the toroidal magnetic surfaces.

With this choice of coordinates, $\partial/\partial r^1$ denotes differentiation *normal* to magnetic surfaces, while $\partial/\partial r^2$ and $\partial/\partial r^3$ denote differentiation *within* magnetic surfaces. The normal derivative appears only in Eqs. (4) and (5), while the surface derivatives appear in each equation of the system. A matrix notation can be conveniently used to separate the normal derivative from the surface derivatives. Introduce a scalar function v and a column vector $\mathbf{w}$,

$$v = E_1 \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} i\omega b_3 \\ E_2 \end{pmatrix}. \tag{7}$$

Equations (3) – (5) can then be cast in the form

$$Av = \mathcal{B}^T \mathbf{w}, \tag{8}$$

$$\frac{\partial}{\partial r^1}\mathbf{w} = \mathcal{D}\mathbf{w} + \mathcal{C}v, \tag{9}$$

where A is a scalar differential operator that is identical to $\mathcal{L}^{11}$,

$$A = D\frac{g^{11}}{|\mathbf{B}|^2}D + \mu_0\rho\omega^2\frac{g^{11}}{|\mathbf{B}|^2}. \tag{10}$$

The coefficients $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are matrix differential operators that contain only derivatives with respect to the magnetic surface coordinates r^2 and r^3,

$$\mathcal{B} = \begin{pmatrix} -B^3\dfrac{\partial}{\partial r^2} + D\dfrac{B_2}{B_3} \\ -\mathcal{L}^{12} + \mu_0\dfrac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} \end{pmatrix}, \qquad \mathcal{C} = \frac{1}{B^3}\begin{pmatrix} \mathcal{L}^{21} + \mu_0\dfrac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} \\ B^3\dfrac{\partial}{\partial r^2} - \dfrac{B_2}{B_3}D \end{pmatrix}, \tag{11}$$

$$\mathcal{D} = \frac{1}{B^3}\begin{pmatrix} D\dfrac{B_1}{B_3} & \mathcal{L}^{22} \\ -1 & \dfrac{B_1}{B_3}D \end{pmatrix}, \tag{12}$$

where $\partial/\partial r^3$ enters via the scalar operators D and $\mathcal{L}^{ik}$. Equations (8) and (9) form a closed system of linear partial differential equations that governs v and $\mathbf{w}$ over the toroidal plasma volume. Equation (8) contains only derivatives with respect to the surface coordinates r^2 and r^3. Therefore, it simply relates $\mathbf{w}$ to v on a magnetic surface labeled by r^1. This relationship yields a formal expression for v in terms of $\mathbf{w}$ with r^1 a fixed parameter,

$$v = A^{-1}\mathcal{B}^T\mathbf{w}, \tag{13}$$

where A^{-1} denotes the inverse of the differential operator A. Substitute Eq. (13) in Eq. (9). The result is an equation that governs $\mathbf{w}$ over the toroidal volume,

$$\frac{\partial}{\partial r^1}\mathbf{w} = \mathcal{C}A^{-1}\mathcal{B}^T\mathbf{w} + \mathcal{D}\mathbf{w}. \tag{14}$$

Equation (13) provides v associated with any solution of Eq. (14). A key issue is the existence of the inverse operator A^{-1}, which is associated with the null space of A over a given magnetic surface. The inverse of A does not exist if its null space is not empty, i.e. if the homogeneous equation,

$$AV^0 = 0 \tag{15}$$

has nontrivial solutions $V^0(r^1, r^2, r^3)$ that satisfy suitable boundary conditions over the toroidal surface labeled by r^1. If nontrivial solutions exist, the magnetic surface is said to be singular, and v and $\mathbf{w}$ have spatial singularities about it. These two specific issues, the existence of solutions of Eq. (15) on a magnetic surface r_0^1 = const and the spatial singularities of v and $\mathbf{w}$ about $r^1 = r_0^1$, form the central themes of this paper.

3. Shear Alfvén Continuum Equation

Equation (15), where the operator A is defined by Eq. (10), governs the singular magnetic surfaces. The operator $D \equiv \mathbf{B}\cdot\nabla = B^3\partial/\partial r^3 = |\mathbf{B}|\mathrm{d}/\mathrm{d}\ell$, where ℓ is the arc length along a magnetic field line, renders it a coordinate invariant ordinary differential equation along that field line,

$$|\mathbf{B}|\frac{\mathrm{d}}{\mathrm{d}\ell}\left(\frac{|\nabla\psi|^2}{|\mathbf{B}|^2}|\mathbf{B}|\frac{\mathrm{d}}{\mathrm{d}\ell}V^0\right) + \mu_0\rho\omega^2\frac{|\nabla\psi|^2}{|\mathbf{B}|^2}V^0 = 0, \tag{16}$$

where the field line coordinates $r^1 = r_0^1$ and $r^2 = r_0^2$ are fixed parameters for the integration with respect to ℓ. In Eq. (16), ω^2 is an eigenvalue parameter that is to be chosen such that the solution V^0 satisfies appropriate boundary conditions. For open-ended magnetic field configurations, see e.g. Refs. [10, 18] for some geophysical applications, boundary conditions are dictated by the physics at the ends of the field lines. In the present analysis, which deals with toroidally closed magnetic surfaces, "most" of which are irrational and support infinitely extended magnetic field lines, different "boundary" conditions are required. Essentially two types may be envisaged.

Type I boundary conditions: on irrational surfaces, and hence on surfaces with infinitely long field lines, $V^0(\ell)$ is bounded; on rational surfaces, and hence on surfaces with closed field lines, $V^0(\ell)$ is periodic. Hameiri, in Ref. [8] advocates this approach quite generally. Here, we consider type I conditions only as an intermediate step. It permits us to obtain information about the spectrum and the eigenfunctions of Eq. (16) that is also relevant for our principal but more restrictive boundary conditions, namely those of type II.

Type II boundary conditions: V^0 is smooth everywhere on singular surfaces, and hence bounded and periodic around poloidally and around toroidally closed paths. These boundary conditions were applied e.g. in Refs. [11, 13, 14] and will be applied here, as well. We thus restrict our analysis to modes which smoothly cover the whole, singular magnetic surface. To make this point clearer, consider an irrational magnetic surface. Equation (16) determines the solution along a field line. The integration starts at some arbitrary point $\ell = \ell_0$, where the values of V^0 and its derivative $\mathrm{d}V^0/\mathrm{d}\ell$ are prescribed, and continues without bound. As l increases the entire surface will eventually be covered. Provided the magnetic field and the geometry of the configuration are smooth enough and ω^2 is chosen properly, a smooth bounded solution $V^0(\ell)$ might exist everywhere along this whole field line. Nevertheless, V^0, when considered as a function of a poloidal and a toroidal angle, is, in general, not smooth on that surface. It may vary significantly from one point on the surface to a neighboring point that is connected to the first one by many toroidal circuits of the field line. Type II boundary conditions exclude this class of non-smooth solutions and also solutions of Eq. (16) that are exponentially "localized" on a field line, see below. Type II conditions, therefore, are more restrictive than type I conditions. By imposing smoothness and periodicity of the solution on the whole surface, however, we are permitted to perform partial integrations wherever this may be useful. From a practical point of view smooth, surface covering solutions may be more relevant than non-smooth solutions as regards e.g. Alfvén wave heating of plasmas. Still, localized modes could after all be of some experimental relevance. Singular surfaces with rational rotational transform, i.e. with all field lines closed but different from each other in 3-D configurations, are excluded from our consideration as well since it is unlikely that a common eigenvalue ω^2 of Eq. (16) could be found that is valid for all those different field lines.

An important issue later in the analysis concerns the existence and number of linearly independent solutions of Eq. (16) that satisfy type II boundary conditions. In order to be doubly periodic the required solution would have to depend on ℓ in the form $V^0(\theta(\ell), \phi(\ell))$, with $V^0(\theta, \phi) = V^0(\theta + 2\pi, \phi) = V^0(\theta, \phi + 2\pi)$, where θ and ϕ are, respectively, poloidal and toroidal angle-like, but otherwise arbitrary, coordinates on the singular magnetic surface. In order to bring out the implications of these conditions more clearly, it is useful to pick e.g. the coordinate θ as an independent variable along the field line and to make use of the freedom in θ and ϕ in such a way that the field line becomes straight in these coordinates. With

$$|\mathbf{B}|\frac{\mathrm{d}}{\mathrm{d}\ell} = B^\theta \frac{\mathrm{d}}{\mathrm{d}\theta}, \qquad \mathrm{d}\phi = q\,\mathrm{d}\theta, \tag{17}$$

along the field line, where $B^\theta = \mathbf{B}\cdot\nabla\theta$, $B^\phi = \mathbf{B}\cdot\nabla\phi$ and $q(\psi) \equiv B^\phi/B^\theta$, Eq. (16) assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(P_0(\theta, q\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}V^0\right) + \mu_0\rho\omega^2\, Q_0(\theta, q\theta)\, V^0 = 0. \tag{18}$$

Here $P_0 \equiv (\nabla\psi)^2 B^\theta/|\mathbf{B}|^2$ and $Q_0 \equiv (\nabla\psi)^2/(B^\theta|\mathbf{B}|^2)$. The coefficients $P_0(\theta,\phi)$ and $Q_0(\theta,\phi)$ are 2π-periodic in each of their two arguments. For irrational values of q, such functions, with $\phi = q\theta$, are commonly denoted as "quasiperiodic" with respect to θ. In this terminology type II boundary conditions require that the solution $V^0(\theta)$ be quasiperiodic with the same periodicities as the coefficients P_0 and Q_0, i.e. $V^0(\theta) = V^0(\theta, q\theta)$ where $V^0(\theta, q\theta)$ is 2π-periodic in both arguments. Equation (18) can be put into a more standardized form. We apply a Liouville transform from the independent and the dependent variables θ and V^0 into x and W,

$$x = \int_{\theta_0}^{\theta}\frac{\mathrm{d}\theta}{\widetilde{B^\theta}}, \qquad W = a(\theta, q\theta)\, V^0(\theta), \tag{19}$$

where $a(\theta, q\theta) \equiv |\nabla\psi|/\widetilde{B}$, $\widetilde{B} \equiv |\mathbf{B}|/B_0$ and $\widetilde{B^\theta} \equiv B^\theta/B_0$. For normalizing purposes, $B_0 = |\mathbf{B}_0|$ is the modulus of the magnetic field at some arbitrarily chosen point on the singular surface. If B^θ goes through zero somewhere along the field line, an analogous transformation with the roles of θ and ϕ reversed should be applied. With these transformations, Eq. (16) becomes a generalized Hill's equation,

$$\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + [\lambda - U(x, qx)]\, W = 0, \tag{20}$$

where

$$\lambda \equiv \frac{\mu_0\rho\omega^2}{B_0^2}, \qquad U(x, qx) \equiv \frac{1}{a}\widetilde{B^\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\widetilde{B^\theta}\frac{\mathrm{d}a(\theta, q\theta)}{\mathrm{d}\theta}\right)\Bigg|_{\theta=\theta(x)} \tag{21}$$

and $U(x, qx)$ is a real 2π quasiperiodic function. Our type II boundary conditions are equivalent to the requirement

$$W(x) = W(x, qx), \tag{22}$$

with $W(x, qx)$ 2π-periodic in both arguments. Equation (20) also corresponds to the Schrödinger equation with potential energy U and total energy $E = \lambda$.

The theory of quasiperiodic differential equations is far more involved than that for differential equations with periodic coefficients. It is often treated in the context of the so-called almost periodic differential equations, which are slightly more general. Two reviews are given e.g. in Ref. [19]. While there does not seem to exist a complete spectral theory for Eq. (20) with boundary conditions of type II, pertinent analytical results are available for large values of ω^2. Indirect evidence from analytical studies with type I boundary conditions and from numerical solutions [13, 14] are also helpful. In Ref. [20], analytic results were obtained which, when specialized to our present situation, are as follows. For sufficiently large values of ω^2 or for sufficiently small nonaxisymmetry, provided q is sufficiently irrational and $U(x, qx)$ is real analytic and bounded, Eq. (20) has solutions of the Bloch form

$$W(x) = e^{i\mu x} P(x, qx), \tag{23}$$

with $P(x, qx)$ 2π-periodic in both arguments and $\mu = \mu(\lambda)$ real in an infinite sequence of λ intervals. The forbidden gaps between the intervals are centered approximately around $\lambda = (m + nq)/2$, where $m, n = 0, \pm 1, \pm 2, \ldots$, etc.. Away from the gaps the functions $W(x)$ and $W^*(x)$, where the asterisk denotes the conjugate complex, are linearly independent solutions to the same eigenvalue λ. These results are the straightforward generalization of the result for Hill's equation with real periodic $U(x)$ [21], where the forbidden gaps, for $|U|$ small enough, are situated approximately around $\lambda = m/2$. In Ref. [22], it was shown that under conditions similar to those given above, the analogy to Hill's equation goes even further. At both edges of the gaps, μ assumes the same value μ_{mn}, where $\mu_{mn} = (m + nq)/2$. One of the solutions at each end is still of the form of Eq. (23), while the second linearly independent solution is secular. For even values of m and n, μ_{mn} assumes the value $M + Nq$, with $M = m/2$ and $N = n/2$ integer, so that the factor $e^{i\mu x}$ becomes properly quasiperiodic, $e^{i\mu x} = e^{i(Mx+Nqx)}$. At each end of "even" gaps, therefore, there exists one quasiperiodic solution with period 2π for x and qx. Thus, although each such solution develops, so to speak, along an ergodic field line, it fits together smoothly across the windings on the whole surface. The two quasiperiodic solutions per gap have the same number of zeros in the poloidal and toroidal directions, respectively, but different eigenvalues, $\lambda = \lambda_{1mn}$ and $\lambda = \lambda_{2mn}$. In axisymmetry, the m-gaps are caused by the equilibrium quantities not being independent of the poloidal angle, either from the bending of a cylindrical configuration into a torus, and/or by non-circularity of the cross section. Analogously, the additional splitting up of the m gaps into (m, n)-gaps comes from the nonsymmetry in the toroidal direction. In the periodic Hill's equation it can happen for very special cases of $U(x)$ that one or more gaps shrinks to zero with the consequence that *two* periodic linearly independent solutions *coexist* [21]. The analogous situation might occur also in the nonaxisymmetric, quasiperiodic case for very special configurations. The likelihood for this to occur, however, seems to be quite small [22]. For the following sections, therefore, it is important to keep in mind that, in the generic weakly nonaxisymmetric case, there is only *one* smooth solution per eigenvalue for our type II boundary conditions.

If the nonaxisymmetry of the configuration is large enough and/or the modulus of the frequency ω small enough, spectral theory for Eq. (20) with type I boundary conditions predicts that in general the type of the spectrum and of the eigenfunctions changes [19]. Instead of the Bloch type solutions of Eq. (23), which exist "undamped" in $-\infty < x < \infty$, localized eigenfunctions occur which decrease exponentially on both sides of finite x intervals and are accompanied by a point spectrum instead of the continuous spectrum of the Bloch eigenstates. In Ref. [14], this transition was observed numerically. A mode satisfying type II boundary conditions, with fixed m and n numbers, was observed with increasing amount of nonaxisymetry, measured, say, by a parameter F. As F increases, the mode develops steeper and steeper ridges until at a particular value of F the mode ceases to smoothly fill the surface and turns into a localized mode along the field line. We speculate that such *localized* modes in the shear Alfvén spectrum, described by Eq. (16), could be signatures of a possibly new class of eigenstates of the full MHD equations. They would be analogous to the *localized* "gap" modes whose footprints can be observed within the gaps of the continuous ballooning spectrum. Analytical [19] and numerical [14] results suggest that the localized states of Eq. (16) usually fill a finite frequency band. This distinguishes them from the gap modes which only occur as isolated points in the spectrum. The issue of possibly new modes is an important one which has potential implications e.g. for rf heating and current drive. It requires a thorough examination. A coexistence of localized and extended shear Alfvén continuum modes in a nonaxisymmetric, though not quasiperiodic, configuration was observed analytically in Ref. [23]. As a further special case, if $q(\psi)$ is irrational but well approximated by rationals ("Liouville number"), the spectrum may still be of another type, called "singular continuous", with eigenstates which have chaotic properties [19]. For our type II boundary conditions these moderately irrational ψ surfaces must also be excluded.

4. Singular Modes

In this section, Eqs. (8) and (9) are solved in the vicinity of a singular magnetic surface labeled by $r^1 = r_0^1(\omega)$. On this surface, by assumption, Eq. (15) has nontrivial period solutions for V^0. Our solution technique is based on a self-consistent ordering scheme for v and $\mathbf{w}$ and their derivatives, as already applied in the axisymmetric case [7]. We introduce a small quantity ϵ that parameterizes the order scheme, and we introduce the notation $\mathrm{O}(\cdots)$ to depict the order of the enclosed quantity with respect to ϵ. Because Eq. (15) is satisfied when $r^1 = r_0^1(\omega)$, we make the ansatz that in the vicinity of $r_0^1(\omega)$, v is order 1, while Av is small quantity of order ϵ,

$$v \sim \mathrm{O}(1) \quad \text{and} \quad Av \sim \mathrm{O}(\epsilon). \tag{24}$$

Equation (8) then implies that $\mathcal{B}^T\mathbf{w}$ must also be of order ϵ,

$$\mathcal{B}^T\mathbf{w} \sim \mathrm{O}(\epsilon). \tag{25}$$

A second ansatz that we make is that the orders of surface derivatives of $\mathbf{w}$ and v are identical to the orders of $\mathbf{w}$ and v respectively. Equation (25) then implies that $\mathbf{w}$ is an $O(\epsilon)$ quantity, while $\partial\mathbf{w}/\partial r^1$ is an order 1 quantity,

$$\mathbf{w} \sim O(\epsilon), \tag{26}$$

$$\frac{\partial \mathbf{w}}{\partial r^1} \sim O(\mathcal{C}v) \sim O(1). \tag{27}$$

Because $\mathbf{w}$ is of order ϵ, Eq. (27) implies that normal differentiation $\partial/\partial r^1$ is of order ϵ^{-1}. The ordering scheme defined by Eqs. (24) – (27) can be made explicit by introducing scaled variables y and $\bar{\mathbf{w}}$,

$$r^1 - r^1_0 = \epsilon y \qquad \text{and} \qquad \mathbf{w} = \epsilon\bar{\mathbf{w}}, \tag{28}$$

where y and $\bar{\mathbf{w}} \sim O(1)$. The derivative of $\mathbf{w}$ with respect to r^1 then becomes,

$$\frac{\partial \mathbf{w}}{\partial r^1} = \frac{\partial \bar{\mathbf{w}}}{\partial y} \sim O(1). \tag{29}$$

Substitute Eqs. (28) and (29) in Eqs. (8) and (9),

$$Av = \epsilon\mathcal{B}^T\bar{\mathbf{w}}, \tag{30}$$

$$\frac{\partial}{\partial y}\bar{\mathbf{w}} = \mathcal{C}v + \epsilon\mathcal{D}\bar{\mathbf{w}}. \tag{31}$$

Assume now power series expansions for v and $\bar{\mathbf{w}}$ with respect to ϵ,

$$v = \mu_0(y)V^0(r^2, r^3) + \epsilon\mu_1(y)V^1(r^2, r^3) + \cdots , \tag{32}$$

$$\bar{\mathbf{w}} = \nu_0(y)\mathbf{W}^0(r^2, r^3) + \epsilon\nu_1(y)\mathbf{W}^1(r^2, r^3) + \cdots , \tag{33}$$

where the factors $\mu_i(y)$, $V^i(r^2, r^3)$ and $\nu_i(y)$, $\mathbf{W}^i(r^2, r^3)$, $i = 0, 1, \cdots$, are to be determined. In addition, we expand the scalar operator A and the matrix operators $(\mathcal{B}^T, \mathcal{C}, \mathcal{D})$ about $y = 0$ in the following form

$$A(r^1, r^2, r^3) = A_0(r^2, r^3) + \epsilon y A_1(r^2, r^3) + \cdots . \tag{34}$$

An analogous notation is adopted for the expansions of $(\mathcal{B}^T, \mathcal{C}, \mathcal{D})$. The spatial variations of v and $\mathbf{w}$ in the normal direction, and therefore the spatial singularities, are contained in the factors $\mu_i(y)$ and $\nu_i(y)$, respectively. Substitute these power series expansions in Eqs. (30) and (31) and order terms according to powers of ϵ. This yields equations that determine the expansion coefficients of v and $\bar{\mathbf{w}}$. The results at order 1 are

$$\mu_0 A_0 V^0 = 0, \tag{35}$$

$$\frac{\mathrm{d}\nu_0}{\mathrm{d}y}\mathbf{W}^0 = \mu_0 \mathcal{C}_0 V^0. \tag{36}$$

If $\mu_0 \neq 0$, Eq. (35) yields $A_0 V^0 = 0$ which is identical with Eq. (15), considered at $r^1 = r_0^1$. This equation was amply discussed in Sections 2 and 3. It implies that the magnetic surface is singular. According to our type II boundary conditions, we accept only such $V^0(r^2, r^3)$ which are smooth and satisfy periodic boundary conditions. Equation (36) can be separated to yield equations that connect $\mathbf{W}^0$ to V^0 and ν_0 to μ_0,

$$\mathbf{W}^0 = \mathcal{C}_0 V^0, \tag{37}$$

$$\frac{\mathrm{d}\nu_0}{\mathrm{d}y} = \mu_0. \tag{38}$$

At order ϵ, the following equations appear,

$$\mu_1 A_0 V^1 + y\mu_0 A_1 V^0 = \nu_0 \mathcal{B}_0^T \mathcal{C}_0 V^0, \tag{39}$$

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}y}\mathbf{W}^1 = \mu_1 \mathcal{C}_0 V^1 + y\mu_0 \mathcal{C}_1 V^0 + \mu_0 \mathcal{D}_0 \mathcal{C}_0 V^0, \tag{40}$$

where Eq. (37) was substituted for $\mathbf{W}^0$. Equation (39) determines V^1 in terms of V^0. However, because the operator A_0 has no inverse, the equation can be solved only if appropriate solvability conditions over the singular magnetic surface are satisfied. The number of solvability conditions agrees with the number of linearly independent solutions of Eq. (16) at fixed eigenvalue ω^2. Introduce the operator A_{0*} that is adjoint to A_0,

$$\Big\langle a, A_0 b \Big\rangle = \Big\langle A_{0*} a, b \Big\rangle, \tag{41}$$

where $\langle a, b\rangle$ designates an inner product over the singular magnetic surface that we define as

$$\Big\langle a, b \Big\rangle \equiv \oint \mathrm{d}S \frac{1}{|\nabla\psi|} a^* b. \tag{42}$$

In Eq. (42), $\mathrm{d}S$ is the differential surface element, and the asterisk denotes complex conjugation. With this definition of the inner product and with type II boundary conditions it is easily seen that the operator $\mathcal{L}^{ik}$ defined by Eq. (6), and therefore the operator $A = \mathcal{L}^{11}$, is self-adjoint, see Appendix A.

As has been shown in Section 3, Eq. (15) has in most cases only one linearly independent solution in the generic three-dimensional case. In the rest of this section we consider this case only. In Appendix C the non-generic case is also briefly investigated. Let U^0 be any null vector of $A_{0*} = A_0$ which need not be identical to V^0. The solvability condition for Eq. (39) is obtained by taking the inner product of Eq. (39) with U^0 and using Eqs. (15) and (41). The following expression results,

$$y\frac{\mathrm{d}\nu_0}{\mathrm{d}y} = \sigma\nu_0, \tag{43}$$

where

$$\sigma = \frac{\left\langle U^0, \mathcal{B}_0^T C_0 V^0\right\rangle}{\left\langle U^0, A_1 V^0\right\rangle}. \tag{44}$$

Equation (43) determines ν_0. If $\sigma \neq 0$, its integral can be expressed as

$$\nu_0 = \frac{c_0}{\sigma} y^\sigma, \tag{45}$$

where c_0 is the free integration constant. If $\sigma = 0$, Eq. (43) reduces to

$$y\frac{\mathrm{d}\nu_0}{\mathrm{d}y} = 0. \tag{46}$$

The implication of Eq. (46) is that $\nu_0(y)$ is a distribution proportional to the delta function, $\nu_0(y) \sim \delta(y)$. In order to obtain a non-distributional solution for $\nu_0(y)$, it is necessary to modify the expansions of v and $\bar{\mathrm{w}}$ [7]. In place of Eqs. (32) and (33), we introduce the generalized expansions,

$$\begin{aligned} v &= \nu'(y)\left[V^0(r^2,r^3) + \epsilon y V^1(r^2,r^3) + \cdots\right] \\ &\quad + \epsilon\nu(y)\left[k^0(r^2,r^3) + \epsilon y k^1(r^2,r^3) + \cdots\right], \end{aligned} \tag{47}$$

$$\begin{aligned} \bar{\mathbf{w}} &= \nu(y)\left[\mathbf{W}^0(r^2,r^3) + \epsilon y \mathbf{W}^1(r^2,r^3) + \cdots\right] \\ &\quad + \nu'(y) y\left[\boldsymbol{\ell}^0(r^2,r^3) + \epsilon y \boldsymbol{\ell}^1(r^2,r^3) + \cdots\right], \end{aligned} \tag{48}$$

where the added terms contain the expansion coefficients $k^i(r^2,r^3)$ and $\boldsymbol{\ell}^i(r^2,r^3)$. If Eqs. (47) and (48), and the matrix expansion defined by Eq. (34), are substituted in Eqs. (30) and (31) and the resulting terms are ordered according to ϵ, in the same manner as in Ref. [7], $\nu(y)$ is found to be a logarithm [7],

$$\nu(y) = c_{00} + c_0 \ln y, \tag{49}$$

where c_{00} and c_0 are free integration constants.

Let β denote the numerator and α denote the denominator of σ in Eq. (44),

$$\beta = \left\langle U^0, \mathcal{B}_0^T C_0 V^0\right\rangle, \tag{50}$$

$$\alpha = \left\langle U^0, A_1 V^0\right\rangle. \tag{51}$$

To simplify the notation the subscript 0 and the superscript 0 in the integrands of Eqs. (50) and (51) will be henceforth omitted. From the definition of the matrices $\mathcal{B}$ and $\mathcal{C}$ in Eq. (11), there results $\beta = \beta_0 + \beta_J$, where

$$\beta_0 \equiv -\Big\langle U^* \Big[B^3 \left(\frac{\partial}{\partial r^2} - \frac{\partial}{\partial r^3} \frac{B_2}{B_3} \right) \frac{1}{B^3} \mathcal{L}^{12} + \mathcal{L}^{12} \left(\frac{\partial}{\partial r^2} - \frac{B_2}{B_3} \frac{\partial}{\partial r^3} \right) \Big] V \Big\rangle, \tag{52}$$

$$\beta_J \equiv -\Big\langle U^* \Big[B^3 \left(\frac{\partial}{\partial r^2} - \frac{\partial}{\partial r^3} \frac{B_2}{B_3} \right) \widehat{\sigma} \frac{\partial}{\partial r^3} - \widehat{\sigma} B^3 \frac{\partial}{\partial r^3} \left(\frac{\partial}{\partial r^2} - \frac{B_2}{B_3} \frac{\partial}{\partial r^3} \right) \Big] V \Big\rangle. \tag{53}$$

Here, $\langle a \rangle$ is defined as $\oint \mathrm{d}S\, a / |\nabla \psi|$ and $\widehat{\sigma} \equiv \mathbf{J} \cdot \mathbf{B} / |\mathbf{B}|^2$. Since $\widehat{\sigma}$ depends only on ψ, see Eq. (B.11), it may be factored out of the surface integration in Eq. (53). The remaining terms in the integrand cancel, resulting in

$$\beta_J = 0. \tag{54}$$

Hence $\beta = \beta_0$.

In order to evaluate β_0 in more detail, it is advantageous to employ Boozer coordinates [24]. In Appendix B some of their relevant properties are summarized. For ease of notation, we denote Boozer coordinates by (ψ, θ, ϕ), although the same symbols are employed in Section 3 in various ways for more generally defined poloidal and toroidal coordinates. In Boozer coordinates, $B_2 = 0$ holds, see Eq. (B.3), so that β simplifies to

$$\beta = -\Big\langle U^* \left(B^3 \frac{\partial}{\partial r^2} \frac{1}{B^3} \mathcal{L}^{12} + \mathcal{L}^{12} \frac{\partial}{\partial r^2} \right) V \Big\rangle. \tag{55}$$

With Eqs. (A.4) and (B.9) the operators $B^3 \partial / \partial r^2$ and $\mathcal{L}^{12}$ in the first and the second terms, respectively, may be transferred to the factors preceding them. This results in

$$\beta = \Big\langle \left(\mathcal{L}^{12} V \right) \frac{\partial U^*}{\partial r^2} - \left(\mathcal{L}^{12} U^* \right) \frac{\partial V}{\partial r^2} \Big\rangle. \tag{56}$$

It has been assumed above that there exists only one linearly independent, doubly periodic solution to $A_0 V^0 = 0$. This implies

$$U^* = c\, V, \tag{57}$$

where c is in general an arbitrary, complex constant. With Eq. (57) there results

$$\beta = 0. \tag{58}$$

We thus find that in nonaxisymmetric toroidal configurations the shear Alfvén continuum is connected in general with a logarithmic singularity.

As mentioned above, non-generic plasma equilibria without symmetry might exist in which the equation $A_0 V^0 = 0$ has two instead of one linearly independent solutions. This case is explored in Appendix C. It is shown that, under these circumstances, β, in general, is not zero, implying thereby a power law singularity with an imaginary exponent σ.

The logarithmic singularity in the general asymmetric case is in contrast with the axisymmetric case [7] where, generically, β does not vanish, and a power law with imaginary exponent σ is normally the rule. In the next section we comment on this distinction between the general 3-D case and the axisymmetric 2-D case from a physical point of view. In Appendices C and D σ in the 2-D case is derived from the fully 3-D expressions.

It follows from Eq. (52) and the definition of $\mathcal{L}^{12}$ in Eq. (6) that β always vanishes if *orthogonal* field line coordinates exist. Orthogonality of coordinates implies that $g^{12} \equiv 0$ and from Eq. (D.5) in Appendix D a shear-free magnetic field. Existence of orthogonal field line coordinates is treated in detail in Ref. [25].

It is pointed out here that Eq. (58) complements a result reported in Ref. [26] in that the exponent μ, which corresponds to $-i\sigma$ in the present study is in most cases zero.

5. Discussion and Conclusions

This investigation is devoted to the existence and the properties of shear Alfvén continuum modes in arbitrary, nonaxisymmetric, toroidal MHD equilibria with zero pressure. A key assumption is that such 3-D equilibria exist. Shear Alfvén continuum modes are commonly characterized by a radial singularity when the distance to a "singular" magnetic surface $\psi = \psi_0$, say, shrinks to zero. We consider this type of behavior, and we assume that the modes smoothly fill in that surface. Our main result is as follows. Such shear Alfvén continuum modes, which are well known from axisymmetric or plane configurations, exist as well in nonaxisymmetric configurations, but they have characteristically different properties.

The first difference concerns the singularity behavior. It is found that the dominant generic dependence of the normal velocity $v_\psi \equiv \mathbf{v} \cdot \nabla\psi$ of the modes, out of the singular surface, is a logarithm in $\psi - \psi_0$, while in the axisymmetric case it, generically, is a power law with imaginary exponent, i.e. an oscillatory singularity [7]. The second difference has to do with the way the modes fill the singular surface. In axisymmetry, no matter how contorted the poloidal cross section may be, the modes always fill in smoothly the resonant surface. With sufficiently strong nonaxisymmetry, some or all modes in a finite frequency interval may lose this property. Instead, they turn into "localized" modes, with an exponential localization along a magnetic field line only, and corresponding infinite gradients within the magnetic surface, perpendicular to the field line. Such cases require a different treatment, which we have not persued here.

At the first glance, it may seem strange that an arbitrarily small amount of nonaxisymmetry should change the radial dependence of the modes discontinuously from $v_\psi \sim (\psi - \psi_0)^{i\tau}$, with τ a real and not necessarily small number, into $v_\psi \sim \ln(\psi - \psi_0)$. The reason behind this qualitative change is the fact that the axisymmetric case is degenerate, with two linearly independent eigensolutions per eigenvalue, while in the nonaxisymmetric case, however small the asymmetry may be, the degeneracy is removed, and only one eigensolution remains. An additional consideration may also be helpful. Both radial dependencies, i.e. the power law and the logarithm, are derived under the assumption of a steady state of the plasma. If one considers an axisymmetric plasma to which a small nonaxisymmetric perturbation is applied, it will take some time, Δt say, until a new steady state and a new, modified set of modes are reached. Changes in the phase of a mode are transmitted with the propagation of the mode itself. Equation (16), with ω^2 replaced by $-\partial^2/\partial t^2$, e.g., is a wave equation that plays an important role in that process. Noticeable effects in the mode structure can be expected if the difference between the new phase and the original one becomes of the order of $\pi/2$. The weaker is the nonaxisymmetry in the configuration, the longer it takes until such a phase difference accumulates during the wave propagation. In that sense the transition from axisymmetry to nonaxisymmetry is not as discontinuous as it first seems since the transition period Δt gets longer and longer the smaller the asymmetry is. An analogous situation is well known e.g. from the theory of Bernstein waves which propagate perpendicularly to a constant magnetic field $\mathbf{B}_0$. However small B_0 may be, there is no Landau damping of the modes while for $B_0 = 0$ Landau damping is present. A qualitative explanation of this effect lies again in the temporal evolution of the modes. For finite B_0, the mode amplitude oscillates periodically in time with period $\sim 1/B_0$. For $B_0 \to 0$, the oscillation period therefore goes to infinity, and it takes an infinite amount of time until a steady state is reached [27].

Our results show that a logarithmic singularity is the genuine generic property of shear Alfvén continuum modes in toroidal configurations of moderate nonaxisymmetry. In such cases the logarithm prevails whether or not the configuration has additional symmetries, such as e.g. stellarator symmetry [28]. The logarithmic law is familiar also from much simpler geometries such as the cylindrical screw pinch. The causes behind this common law in the three-dimensional case and in the one-dimensional case, however, are quite different. In configurations with less than the full 3-D topology, as in axisymmetric or in straight configurations, the eigenvalues are degenerate, and this, in general, leads to an oscillatory singularity [7] of the form $(\psi - \psi_0)^{i\tau}$, where the value of τ depends on the form of the poloidal plasma cross section. In special cases, in particular if the poloidal cross section is deformed continuously into an up/down symmetric shape, τ continuously goes to zero and ultimately the logarithm emerges. In that latter case the logarithm is not genuine, but simply the result of a particular symmetry [7].

Appendix A: Self-Adjoint Operator $\mathcal{L}^{ik}$

The operator $\mathcal{L}^{ik}$ defined by Eq. (6) can be written in the form

$$\mathcal{L}^{ik} = \mathbf{B}\cdot\nabla p^{ik}\mathbf{B}\cdot\nabla + q^{ik}, \tag{A.1}$$

where $p^{ik} = g^{ik}/|\mathbf{B}|^2$ and $q^{ik} = \mu_0\rho\omega^2 p^{ik}$. In arbitrary coordinates (r^1, r^2, r^3) the covariant expression for the surface element $\mathrm{d}S$ on the $r^1 = \text{const}$ surface is $\mathrm{d}S = \sqrt{g}\,|\nabla r^1|\,\mathrm{d}r^2\,\mathrm{d}r^2$, where $1/\sqrt{g} = [\nabla r^1 \times \nabla r^2]\cdot\nabla r^3$. On irrational magnetic surfaces, field line coordinates (r^1, r^2) are not convenient to apply periodic boundary conditions and to carry out integrals over magnetic surfaces. For these purposes a more convenient set is given by arbitrary angle-like poloidal and toroidal coordinates θ and ϕ, such as were already used in Section 3. In order to avoid confusion with the coordinates (r^1, r^2, r^3), we introduce the notation

$$[\nabla\psi\times\nabla\theta]\cdot\nabla\phi = \frac{1}{\sqrt{g_B}}. \tag{A.2}$$

The index "B", which in this appendix has no real significance, is chosen, for later purposes, to refer to Boozer coordinates, see Appendix B. Since $\operatorname{div}\mathbf{B} = 0$ one may write

$$\begin{aligned}\left\langle a, \mathcal{L}^{ik}b\right\rangle &= \oint \mathrm{d}S\frac{1}{|\nabla\psi|}\,a^*\mathcal{L}^{ik}b = \oint \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g_B}\,a^*\left[\mathbf{B}\cdot\nabla p^{ik}\mathbf{B}\cdot\nabla b + q^{ik}b\right] = \\ &= \oint \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g_B}\,a^*\left[\operatorname{div}\left(\mathbf{B}p^{ik}\mathbf{B}\cdot\nabla b\right) + q^{ik}b\right] = \\ &= \oint \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g_B}\operatorname{div}\left[\mathbf{B}a^*p^{ik}\mathbf{B}\cdot\nabla b\right] + \oint \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g_B}\left[-p^{ik}\left(\mathbf{B}\cdot\nabla a^*\right)\left(\mathbf{B}\cdot\nabla b\right) + a^*q^{ik}b\right].\end{aligned} \tag{A.3}$$

Taking into account the periodicity of the integrand in θ and ϕ the first term in the last line of Eq. (A.3) vanishes since $\sqrt{g_B}\operatorname{div}u = \partial(\sqrt{g_B}\,u)/\partial\theta + \partial(\sqrt{g_B}\,u)/\partial\phi$, implying that the integrand consist of total differentials in θ and ϕ. If now the same procedure is inverted, but with the roles of a^* and b interchanged, there results

$$\left\langle a, \mathcal{L}^{ik}b\right\rangle = \oint \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g_B}\left[\mathbf{B}\cdot\nabla p^{ik}\mathbf{B}\cdot\nabla a^* + q^{ik}a^*\right]b = \left\langle \mathcal{L}^{ik}a, b\right\rangle. \tag{A.4}$$

This proves $\mathcal{L}^{ik}_* = \mathcal{L}^{ik}$.

Appendix B: Boozer Coordinates

Boozer [17, 24] has shown that coordinates (r^1, r^2, r^3) exist such that MHD equilibrium magnetic fields can be written in both, contravariant and covariant form as

$$\mathbf{B} = [\nabla r^1 \times \nabla r^2] = \gamma \, \nabla r^1 + \nabla r^3, \tag{B.1}$$

where $\gamma(\mathbf{r})$ is determined by plasma currents and need not interest us here. From (r^1, r^2, r^3) it is possible to transform furthermore to coordinates (ψ, θ, ϕ) such that the coefficients in the transformation are purely functions of ψ [24], according to

$$\begin{aligned} r^1 &= \psi, \\ r^2 &= \theta - q^{-1}(\psi)\phi, \\ r^3 &= I(\psi)\theta + G(\psi)\phi. \end{aligned} \tag{B.2}$$

In Eq. (B.2), θ and ϕ are, respectively, poloidal and toroidal coordinates normalized to the interval $[0, 2\pi]$. They constitute a particular example of angular coordinates, in terms of which the magnetic field lines are straight, as mentioned in Section 3. The functions $I(\psi)$ and $G(\psi)$ correspond, respectively, to the equilibrium toroidal and poloidal currents, and $q(\psi)$ is the safety factor. It readily follows that $B^1 = B^2 = 0$ and

$$B_1 = \gamma, \quad B_2 = 0, \quad B_3 = 1, \tag{B.3}$$

so that

$$B^3 = [\nabla r^1 \times \nabla r^2] \cdot \nabla r^3 = \frac{1}{\sqrt{g}} = |\mathbf{B}|^2 = B^3 B_3. \tag{B.4}$$

From Eqs. (B.2) it readily follows that

$$\frac{\partial}{\partial r^1} = \frac{\partial}{\partial \psi}, \quad \frac{\partial}{\partial r^2} = \frac{1}{R}\left(G\frac{\partial}{\partial \theta} - I\frac{\partial}{\partial \phi}\right), \quad \frac{\partial}{\partial r^3} = \frac{1}{qR}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \phi}\right), \tag{B.5}$$

where $R(\psi) \equiv G(\psi) + I(\psi)/q(\psi)$. Similarly, it is straightforward to show that

$$B^\psi = 0, \quad B^\theta = \frac{1}{q\sqrt{g_B}}, \quad B^\phi = \frac{1}{\sqrt{g_B}}, \quad B_\theta = G, \quad B_\phi = I, \tag{B.6}$$

so that

$$|\mathbf{B}|^2 = B^\theta B_\theta + B^\phi B_\phi = \frac{R}{\sqrt{g_B}}, \tag{B.7}$$

which relates $\sqrt{g_B}$ with the modulus of the magnetic field, and

$$\mathbf{B} \cdot \nabla = B^\theta \frac{\partial}{\partial \theta} + B^\phi \frac{\partial}{\partial \phi} = \frac{1}{q\sqrt{g_B}}\left(\frac{\partial}{\partial \theta} + q\frac{\partial}{\partial \phi}\right). \tag{B.8}$$

It is easily seen in Boozer coordinates that, for arbitrary doubly periodic functions $a(\theta,\phi)$ and $b(\theta,\phi)$,

$$\left\langle aB^3 \frac{\partial}{\partial r^2} b \right\rangle = -\left\langle bB^3 \frac{\partial}{\partial r^2} a \right\rangle. \tag{B.9}$$

This follows from

$$\left\langle aB^3 \frac{\partial}{\partial r^2} b \right\rangle = \oint \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g_B}\,aB^3 \frac{\partial}{\partial r^2} b = \oint \mathrm{d}\theta\,\mathrm{d}\phi\,a\frac{1}{R}\left(G\frac{\partial}{\partial\theta} - I\frac{\partial}{\partial\phi}\right) b =$$
$$= -\oint \mathrm{d}\theta\,\mathrm{d}\phi\,b\frac{1}{R}\left(G\frac{\partial}{\partial\theta} - I\frac{\partial}{\partial\phi}\right) a = -\left\langle bB^3 \frac{\partial}{\partial r^2} a \right\rangle, \tag{B.10}$$

where Eq. (B.4), the periodicity of the integrand and the pure ψ dependence of the coefficients were used.

A further relevant relation for the present pressureless case is most easily evident in Boozer coordinates, see Eq. (26) in Ref. [24], namely

$$\frac{\partial}{\partial r^2}\frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} = \frac{\partial}{\partial r^3}\frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2} = 0. \tag{B.11}$$

Appendix C: Two Eigensolutions per Eigenvalue

Here, we show how the method of Section 3 to determine the singular ψ dependence has to be modified when the spectral equation $A_0V^0 = 0$, Eq. (35), with real operator A_0, has two instead of one linearly independent doubly periodic solutions per eigenvalue. For plasma configurations without axial symmetry, this would be a non-generic case. Let $U_1(r^1, r^2)$ and $U_2(r^1, r^2)$ be two linearly independent doubly periodic solutions to

$$A_{0*}U = A_0U = 0. \tag{C.1}$$

In the general case these two functions would not be related to each other in any obvious way. Of course, V^0 in Eq. (39) of Section 3 is a linear combination of U_1 and U_2,

$$V^0 = a_1U_1 + a_2U_2, \tag{C.2}$$

with constant coefficients a_1 and a_2. We take the inner product of Eq. (39) with U_1 and U_2. As a consequence of Eq. (C.1), terms containing A_0V^1 vanish and, with Eq. (C.2), a system of two homogeneous equations for the coefficients a_1 and a_2 results. The determinant of this system must vanish. For $i,\ k = 1,\ 2$, we define

$$\alpha_{ik} \equiv \left\langle U_i, A_1U_k \right\rangle, \quad \beta_{ik} \equiv \left\langle U_i, B_0^T C_0 U_k \right\rangle. \tag{C.3}$$

With $\sigma \equiv (\mathrm{d}\nu_0/\mathrm{d}y)y/\nu_0$, see Eq. (43), a quadratic equation for σ emerges,

$$\sigma^2 \det\{\alpha_{ik}\} - \sigma\,Q + \det\{\beta_{ik}\} = 0, \tag{C.4}$$

where

$$Q \equiv \alpha_{11}\beta_{22} + \alpha_{22}\beta_{11} - \alpha_{21}\beta_{12} - \alpha_{12}\beta_{21}. \tag{C.5}$$

The two solutions for σ are

$$\sigma_{\pm} = \frac{1}{2\det\{\alpha_{ik}\}}\left[Q \pm \left(Q^2 - 4\det\{\alpha_{ik}\}\det\{\beta_{ik}\}\right)^{1/2}\right]. \tag{C.6}$$

Repeating the steps from Eq. (55) to Eq. (56) for β_{ik}, one finds

$$\beta_{ik} = \Big\langle \left(\mathcal{L}^{12}U_k\right)\frac{\partial U_i^*}{\partial r^2} - \left(\mathcal{L}^{12}U_i^*\right)\frac{\partial U_k}{\partial r^2}\Big\rangle. \tag{C.7}$$

This shows that $\beta_{ik} = -\beta_{ki}^*$, which implies that β_{11} and β_{22} are imaginary. Also, keeping in mind that the operator A_1 is the ψ derivative of the operator A taken at $r^1 = r_0^1$, it follows from the self-adjointness of the operator A that $\alpha_{ik} = \alpha_{ki}^*$, implying that α_{11} and α_{22} are real . With these relations, Q, $\det\{\alpha_{ik}\}$ and $\det\{\beta_{ik}\}$ become

$$Q = i\left[\alpha_{11}\,\Im\beta_{22} + \alpha_{22}\,\Im\beta_{11} - 2\Im(\alpha_{12}^*\beta_{12})\right], \tag{C.8}$$

$$\det\{\alpha_{ik}\} = \alpha_{11}\alpha_{22} - |\alpha_{12}|^2, \quad \det\{\beta_{ik}\} = -(\Im\beta_{11})\,(\Im\beta_{22}) + |\beta_{12}|^2, \tag{C.9}$$

where $\Im$ denotes the imaginary part. If Q does not vanish, it is an imaginary quantity, while $\det\{\alpha_{ik}\}$ and $\det\{\beta_{ik}\}$, are real quantities. This renders the first term in Eq. (C.6) imaginary and the first term in the radicand negative. The properties of the second term in the radicand, however, are not obvious in general. Without going into further details, it cannot be excluded, therefore, that the exponents $\sigma_{\pm}$, which governs the spatial dependence of $\mathbf{w} \sim (\psi - \psi_0)^\sigma$, have both real and imaginary parts. This leaves open the possibility that nonaxisymmetric configurations of the non-generic type which are discussed here might show a combination of a power low with a real exponent together with an oscillatory behavior, in the vicinity of the singular surface.

An example for which two linearly independent solutions U_1 and U_2 exist and for which $\sigma_{\pm}$ can be fully evaluated is the axisymmetric torus. The result for this configuration is already known [7]. The imaginary exponent σ from Ref. [7] is recovered with the present technique in the following way. Introduce coordinates (ψ, θ, ϕ), where θ is an arbitrary poloidal coordinate and ϕ is the toroidal angle. Because of axisymmetry, Fourier modes in the toroidal direction decouple, and the ansatz $\sim \mathrm{e}^{\pm in\phi}$, with n denoting the toroidal Fourier mode number, can be made. Let

$$V^0(\theta, \phi) = V^0(\theta)\,\mathrm{e}^{in\phi} \tag{C.10}$$

denote a doubly periodic solution to Eq. (C.1). Since A and hence A_0 are real operators the conjugate complex function $V^{0*}(\theta, \phi) = V^{0*}(\theta)\,\mathrm{e}^{-in\phi}$ of V^0 must satisfy $A_0\,V^{0*} = 0$

as well. The functions $e^{in\phi}$ and $e^{-in\phi}$ are, however, linearly independent provided $n \neq 0$. This proves that in the axisymmetric case, for $n \neq 0$, there always exist two linearly independent solutions, V^0 and V^{0*}, to the same eigenvalue(s) ω^2 of the *real* operator A. The gaps in the spectrum, which are opened up by nonaxisymmetry, see Section 3, shrink to zero in the axisymmetric case, and the corresponding pairs of separate eigenvalues degenerate into double eigenvalues. Possible choices for U_1 and U_2 thus are $U_1 = V^0(\theta)\, e^{in\phi}$ and $U_2 = V^{0*}(\theta)\, e^{-in\phi}$. Since $\oint d\phi\, e^{\pm 2in\phi} = 0$ for $n \neq 0$, it follows that α_{ik} and β_{ik} vanish if $i \neq k$. Furthermore, by inspection, it is clear that $\alpha_{22} = \alpha_{11}$ and $\beta_{22} = -\beta_{11}$. As a result, Eq. (C.6) simplifies to

$$\sigma_\pm = \pm\frac{\beta_{11}}{\alpha_{11}} = \mp\frac{\beta_{22}}{\alpha_{22}}. \tag{C.11}$$

Hence σ_+ and σ_- are imaginary with $\sigma_+ = -\sigma_-$. This comes from the fact that the functions U_1 and U_2 together include the pair of mode numbers $(n, -n)$. In the pressureless axisymmetric case, opposite signs of n correspond to opposite signs of σ, as follows from Eq. (D.6) below, or from Ref. [7].

In the present case of axisymmetry one can proceed also in a different manner, which treats modes with one particular sign of n only. Details of this alternate method are in Appendix D.

Appendix D: The Axisymmetric Case

With (ψ, θ, ϕ) coordinates the operator A in Eq. (10), which governs the shear Alfvén continuum, can be written in the form

$$A = \left(B^\theta \frac{\partial}{\partial\theta} + B^\phi \frac{\partial}{\partial\phi}\right) \frac{g^{11}}{|\mathbf{B}|^2} \left(B^\theta \frac{\partial}{\partial\theta} + B^\phi \frac{\partial}{\partial\phi}\right) + \mu_0 \rho \omega^2 \frac{g^{11}}{|\mathbf{B}|^2}, \tag{D.1}$$

where θ is an arbitrary poloidal coordinate and ϕ is the toroidal angle. We make the ansatz that the modes are proportional to $e^{in\phi}$, with n denoting now a fixed toroidal Fourier mode number. As in Eq. (C.10), a doubly periodic solution $V^0(\theta, \phi)$ of the spectral equation (35) is again written in the form $V^0(\theta, \phi) = V^0(\theta)\, e^{in\phi}$. With this ansatz, the operator A can be cast into an alternate form, $A \to \widetilde{A}$, where

$$\widetilde{A} \equiv \left(B^\theta \frac{\partial}{\partial\theta} + inB^\phi\right) \frac{g^{11}}{|\mathbf{B}|^2} \left(B^\theta \frac{\partial}{\partial\theta} + inB^\phi\right) + \mu_0 \rho \omega^2 \frac{g^{11}}{|\mathbf{B}|^2}. \tag{D.2}$$

For $n \neq 0$, $\widetilde{A}$ is a *complex* operator. For periodic boundary conditions, $\widetilde{A}$ and the other analogously defined operators $\widetilde{\mathcal{L}}^{ik}$ are still self-adjoint, $\widetilde{\mathcal{L}}^{ik}_* = \widetilde{\mathcal{L}}^{ik}$. This readily follows e.g. with Boozer coordinates, analogous to the derivation of Eq. (B.9) in Appendix B. Since $\widetilde{A}$ is complex, V^{0*} is not an eigenfunction of $\widetilde{A}_0$, in general. At a fixed value of n, the operator $\widetilde{A}_0$, generically, has only one periodic eigensolution with respect to θ and ϕ per eigenvalue ω^2. In contrast, A_0 has two independent eigensolutions corresponding to n and $-n$. We now sketch a derivation of β using the single eigensolution of $\widetilde{A}$. The known result, with pressure neglected, will be recovered [7]. In the axisymmetric case, Eq. (55), with $\mathcal{L}^{12}$ replaced by $\widetilde{\mathcal{L}}^{12}$, is still valid, but the operator $\widetilde{\mathcal{L}}^{12}$ is now complex, for $n \neq 0$. We then have, in place of Eq. (56),

$$\beta = \Big\langle \left(\widetilde{\mathcal{L}}^{12} V\right) \frac{\partial U^*}{\partial r^2} - \left(\widetilde{\mathcal{L}}^{12*} U^*\right) \frac{\partial V}{\partial r^2} \Big\rangle = 2\,i\,\Im \Big\langle \left(\widetilde{\mathcal{L}}^{12} V\right) \frac{\partial V^*}{\partial r^2} \Big\rangle. \tag{D.3}$$

Here, $U = cV$ was taken, since U, the solution of the adjoint equation $\widetilde{A}_* U = \widetilde{A} U = 0$, is linearly dependent on V. Without loss of generality, the arbitrary complex constant c has been set equal to unity since it cancels in $\sigma = \beta/\alpha$. Equation (D.3) contains terms with ω^2 in $\widetilde{\mathcal{L}}^{12}$. They can be eliminated by making use of the relation $\widetilde{\mathcal{L}}^{11} V = \widetilde{A} V = 0$ which also contains ω^2. We then find,

$$\beta = -2i\Im \Big\langle \frac{g^{11}}{|\mathbf{B}|^2}\, s\, (DV) \frac{\partial V^*}{\partial r^2} \Big\rangle, \tag{D.4}$$

where

$$s \equiv -D \frac{g^{12}}{g^{11}}. \tag{D.5}$$

Evaluation of the local shear $\hat{s} \equiv [\nabla\psi \times \mathbf{B}] \cdot \mathrm{curl}\, [\nabla\psi \times \mathbf{B}] / |\nabla\psi|^4$ [29] in field line coordinates shows that s is identical to $\hat{s}$. Transforming to coordinates in which the field lines are straight, e.g. with Eqs. (B.5) and (B.8), one obtains

$$\beta = i\,n \Big\langle \frac{1}{\sqrt{g_B}} |V(\theta)|^2 \frac{\partial}{\partial\theta} \frac{g^{11} s}{|\mathbf{B}|^2} \Big\rangle. \tag{D.6}$$

After a tedious but straightforward evaluation of the terms under the θ derivative, it is found that Eq. (D.6) agrees identically with Eq. (57) in Ref. [7].

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