

Darío Correa-Restrepo and Horst Wobig

**Resistive Magnetohydrodynamic Equations,  
Stability and Nonlocal Perturbations in  
Three-dimensional Geometry**

IPP III/267

Oktober 2000

Darío Correa-Restrepo and Horst Wobig

**Resistive Magnetohydrodynamic Equations,  
Stability and Nonlocal Perturbations in  
Three-dimensional Geometry**

IPP III/267

Oktober 2000

"Dieser IPP-Bericht ist als Manuskript des Autors gedruckt. Die Arbeit entstand im Rahmen der Zusammenarbeit zwischen dem IPP und EURATOM auf dem Gebiet der Plasmaphysik. Alle Rechte vorbehalten."

"This IPP-Report has been printed as author's manuscript elaborated under the collaboration between the IPP and EURATOM on the field of plasma physics. All rights reserved."

# Resistive magnetohydrodynamic equations, stability and nonlocal perturbations in three-dimensional geometry

Darío Correa-Restrepo and Horst Wobig

*Max-Planck-Institut für Plasmaphysik EURATOM Association, D-85748 Garching, Germany*

## Abstract

A simplified set of resistive MHD (magnetohydrodynamic) equations is derived which describes the joint effect of pressure-driven (interchange and ballooning) and current-driven (tearing) perturbations in arbitrary three-dimensional geometry. These equations are shown to contain, as special cases, several sets of previously derived equations. Stability is investigated by introducing appropriate Liapunov functionals and studying their time dependence. In particular, it is shown that toroidal configurations (tokamaks and stellarators) are always unstable (though not necessarily exponentially unstable) with respect to ballooning perturbations, and that these perturbations do not have to be radially localized, but can extend over large plasma regions.

# I. INTRODUCTION

The purpose of the present paper is to study the stability of general toroidal configurations in the presence of resistivity, focusing primarily on nonlocal perturbations which can extend over a considerable region of the plasma. Here, the region of interest is the outer plasma, where the plasma beta,  $\beta = 8\pi p/B^2$ , with  $p$  the plasma pressure and  $B$  the magnitude of the magnetic field, is small. While for two-dimensional configurations there are already high-developed numerical methods for studying linear and nonlinear dissipative instabilities, and the related transport, this is not yet the case for three-dimensional configurations, in particular as far as nonlocal perturbations are concerned. Although there are promising approaches to fully integrate the three-dimensional geometry in the calculations [1], it is especially for three-dimensional configurations, and, in particular, for nonlocal perturbations, that analytical investigations such as the present one can provide important results. In this sense, the results concerning two-dimensional configurations are a kind of by-product of the general calculations.

As it is well known, resistivity decouples the plasma from the magnetic field and removes the stabilizing effect of “frozen” magnetic field lines. This leads to tearing mode instability of a plasma with finite toroidal current [2, 3, 4], and to resistive interchange [4] and ballooning modes [5]-[10]. The usual ballooning mode theory is characterized by small perpendicular wave lengths; in this paper it will be shown that this is no restriction, and that the plasma may become unstable for any perturbation initially localized to the region of unfavorable curvature.

In stellarators experiments without net toroidal current, tearing modes and disruptions do not occur in the beta regimes explored so far. However, there are local parallel currents which grow with increasing plasma beta, and these currents could lead to tearing instabilities. In the following theory, tearing mode equations will be derived which are valid in general three-dimensional equilibria with finite parallel currents.

In Sec. II, the basic resistive equations used are presented. In Sec. III, these equations are linearized, and a basic set of simplified equations is derived by making plausible approximations. In Sec. IV A, useful operators are introduced and their properties are investigated. By making use of these operators, the basic equations are written in a concise form in Sec. IV B. In Sec. IV C, global integral relations describing the time evolution of the perturbations are obtained. The stability of toroidal equilibria with respect to *nonlocal*, pressure-driven perturbations is studied in Sec. V A. In Sec. V B, approximate equations describing *localized*, pressure-driven, resistive and ideal interchange and ballooning perturbations are derived. The equations describing tearing perturbations in three-dimensional configurations are obtained and discussed in Sec. V C. The results are summarized in Sec. VI.

## II. RESISTIVE EQUATIONS

The equations given in this section are well-known. However, since they are the basis of the following investigations, it is useful to present them here in a concise way. An exhaustive discussion of these and many other related equations can be found in a recent book by A. B. Mikhailovskii [10].

Expressed in Gaussian units using the standard notation, and assuming that resistivity is small, the equations of resistive MHD considered here are the following. The rate of change of the velocity is given by the equation of motion

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \frac{1}{c} \mathbf{j} \times \mathbf{B} . \quad (1)$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 . \quad (2)$$

The rate of change of the magnetic field  $\mathbf{B}$  is given by Faraday's law

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} . \quad (3)$$

Since  $\nabla \cdot \mathbf{B} = 0$ , Eq. (3) yields

$$\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} = -\nabla \Phi , \quad (4)$$

with  $\mathbf{A}$  and  $\Phi$  the vector potential and the electric potential, respectively. The current density  $\mathbf{j}$  is given by Ampère's law

$$\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B} . \quad (5)$$

The pressure  $p$  and the mass density  $\rho$  are related by the equation of state

$$\frac{d}{dt} \left[ \frac{p}{\rho^{\gamma_H}} \right] = 0 , \quad (6)$$

with  $\gamma_H$  the ratio of the specific heats. By combining Eqs. (2) and (6) one obtains the equation for the evolution of the pressure

$$\frac{dp}{dt} + \gamma_H p \nabla \cdot \mathbf{u} = 0 . \quad (7)$$

The relation between the current density and the total electric field is given by Ohm's law

$$\eta \mathbf{j} = \mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} . \quad (8)$$

Note that resistivity has been kept only in Ohm's law, for reasons explained below. For simplicity,  $\eta$  is considered to be constant. It has also been assumed that it is so small that the dissipative equilibrium flows associated with it can be neglected.

The component of the current perpendicular to  $\mathbf{B}$  can be obtained from Eq. (1). By inserting the resulting expression into  $\nabla \cdot \mathbf{j} = \nabla \cdot [\mathbf{j}_\perp + \mathbf{j}_\parallel] = 0$ , one obtains

$$\nabla \cdot \left[ c \frac{\mathbf{B}}{B^2} \times \left[ \nabla p + \rho \frac{d\mathbf{u}}{dt} \right] + \left[ \mathbf{j} \cdot \frac{\mathbf{B}}{B^2} \right] \mathbf{B} \right] = 0. \quad (9)$$

Scalar multiplication of Eq. (8) by  $\mathbf{B}$  and insertion of  $\mathbf{E}$  from Eq. (4) yields the parallel component of Ohm's law,

$$\eta \mathbf{j} \cdot \mathbf{B} = -\mathbf{B} \cdot \left[ \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \right]. \quad (10)$$

An expression giving the perpendicular velocity  $\mathbf{u}_\perp$  as a function of the electromagnetic fields is obtained from Ohm's law by crossing Eq. (8) with  $\mathbf{B}$ . This yields

$$\mathbf{u}_\perp = c \frac{\mathbf{B}}{B^2} \times \left[ \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \right] + c \eta \frac{\mathbf{B}}{B^2} \times \mathbf{j}. \quad (11)$$

Equations (7) and (9)-(11) are the basic equations used in the following analysis.

### III. LINEAR EQUATIONS

#### A. Basic equations

All quantities are now expressed as a sum of equilibrium and fluctuating contributions by making the replacement  $\mathbf{B} \rightarrow \mathbf{B} + \delta\mathbf{B}$ , etc.. It is assumed that the equilibrium is static, i.e.  $\mathbf{u} = 0$ , and that it is described by the pressure  $p$ , the mass density  $\rho$  and the fields  $\mathbf{B}$  and  $\mathbf{j}$ . By keeping only the linear terms and taking the relation  $\nabla \cdot \mathbf{j} = 0$  into account, Eq.(9) yields  $\nabla \cdot [(\delta\mathbf{j})_\perp + (\delta\mathbf{j})_\parallel] = 0$  in the form

$$\nabla \cdot \left[ c \frac{\mathbf{B}}{B^2} \times \left( \nabla \delta p + \rho \frac{\partial (\delta\mathbf{u})_\perp}{\partial t} \right) - \frac{\mathbf{B}}{B^2} \times (\mathbf{j} \times \delta\mathbf{B}) + \left( \delta\mathbf{j} \cdot \frac{\mathbf{B}}{B^2} \right) \mathbf{B} \right]. \quad (12)$$

Linearization of Eqs. (10) and (11) yields the parallel component of Ohm's law

$$\eta \delta \mathbf{j} \cdot \mathbf{B} = -\mathbf{B} \cdot \left[ \frac{1}{c} \frac{\partial (\delta\mathbf{A})_\parallel}{\partial t} + \nabla \delta \Phi \right] \quad (13)$$

and the perpendicular perturbed velocity

$$(\delta \mathbf{u})_{\perp} = \frac{c}{B^2} \mathbf{B} \times \nabla \delta \Phi + \frac{1}{B^2} \frac{\partial}{\partial t} [\mathbf{B} \times (\delta \mathbf{A})_{\perp}] + c\eta \frac{\mathbf{B}}{B^2} \times \delta \mathbf{j} , \quad (14)$$

respectively.

The evolution equation for the perturbed pressure is obtained from Eq. (7). Since the equilibrium pressure is constant along  $\mathbf{B}$ , and thus  $((\delta \mathbf{u})_{\parallel} \cdot \nabla) p = 0$ , one obtains

$$\frac{\partial \delta p}{\partial t} + ((\delta \mathbf{u})_{\perp} \cdot \nabla) p + \gamma_H p \nabla \cdot \delta \mathbf{u} = 0 . \quad (15)$$

Scalar multiplication of the equation of motion by  $\mathbf{B}$  yields the equation

$$\rho B \frac{\partial \delta u_{\parallel}}{\partial t} = -\mathbf{B} \cdot \nabla \delta p - \nabla p \cdot \delta \mathbf{B} . \quad (16)$$

Derivation of this equation with respect to time, subsequent multiplication by  $1/B^2$  and derivation along the equilibrium magnetic field yields the equation of the perturbed motion along  $\mathbf{B}$  in the form

$$\begin{aligned} \rho \frac{\partial^2}{\partial t^2} [\delta D - \nabla \cdot (\delta \mathbf{u})_{\perp}] &= \mathbf{B} \cdot \nabla \left[ \frac{1}{B^2} \left[ \mathbf{B} \cdot \nabla [(\delta \mathbf{u})_{\perp} \cdot \nabla p] - \frac{\partial}{\partial t} (\delta \mathbf{B} \cdot \nabla p) \right] \right] \\ &\quad + \gamma_H p \mathbf{B} \cdot \nabla \left[ \frac{1}{B^2} \mathbf{B} \cdot \nabla \delta D \right] , \end{aligned} \quad (17)$$

with

$$\delta D := \nabla \cdot \delta \mathbf{u} , \quad (18)$$

and where the relation  $\nabla \cdot \delta \mathbf{u} = \nabla \cdot (\delta \mathbf{u})_{\perp} + (\mathbf{B} \cdot \nabla) [\delta u_{\parallel}/B]$  and Eq.(15) have been used.

## B. Simplified equations

Considerable simplification of Eqs. (12)-(17) is achieved with the following approximations.

The (small) resistivity is taken into account only in the parallel component of Ohm's law, Eq. (13), to make a parallel electric field  $(\delta \mathbf{E})_{\parallel}$  possible. This is the basic mechanism to decouple the plasma from the magnetic field and to allow for breaking and reconnection of magnetic field lines. In the perpendicular component of Ohm's law, Eq. (14),  $\eta$  is neglected since a perpendicular electric field  $(\delta \mathbf{E})_{\perp}$  is possible even with vanishing resistivity.

It is further assumed that the perturbed vector potential  $\delta \mathbf{A}$  is approximately given by

$$\delta \mathbf{A} = \delta A \frac{\mathbf{B}}{B} \quad (19)$$



since the parallel component of the perturbed vector potential is the main factor generating the perturbed perpendicular magnetic field, and this is much larger than the perturbed parallel magnetic field if  $p/B^2$  is not large. The perturbed perpendicular vector potential, which is the main factor generating the small perturbed parallel magnetic field, is ignored here. With these approximations, Eq. (14) reduces to

$$(\delta \mathbf{u})_{\perp} = \frac{c}{B^2} \mathbf{B} \times \nabla \delta \Phi . \quad (20)$$

The perturbed magnetic field  $\delta \mathbf{B} = \nabla \times \delta \mathbf{A}$  and parallel current density  $\mathbf{B} \cdot \delta \mathbf{j} = (c/4\pi) \mathbf{B} \cdot \nabla \times \delta \mathbf{B}$  are then

$$\delta \mathbf{B} = \left[ \nabla \left( \frac{\delta A}{B} \right) \right] \times \mathbf{B} + \frac{4\pi}{c} \frac{\delta A}{B} \mathbf{j} \quad (21)$$

and

$$\mathbf{B} \cdot \delta \mathbf{j} = -\frac{c}{4\pi} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] + c \left( \frac{\delta A}{B} \right) \left[ \Delta p + \frac{4\pi}{c^2} \mathbf{j}^2 \right] , \quad (22)$$

with  $\nabla_{\perp} := \nabla - \mathbf{e}_B (\mathbf{e}_B \cdot \nabla)$ ,  $\mathbf{e}_B = \mathbf{B}/B$ .

The second term in Eq. (21) is small  $\sim p/B^2$  as compared with the first one; however, it guarantees that  $\delta \mathbf{B}$  is divergence-free. The terms containing  $\Delta p$  and  $\mathbf{j}^2$  in Eq. (22) will be neglected since they are small compared with the first term,  $(\mathbf{B} \cdot \delta \mathbf{j})_1 := -(c/4\pi) \nabla \cdot [B^2 \nabla_{\perp} (\delta A/B)]$ . Estimation of the orders of magnitude yields

$$[c (\delta A/B) \Delta p] / (\mathbf{B} \cdot \delta \mathbf{j})_1 \sim \frac{\beta}{2} \frac{l^2}{L_p^2} , \quad (23)$$

$$[(4\pi \delta A/cB) \mathbf{j}_{\perp}^2] / (\mathbf{B} \cdot \delta \mathbf{j})_1 \sim \frac{\beta^2}{4} \frac{l^2}{L_p^2} , \quad (24)$$

and

$$[(4\pi \delta A/cB) \mathbf{j}_{\parallel}^2] / (\mathbf{B} \cdot \delta \mathbf{j})_1 \sim \frac{\beta^2}{4} \frac{l^2}{L_p^2} \frac{L_B^2}{L_p^2} , \quad (25)$$

with  $\mathbf{j}_{\perp} = (c\mathbf{B} \times \nabla p) / B^2$ ,  $\mathbf{j}_{\parallel} = (\mathbf{j} \cdot \mathbf{e}_B) \mathbf{e}_B$ , and  $l$ ,  $L_p$  and  $L_B$  the characteristic lengths of the fluctuation, the equilibrium pressure and the equilibrium gradients along  $B$ , respectively.

With these approximations, and by deriving Eq. (12) with respect to  $t$ , Eqs. (12), (13) and (17) yield three coupled scalar equations for  $\delta D$  and the perturbed potentials  $\delta \Phi$  and  $\delta A$ ,

$$\nabla \cdot \left[ \frac{c^2 \rho}{B^2} \nabla_{\perp} \frac{\partial^2 \delta \Phi}{\partial t^2} + \frac{c^2}{B^2} [(\mathbf{B} \times \nabla \delta \Phi) \cdot \nabla p] \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right]$$

$$\begin{aligned}
& + \frac{1}{B} \frac{\partial \delta A}{\partial t} \nabla \times \mathbf{j}_{\parallel} + \frac{c}{4\pi} \frac{\mathbf{B}}{B^2} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \right] \\
& = -\nabla \cdot \left[ c \gamma_H p \delta D \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right], \tag{26}
\end{aligned}$$

$$\eta \frac{c}{4\pi} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] = \mathbf{B} \cdot \nabla \delta \Phi + \frac{B}{c} \frac{\partial \delta A}{\partial t}, \tag{27}$$

$$\begin{aligned}
\rho \frac{\partial^2 \delta D}{\partial t^2} - \rho c \left[ \nabla \frac{\partial^2 \delta \Phi}{\partial t^2} \right] \cdot \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) &= -\mathbf{B} \cdot \nabla \left[ \frac{1}{B^2} \frac{\eta c}{4\pi} \nabla \cdot \left[ \mathbf{j}_{\perp} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] \right] \right] \\
&+ \gamma_H p \mathbf{B} \cdot \nabla \left[ \frac{1}{B^2} \mathbf{B} \cdot \nabla \delta D \right]. \tag{28}
\end{aligned}$$

In deriving the last equation use has been made of Eq. (27) in the relation

$$\begin{aligned}
\mathbf{B} \cdot \nabla [(\delta \mathbf{u})_{\perp} \cdot \nabla p] - \frac{\partial}{\partial t} (\delta \mathbf{B} \cdot \nabla p) &= \nabla \cdot \left[ \nabla p \times \left[ \mathbf{B} \times (\delta \mathbf{u})_{\perp} + \frac{\partial \delta \mathbf{A}}{\partial t} \right] \right] \\
&= -\frac{\eta c}{4\pi} \nabla \cdot \left[ \mathbf{j}_{\perp} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] \right]. \tag{29}
\end{aligned}$$

The five terms in Eq. (26) represent the inertia, the interaction between pressure gradient and curvature (interchange and ballooning), the current-driven tearing perturbations, the stabilizing effect of magnetic field line bending, and the plasma compression term, respectively. The terms in Eq. (27) express the balance between the parallel current and the parallel electric field. Equation (28) describes the perturbed motion along the equilibrium magnetic field.

When the sound waves can be neglected ( $c_s^2 = \gamma_H p / \rho \rightarrow 0$ , the RHS (right-hand side) of Eq.(26) becomes small and Eqs. (26), and (27) decouple from Eq. (28). However, sound propagation is not negligible if the perturbations grow slowly and  $(l_{\parallel} \partial / \partial t)^2 / c_s^2 \rightarrow 0$ , where  $l_{\parallel}$  is the characteristic gradient length of the perturbation along  $\mathbf{B}$ . Equation (28) then yields, to leading order,

$$\delta D = 0. \tag{30}$$

In both cases, the system of equations (26)-(28) reduces to Eq. (27) and

$$\begin{aligned}
& \nabla \cdot \left[ \frac{c^2 \rho}{B^2} \nabla_{\perp} \frac{\partial^2 \delta \Phi}{\partial t^2} + \frac{c^2}{B^2} [(\mathbf{B} \times \nabla \delta \Phi) \cdot \nabla p] \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right. \\
& \left. + \frac{1}{B} \frac{\partial \delta A}{\partial t} \nabla \times \mathbf{j}_{\parallel} + \frac{c}{4\pi} \frac{\mathbf{B}}{B^2} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \right] \right] = 0. \tag{31}
\end{aligned}$$

The two coupled equations (27) and (31) for  $\delta\Phi$  and  $\delta A$  are the basis of the following investigation.

Equation (27) is a diffusion equation for the magnitude  $\delta A$  of the perturbed vector potential. The driving term is  $\mathbf{B} \cdot \nabla \delta\Phi$ . Neglecting the resistivity changes the mathematical character of the equation completely and eliminates the diffusion process. With vanishing resistivity, the time derivative  $\partial\delta A/\partial t$  can be replaced in Eq. (31), which becomes a self-adjoint equation for the perturbed electric potential  $\delta\Phi$ , as shown explicitly in Sec. IV B.

## IV. GLOBAL RELATIONS

Relations describing the time evolution of the global perturbation energy can be derived from Eqs. (27) and (31). Here, the investigations are restricted to those perturbations which leave the plasma surface unchanged and do not yield boundary contributions. This is guaranteed by requiring  $\delta\Phi = \text{const.}$ ,  $\delta A/B = \text{const.}$  on the plasma surface. Since the actual value of these constants is irrelevant, it is then convenient to consider a Hilbert space of functions which vanish on the boundary. Thus,

$$\delta\Phi = 0 \quad , \quad \delta A = 0 \quad \text{on } \partial\Omega \quad , \quad (32)$$

where  $\partial\Omega$  represents the boundary. For any two functions  $g, f$  of this space, a scalar product is defined by the relation

$$(g, f) = \int g f d^3x \quad . \quad (33)$$

In the function space considered, useful operators are introduced in a standard way [11]. To avoid any confusion with vector or scalar quantities, these operators will be designated by calligraphic letters. When an operator  $\mathcal{O}$  is applied to a function  $f$ , the mapped function  $\mathcal{O}f$  is obtained. The scalar products  $(g, \mathcal{O}f)$  are functionals in the space in question. Some of these functionals can be used as Liapunov functionals [12] to decide on the stability of the system. Thus, it is possible to obtain general results without having to solve the equations for the perturbations in full detail.

### A. Operators and their properties

In the Hilbert space considered, let  $\mathcal{I}$  be the operator defined by

$$\mathcal{I} = -\nabla \cdot \left[ \frac{c^2 \rho}{B^2} \nabla_{\perp} \cdots \right] \quad , \quad (34)$$

where the  $\dots$  represent the function on which the operator is applied. This operator is related to the inertia term. The scalar product of  $\mathcal{I}f$  with the functions  $g$  and  $f$  then yields

$$\begin{aligned}(g, \mathcal{I}f) &= - \int g \nabla \cdot \left[ \frac{c^2 \rho}{B^2} \nabla_{\perp} f \right] d^3x \\ &= - \int f \nabla \cdot \left[ \frac{c^2 \rho}{B^2} \nabla_{\perp} g \right] d^3x \\ &= (f, \mathcal{I}g) ,\end{aligned}\tag{35}$$

and

$$(f, \mathcal{I}f) = \int \frac{c^2 \rho}{B^2} [\nabla_{\perp} f]^2 d^3x \geq 0 ,\tag{36}$$

respectively. Thus,  $\mathcal{I}$  is Hermitian and positive. The operator  $\mathcal{F}$ , defined by

$$\mathcal{F} = -\nabla \cdot [B^2 \nabla_{\perp} \dots]\tag{37}$$

is also Hermitian and positive,

$$(g, \mathcal{F}f) = (f, \mathcal{F}g) \quad , \quad (f, \mathcal{F}f) = \int B^2 [\nabla_{\perp} f]^2 d^3x \geq 0 .\tag{38}$$

The operator  $\mathcal{F}$  appears in Eq. (31) in the line bending term, and in Eq. (27) in the magnetic diffusion term.

The operator

$$\mathcal{D} = \mathbf{B} \cdot \nabla = \nabla \cdot [\mathbf{B} \dots]\tag{39}$$

has the property

$$(g, \mathcal{D}f) = - (f, \mathcal{D}g) \quad , \quad (f, \mathcal{D}f) = 0 ,\tag{40}$$

and is therefore anti-Hermitian. This is also the case for the operator  $\mathcal{J}$ , defined by

$$\mathcal{J} = \nabla \cdot [\dots \nabla \times \mathbf{j}_{\parallel}]\tag{41}$$

since

$$\begin{aligned}(g, \mathcal{J}f) &= \int g \nabla \cdot [f \nabla \times \mathbf{j}_{\parallel}] d^3x \\ &= - \int [f (\nabla \times \mathbf{j}_{\parallel})] \cdot \nabla g d^3x \\ &= - (f, \mathcal{J}g) ,\end{aligned}\tag{42}$$

$$(f, \mathcal{J}f) = 0 .\tag{43}$$

The scalar product  $(g, \mathcal{J}f)$  can also be written as

$$(g, \mathcal{J}f) = \int (\nabla_{\perp} g \times \nabla_{\perp} f) \cdot \mathbf{j}_{\parallel} d^3x . \quad (44)$$

Note that

$$\nabla \times \mathbf{j}_{\parallel} = \nabla_{\perp} \left( \frac{j_{\parallel}}{B} \right) \times \mathbf{B} + \left( \frac{j_{\parallel}}{L} \right)_{\text{char}} O(\beta) \quad (45)$$

where  $(j_{\parallel}/L)_{\text{char}}$  is some characteristic value for the perpendicular gradients of the parallel current density.

In order to describe the second (curvature-driven) term in Eq. (31) it is convenient to introduce the operator  $\mathcal{M}$ ,

$$\mathcal{M} = \nabla \cdot \left[ \mathbf{j}_{\perp} \cdot \nabla \dots \right] c \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) . \quad (46)$$

The scalar product  $(g, \mathcal{M}f)$  yields

$$\begin{aligned} (g, \mathcal{M}f) &= \int g \nabla \cdot \left[ \mathbf{j}_{\perp} \cdot \nabla f \right] c \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) d^3x \\ &= - \int \left[ \nabla g \cdot c \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] \mathbf{j}_{\perp} \cdot \nabla f d^3x \\ &= \int f \nabla \cdot \left[ \mathbf{j}_{\perp} \left[ \nabla g \cdot c \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] \right] d^3x \\ &= (\mathcal{M}^+ g, f) , \end{aligned} \quad (47)$$

with the adjoint operator  $\mathcal{M}^+$  given by the expression

$$\mathcal{M}^+ = \nabla \cdot \left[ \mathbf{j}_{\perp} \left[ \left[ c \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] \cdot \nabla \dots \right] \right] . \quad (48)$$

The operator  $\mathcal{M}$  can be expressed as a sum of a symmetric (Hermitian) part  $\mathcal{M}_s$  and an anti-Hermitian part  $\mathcal{M}_a$  in the usual way,  $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_a$ , with

$$\mathcal{M}_s = (1/2) [\mathcal{M} + \mathcal{M}^+] , \quad \mathcal{M}_a = (1/2) [\mathcal{M} - \mathcal{M}^+] . \quad (49)$$

The relevant scalar products are then

$$\begin{aligned} (g, \mathcal{M}_s f) &= -\frac{c}{2} \int \left[ \left[ \nabla g \cdot \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] (\mathbf{j}_{\perp} \cdot \nabla f) \right. \\ &\quad \left. + \left[ \nabla f \cdot \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] (\mathbf{j}_{\perp} \cdot \nabla g) \right] d^3x , \end{aligned} \quad (50)$$

$$\begin{aligned}
(f, \mathcal{M}_s f) &= -c \int \left[ \nabla f \cdot \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] (\mathbf{j}_\perp \cdot \nabla f) d^3x \\
&= -c \int (\mathbf{j}_\perp \cdot \nabla f) \left[ \left[ \frac{4\pi}{c} \frac{\mathbf{j}}{B^2} - \mathbf{B} \times \nabla_\perp \left( \frac{1}{B^2} \right) \right] \cdot \nabla f \right] d^3x \\
&= c \int (\mathbf{j}_\perp \cdot \nabla f) \left[ \left[ \mathbf{B} \times \nabla_\perp \left( \frac{1}{B^2} \right) + O(\beta) \right] \cdot \nabla f \right] d^3x . \quad (51)
\end{aligned}$$

Note that the higher-order terms in  $O(\beta)$  must be neglected in order to be consistent with the assumptions of Sec. III B. With the perpendicular differential operators  $\partial_{\perp v}$  and  $\partial_{\perp Bv}$  defined by the equations

$$\partial_{\perp v} := \frac{\nabla v}{|\nabla v|} \cdot \nabla , \quad \partial_{\perp Bv} := \frac{(\mathbf{B} \times \nabla v)}{B |\nabla v|} \cdot \nabla , \quad (52)$$

where  $v$  is the volume enclosed by a magnetic surface, and expressing the curvature of the magnetic lines as

$$\begin{aligned}
\kappa &= -\frac{B^2}{2} \nabla_\perp \left( \frac{1}{B^2} \right) + 4\pi \dot{p} \frac{\nabla v}{B^2} \\
&= \kappa_0 + O(\beta) , \quad (53)
\end{aligned}$$

with

$$\kappa_0 := -\frac{B^2}{2} \nabla_\perp \left( \frac{1}{B^2} \right) . \quad (54)$$

Eq. (51) can then written as

$$\begin{aligned}
(f, \mathcal{M}_s f) &= -\frac{c^2}{4\pi} \int \beta \frac{\dot{p}}{p} |\nabla v| (\partial_{\perp Bv} f) [\kappa_{0n} (\partial_{\perp Bv} f) \\
&\quad - \kappa_{0g} (\partial_{\perp v} f)] d^3x + O(\beta^2) . \quad (55)
\end{aligned}$$

Here, dots mean derivatives with respect to the volume  $v$ .  $\kappa_{0n} = \kappa_0 \cdot \nabla v / |\nabla v|$  and  $\kappa_{0g} = \kappa_0 \cdot (\mathbf{B} \times \nabla v) / B |\nabla v|$  are the leading terms (in  $\beta$ ) of the normal and the geodetic curvature, respectively.

The anti-Hermitian part  $\mathcal{M}_a$  of the operator  $\mathcal{M}$  is

$$\begin{aligned}
\mathcal{M}_a &= -\frac{c}{2} \nabla \cdot \left[ (\nabla \cdots) \times \left[ \mathbf{j}_\perp \times \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] \right] \\
&= \frac{c}{2} (\nabla \cdots) \cdot \nabla \times \left[ \mathbf{j}_\perp \times \left[ \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] \right] . \quad (56)
\end{aligned}$$

By taking into account the relations  $\nabla \cdot \mathbf{j}_\perp + \nabla \cdot \mathbf{j}_\parallel = 0$ ,  $\nabla \cdot \mathbf{j}_\perp = B^2 \mathbf{j}_\perp \cdot \nabla (1/B^2)$  and  $\nabla \cdot \mathbf{j}_\parallel = \mathbf{B} \cdot \nabla (j_\parallel/B)$ , it can be shown that the term  $\nabla \times [\mathbf{j}_\perp \times [\nabla \times (\mathbf{B}/B^2)]]$

depends only on the perpendicular gradient of the parallel equilibrium current density:

$$\begin{aligned}
\mathbf{j}_\perp \times \left[ \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] &= \frac{1}{B^2} \left[ -B^2 \left( \mathbf{j}_\perp \cdot \nabla \frac{1}{B^2} \right) \mathbf{B} + 4\pi \frac{j_\parallel}{B} \nabla p \right] \\
&= \frac{1}{B^2} \left[ \left( \mathbf{B} \cdot \nabla \frac{j_\parallel}{B} \right) \mathbf{B} + 4\pi \frac{j_\parallel}{B} \nabla p \right] \\
&= \nabla \frac{j_\parallel}{B} - \nabla_\perp \frac{j_\parallel}{B} + 4\pi \frac{1}{B^2} \frac{j_\parallel}{B} \nabla p .
\end{aligned} \tag{57}$$

The last term in this equation is  $O(\beta^2)$ . Therefore

$$\nabla \times \left[ \mathbf{j}_\perp \times \left[ \nabla \times \left( \frac{\mathbf{B}}{B^2} \right) \right] \right] = -\nabla \times \left[ \nabla_\perp \frac{j_\parallel}{B} \right] + O(\beta^2) . \tag{58}$$

## B. Equations in operator form

With the help of the operators defined in Sec. IV A, Eqs. (31) and (27) can be expressed as

$$\mathcal{I} \frac{\partial^2 \delta \Phi}{\partial t^2} + [\mathcal{M}_s + \mathcal{M}_a] \delta \Phi - \mathcal{J} \left[ \frac{1}{B} \frac{\partial \delta A}{\partial t} \right] + \mathcal{D} \left[ \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \left[ \frac{1}{B} \frac{\partial \delta A}{\partial t} \right] \right] = 0 , \tag{59}$$

and

$$\frac{c}{4\pi} \eta \mathcal{F} \left[ \frac{\delta A}{B} \right] = -\mathcal{D} \delta \phi - \frac{B}{c} \frac{\partial \delta A}{\partial t} , \tag{60}$$

respectively. These are the basic equations used in the following analysis.

When resistivity vanishes, Eq. (60) yields

$$\frac{1}{B} \frac{\partial \delta A}{\partial t} = -\frac{c}{B^2} \mathcal{D} \delta \phi , \tag{61}$$

and, therefore

$$\begin{aligned}
-\mathcal{J} \left[ \frac{1}{B} \frac{\partial \delta A}{\partial t} \right] &= \mathcal{J} \left[ \frac{c}{B^2} \mathcal{D} \delta \Phi \right] \\
&= \nabla \cdot \left[ \frac{c}{B^2} (\mathbf{B} \cdot \nabla \delta \Phi) \nabla \times \mathbf{j}_\parallel \right] \\
&:= \mathcal{N} \delta \Phi ,
\end{aligned} \tag{62}$$

where the operator  $\mathcal{N}$  has been defined as

$$\mathcal{N} := \nabla \cdot \left[ \frac{c}{B^2} (\mathbf{B} \cdot \nabla \dots) \nabla \times \mathbf{j}_\parallel \right] . \tag{63}$$

The adjoint operator  $\mathcal{N}^+$  is obtained in the usual way from the scalar product  $(g, \mathcal{N}f) = (\mathcal{N}^+g, f)$ ,

$$\begin{aligned}(g, \mathcal{N}f) &= \int g \nabla \cdot \left[ \frac{c}{B^2} (\mathbf{B} \cdot \nabla f) \nabla \times \mathbf{j}_{\parallel} \right] d^3x \\ &= - \int \nabla f \cdot \left[ (\nabla g \cdot \nabla \times \mathbf{j}_{\parallel}) \frac{c}{B^2} \mathbf{B} \right] d^3x \\ &= \int f \nabla \cdot \left[ (\nabla g \cdot \nabla \times \mathbf{j}_{\parallel}) \frac{c}{B^2} \mathbf{B} \right] d^3x.\end{aligned}\quad (64)$$

Therefore

$$\mathcal{N}^+ = \nabla \cdot \left[ [(\nabla \times \mathbf{j}_{\parallel}) \cdot \nabla \cdots] \frac{c}{B^2} \mathbf{B} \right]. \quad (65)$$

The operator  $\mathcal{N}$  can be expressed as a sum of its symmetric part,  $\mathcal{N}_s = (\mathcal{N} + \mathcal{N}^+)/2$ , and of its antisymmetric part  $\mathcal{N}_a = (\mathcal{N} - \mathcal{N}^+)/2$ . Explicitly, the antisymmetric contribution is

$$\begin{aligned}\mathcal{N}_a &= \frac{1}{2} (\mathcal{N} - \mathcal{N}^+) \\ &= \frac{c}{2} \nabla \cdot \left[ \frac{1}{B^2} [(\mathbf{B} \cdot \nabla \cdots) \nabla \times \mathbf{j}_{\parallel} - [(\nabla \times \mathbf{j}_{\parallel}) \cdot \nabla \cdots] \mathbf{B}] \right] \\ &= \frac{c}{2} \nabla \cdot \left[ \frac{\nabla \cdots}{B^2} \times [(\nabla \times \mathbf{j}_{\parallel}) \times \mathbf{B}] \right].\end{aligned}\quad (66)$$

Since  $(\nabla \times \mathbf{j}_{\parallel}) \times \mathbf{B} = \mathbf{B} \times [\mathbf{B} \times \nabla (j_{\parallel}/B)] + 4\pi (j_{\parallel}/B) \nabla p$ , one obtains

$$\mathcal{N}_a = \frac{c}{2} \nabla \cdot \left[ \frac{\nabla \cdots}{B^2} \times \left[ (\mathbf{B} \cdot \nabla \frac{j_{\parallel}}{B}) \mathbf{B} + 4\pi \frac{j_{\parallel}}{B} \nabla p \right] \right]. \quad (67)$$

Comparison with Eqs. (56) and (57) shows that the antisymmetric operators  $\mathcal{M}_a$  and  $\mathcal{N}_a$  exactly cancel. The operator  $\mathcal{D}[(c/4\pi B^2) \mathcal{F}[(c/B^2) \mathcal{D} \cdots]]$  can easily be shown to be Hermitian. Therefore, in the ideal case ( $\eta = 0$ ), Eqs. (59) and (60) reduce to the self-adjoint equation

$$\mathcal{I} \frac{\partial^2 \delta \Phi}{\partial t^2} + \mathcal{M}_s \delta \Phi + \mathcal{N}_s \delta \Phi - \mathcal{D} \left[ \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \left[ \frac{c}{B^2} \mathcal{D} \delta \Phi \right] \right] = 0. \quad (68)$$

It is thus shown that the approximations made preserve the self-adjointness of the complete ideal MHD stability operator.

### C. Integral relations

Within the framework of ideal MHD stability analysis, the energy principle is an important tool which is based on the self-adjoint property of the stability operator. In the presence of finite resistivity, the energy of a perturbation is no longer



conserved. By making use of Liapunov functionals, it will be shown in the following that resistivity leads to growing perturbations if the initial perturbation is localized to regions of unfavorable curvature.

Since the operators  $\mathcal{I}$  and  $\mathcal{M}_s$  are Hermitian, the following relations obtain

$$\left( \frac{\partial \delta \phi}{\partial t}, \mathcal{I} \frac{\partial^2 \delta \Phi}{\partial t^2} \right) = \frac{1}{2} \frac{d}{dt} \left( \frac{\partial \delta \phi}{\partial t}, \mathcal{I} \frac{\partial \delta \Phi}{\partial t} \right), \quad (69)$$

and

$$\left( \frac{\partial \delta \Phi}{\partial t}, \mathcal{M}_s \delta \Phi \right) = \frac{1}{2} \frac{d}{dt} (\delta \Phi, \mathcal{M}_s \delta \Phi). \quad (70)$$

By taking into account Eqs. (38), (40) and (60), one obtains

$$\begin{aligned} \left( \frac{\partial \delta \Phi}{\partial t}, \mathcal{D} \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) &= - \left( \mathcal{D} \frac{\partial \delta \Phi}{\partial t}, \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \\ &= \left( \frac{c}{4\pi} \eta \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t}, \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \\ &\quad + \left( \frac{B^2}{c} \frac{1}{B} \frac{\partial^2 \delta A}{\partial t^2}, \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \\ &= \frac{1}{2} \frac{d}{dt} \frac{1}{4\pi} \left( \frac{1}{B} \frac{\partial \delta A}{\partial t}, \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \\ &\quad + \eta \left[ \frac{c}{4\pi} \right]^2 \left( \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t}, \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right). \end{aligned} \quad (71)$$

Multiplication of Eq. (59) by  $\partial \delta \Phi / \partial t$  and subsequent integration  $\int \cdots d^3x$  yields

$$\begin{aligned} \frac{dW}{dt} - \frac{c}{2} \int \frac{\partial \delta \Phi}{\partial t} \nabla \delta \Phi \cdot \nabla \times \left[ \nabla_{\perp} \frac{j_{\parallel}}{B} \right] d^3x \\ + \int \left[ \frac{1}{B} \frac{\partial \delta A}{\partial t} \nabla_{\perp} \frac{j_{\parallel}}{B} \times \mathbf{B} \right] \cdot \nabla \frac{\partial \delta \Phi}{\partial t} d^3x = -\eta C_t^2, \end{aligned} \quad (72)$$

with  $W$  and  $C_t^2$  defined by the relations

$$\begin{aligned} W &:= \frac{1}{2} \left[ \left( \frac{\partial \delta \phi}{\partial t}, \mathcal{I} \frac{\partial \delta \Phi}{\partial t} \right) + \frac{1}{4\pi} \left( \frac{1}{B} \frac{\partial \delta A}{\partial t}, \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) + (\delta \Phi, \mathcal{M}_s \delta \Phi) \right] \\ &= \frac{1}{2} \int \left[ \frac{c^2 \rho}{B^2} \left( \nabla_{\perp} \frac{\partial \delta \Phi}{\partial t} \right)^2 + \frac{B^2}{4\pi} \left[ \nabla_{\perp} \left( \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \right]^2 \right] d^3x \\ &\quad + \frac{1}{2} (\delta \Phi, \mathcal{M}_s \delta \Phi) \end{aligned} \quad (73)$$

and

$$\begin{aligned}
C_t^2 &:= \left[ \frac{c}{4\pi} \right]^2 \left( \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t}, \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \\
&= \left[ \frac{c}{4\pi} \right]^2 \int \frac{1}{B^2} \left[ \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) \right] \right]^2 d^3x \\
&= \int \left[ \frac{\partial \delta j_{\parallel}}{\partial t} \right]^2 d^3x,
\end{aligned} \tag{74}$$

respectively. Multiplication of Eq. (59) by  $\delta\Phi$  and subsequent integration over the plasma volume yields

$$\left( \delta\Phi, \mathcal{I} \frac{\partial^2 \delta\Phi}{\partial t^2} \right) + (\delta\Phi, \mathcal{M}_s \delta\Phi) - \left( \delta\Phi, \mathcal{J} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) - \left( \mathcal{D} \delta\Phi, \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right) = 0. \tag{75}$$

Note that only the symmetric part of  $\mathcal{M}$  survives integration, and that the anti-Hermitian property of  $\mathcal{D}$  has been used. With  $\mathcal{D}\delta\Phi$  taken from Eq. (60), and transforming the time derivatives taking into account that the operators  $\mathcal{I}$  and  $\mathcal{F}$  are Hermitian, Eq. (75) can be written as

$$\frac{d\Lambda}{dt} + \int \left[ \frac{1}{B} \frac{\partial \delta A}{\partial t} \nabla_{\perp} \frac{j_{\parallel}}{B} \times \mathbf{B} \right] \cdot \nabla \delta\Phi d^3x = 2W_1 - 2W, \tag{76}$$

with the following definitions:

$$\Lambda := \left( \delta\Phi, \mathcal{I} \frac{\partial \delta\Phi}{\partial t} \right) + \frac{1}{2} \eta C^2, \tag{77}$$

$$\begin{aligned}
\left( \delta\Phi, \mathcal{I} \frac{\partial \delta\Phi}{\partial t} \right) &= \frac{1}{2} \frac{d}{dt} \int \frac{c^2 \rho}{B^2} (\nabla_{\perp} \delta\Phi)^2 d^3x \\
&= \frac{d}{dt} \int \frac{1}{2} \rho (\delta \mathbf{v}_E)^2 d^3x,
\end{aligned} \tag{78}$$

where  $\delta \mathbf{v}_E$  is the perturbation of the electric drift velocity, and

$$\begin{aligned}
C^2 &:= \left[ \frac{c}{4\pi} \right]^2 \int \frac{1}{B^2} \left[ \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] \right]^2 d^3x \\
&= \int (\delta j_{\parallel})^2 d^3x,
\end{aligned} \tag{79}$$

$$W_1 := \left( \frac{\partial \delta\Phi}{\partial t}, \mathcal{I} \frac{\partial \delta\Phi}{\partial t} \right) = \int \frac{c^2 \rho}{B^2} \left( \nabla_{\perp} \frac{\partial \delta\Phi}{\partial t} \right)^2 d^3x \geq 0. \tag{80}$$

Equations (27) and (31) (or, equivalently, (59) and (60)), and (72) and (76) are the basis of the following stability analysis.

## V. STABILITY ANALYSIS

### A. Global perturbations driven by the pressure gradient

The subject of interest in this section are the perturbations which are driven by the interaction of the unfavorable field line curvature with the pressure gradient. It should be stressed that no condition is imposed requiring that the perturbations be localized to a small region of space. On the other hand, it is assumed that tearing modes play no role, either because their absence is inherent to the type of equilibrium treated, as is usually the case in stellarators or, in the case of tokamaks, because the current density profiles of the configurations of interest have already been optimized with respect to tearing modes. Therefore, it is assumed that the driving force responsible for these modes, i.e. the perpendicular gradients of the parallel current density,  $\nabla_{\perp} (j_{\parallel}/B)$ , can be neglected. In this case, the contributions of  $\mathcal{J}$ , Eq. (41) and  $\mathcal{M}_a$ , Eqs. (49), (56) in Eq. (59) can also be neglected, and the operator  $\mathcal{M}$  contributes only through its Hermitian part, which is explicitly given by Eq. (55).

Then, by taking into account Eq. (57), Eqs. (72) and (76) reduce to

$$\frac{dW}{dt} = -\eta C_t^2, \quad (81)$$

$$\frac{d\Lambda}{dt} = 2W_1 - 2W. \quad (82)$$

The functional  $W$  is related to the perturbation energy; its dimension is energy divided by  $t^2$ .

It should be noted that the perpendicular gradients of the perturbed quantities contained in these equations can be large in some regions of space, and this can have a substantial influence on the growth rates. Although Eqs. (81) and (82) are useful for deriving general properties concerning the time evolution of the perturbations, they do not yield, in general, their explicit time dependence nor the relation between this time dependence (the exponential growth rate  $\gamma$  in the case of unstable eigenmodes) and the resistivity  $\eta$ . To obtain these relations, it is necessary to explicitly solve Eqs. (59) and (60). For instance, in the case of unstable modes, growth rates proportional to fractional powers of the resistivity can be found, whereas a superficial estimation based on Eqs. (81) and (82) might mislead to conclude that the relation between  $\partial/\partial t$  and  $\eta$  is linear. Regarding this point, cf. the last paragraph of Sec. III in Ref. 24.

## Initial value problem

Equations (59) and (60) are of second order in  $t$  for  $\delta\Phi$ , and of first order for  $\delta A$ . Therefore, the initial conditions  $\delta\Phi(\mathbf{x}, t = t_0)$ ,  $\partial\delta\Phi/\partial t(\mathbf{x}, t = t_0)$  and  $\delta A(\mathbf{x}, t = t_0)$  can be arbitrarily chosen.

It can now be shown that these configurations are resistive unstable whenever there is a region of bad curvature. When there is such a region, a perturbation  $\delta\Phi(\mathbf{x}, t = t_0)$  can always be found which makes the functional  $(\delta\Phi, \mathcal{M}\delta\Phi)$  negative at the initial time  $t = t_0$ . This is achieved by choosing  $\delta\Phi(\mathbf{x}, t = t_0) \neq 0$  only in the region of bad curvature. The initial perturbation  $\delta A(\mathbf{x}, t = t_0)$  is chosen as the solution of the equation

$$\frac{c}{4\pi}\eta\mathcal{F}\left(\frac{\delta A}{B}\right) = -\mathcal{D}\delta\phi(\mathbf{x}, t = t_0) . \quad (83)$$

This implies  $\partial\delta A/\partial t(\mathbf{x}, t = t_0) = 0$ , according to Eq. (60). Therefore,

$$(\delta\Phi, \mathcal{M}\delta\Phi)(t = t_0) < 0, \quad \frac{\partial\delta A}{\partial t}(\mathbf{x}, t = t_0) = 0 . \quad (84)$$

With the further initial condition

$$\frac{\partial\delta\Phi}{\partial t}(\mathbf{x}, t = t_0) = 0 , \quad (85)$$

it follows from Eq. (73) that the energy functional  $W(t = t_0)$  is negative.

Since  $W$  is initially negative, it remains so at all later times owing to Eq. (81).  $W_1$ , Eq. (80), is always positive. Then, according to Eq. (82), the functional  $\Lambda$  is a monotonic increasing function of time. One obtains

$$\begin{aligned} \Lambda(t) = & \Lambda(t = t_0) + 2 \int_{t_0}^t W_1(\hat{t}) d\hat{t} + 2 |W(t = t_0)| (t - t_0) \\ & + 2\eta \int_{t_0}^t \left[ \int_{t_0}^{\hat{t}} C_t^2(\tilde{t}) d\tilde{t} \right] d\hat{t} . \end{aligned} \quad (86)$$

Therefore, owing to Eqs. (77)-(79), either the perturbed plasma velocity or the perturbed parallel current, or both, must grow unlimitedly in time.

The choice of initial conditions is based on the freedom to choose arbitrary permissible test functions at the initial time  $t = t_0$ . The linear time evolution of the perturbations is determined by solving Eqs. (59) and (60) with the appropriate boundary conditions. Such a detailed treatment is not the subject of this paper. The results obtained here are of a different nature: in the case considered, a Liapunov functional exists which contains the perturbed plasma velocity and the perturbed

plasma current, and which, within the framework of linear theory, grows unbounded in time.

Note that a similar proof for exponential growth cannot be done in the same way since, for eigenmodes,  $\delta\Phi(\mathbf{x}, t) = \delta\Phi(\mathbf{x}) \exp(\gamma t)$ ,  $\delta A(\mathbf{x}, t) = \delta A(\mathbf{x}) \exp(\gamma t)$ , and  $\delta\Phi(\mathbf{x})$  and  $\delta A(\mathbf{x})$  cannot be chosen independent of each other, but are related by Eq. (60) at all times. Here, on the contrary, the *initial conditions* for  $\delta\Phi$ ,  $\partial\delta\Phi/\partial t$  and  $\delta A$  can be arbitrarily chosen.

These unstable perturbations can extend over large plasma regions, even though, owing to the small resistivity, the main destabilizing contributions of the resistive term  $-\eta \int [\partial\delta j_{\parallel}/\partial t]^2 d^3x$  (cf. Eqs. (72) and (74)) originate only in the regions of large transversal gradients, where the perturbed parallel current is large.

It should be noted here that nonlocal, exponential, resistive *drift* instabilities were found in Ref. 1, where the two coupled equations describing the perturbed density and perturbed electric potential of resistive drift modes were numerically solved for a certain stellarator model, without making further approximations with respect to either the equilibrium geometry or the mode structure.

## B. Localized perturbations driven by the pressure gradient

Perturbations which are localized, either near a particular magnetic surface or around a closed field line, have large gradients across the localization surface or transversally to the localization line. When applied to these kind of perturbations, the operators  $\mathcal{J}$ , Eq. (41), and  $\mathcal{M}_a$ , Eq. (56) can always be neglected as compared to  $\mathcal{I}$ , Eq. (34),  $\mathcal{F}$ , Eq. (37), and  $\mathcal{M}_s$ , Eqs. (46)-(49) since the former contribute with only one large transversal derivative and the latter with two of them. In this case, Eqs. (59) and (60) reduce to

$$\mathcal{I} \frac{\partial^2 \delta\Phi}{\partial t^2} + \mathcal{M}_s \delta\Phi + \mathcal{D} \left[ \frac{c}{4\pi} \frac{1}{B^2} \mathcal{F} \frac{1}{B} \frac{\partial \delta A}{\partial t} \right] = 0, \quad (87)$$

and

$$\frac{c}{4\pi} \eta \mathcal{F} \left( \frac{\delta A}{B} \right) = -\mathcal{D} \delta\phi - \frac{B}{c} \frac{\partial \delta A}{\partial t}. \quad (88)$$

The approximations made here taken into account, these equations can be shown to be the resistive ballooning mode equations of Refs. 6 and 7 (Eqs. (13), (18) and (19) of Ref. 6, with  $\gamma_{HP} \delta D \rightarrow 0$ ). With the effect of sound waves propagation included,  $\gamma_{HP} \delta D \neq 0$ , the stability of localized modes, which depend on time as  $\sim \exp \gamma t$ , can be studied with the general methods developed in those papers, or, more recently, in Ref. 24. For general perturbations, the stability can be studied as in Sec. V A. as an initial value problem. In the case of localized perturbations, the operator  $\mathcal{M}$  is given by its Hermitian part, Eq. (55).

By taking advantage of the fact that the transverse derivatives of the perturbations are large, and by Fourier-transforming the dependence along the field lines,  $\delta A$  can be eliminated from Eqs. (87) and (88), which then reduce to an ordinary differential equation for the transformed  $\delta\Phi$  alone. Detailed calculations can be found in Refs. 6, 7 and 24. Again, *with resistivity*, instability can be shown whenever there is a region of bad curvature.

### Ideal localized perturbations

In the limit of vanishing resistivity, Eq. (88) reduces to

$$\mathcal{D}\delta\phi + \frac{B}{c} \frac{\partial\delta A}{\partial t} = 0, \quad (89)$$

and one does not have now the freedom to choose  $\delta A(\mathbf{x}, t = t_0)$  so as to make  $\partial\delta A/\partial\delta t(\mathbf{x}, t = t_0) = 0$ . In this case, Eqs. (87) and (88) yield the ideal ballooning modes equation

$$\mathcal{I} \frac{\partial^2 \delta\Phi}{\partial t^2} + \mathcal{M}_s \delta\Phi - \mathcal{D} \left[ \frac{c^2}{4\pi} \frac{1}{B^2} \mathcal{F} \left[ \frac{1}{B^2} \mathcal{D}\delta\Phi \right] \right] = 0, \quad (90)$$

from which the conservation of energy follows

$$\frac{dW}{dt} = \frac{d}{dt} \frac{1}{2} \left[ \left( \frac{\partial\delta\phi}{\partial t}, \mathcal{I} \frac{\partial\delta\Phi}{\partial t} \right) + \frac{c^2}{4\pi} \left( \frac{1}{B^2} \mathcal{D}\delta\Phi, \mathcal{F} \frac{1}{B^2} \mathcal{D}\delta\Phi \right) + (\delta\Phi, \mathcal{M}_s \delta\Phi) \right] = 0. \quad (91)$$

The second, stabilizing line-bending term can be made small with flute-like perturbations, which have a small variation along  $\mathbf{B}$ , i.e.  $\mathcal{D}\delta\Phi \approx 0$ . These perturbations can be stabilized by a magnetic well. For ballooning modes proper,  $\mathcal{D}\delta\Phi$  is not small, and there is a threshold for the pressure gradient.

### C. Tearing modes

The basic theory of tearing modes was developed in Ref. 2, which treated the finite resistivity instability of a sheet pinch. Here, these modes will be only briefly discussed. The equations obtained are valid in general geometry.

Equations (27) and (31) describe the combined effect of ballooning and tearing modes. By neglecting the perturbed pressure in these equations (the term proportional to  $(\mathbf{B} \times \nabla \delta\Phi) \cdot \nabla p$ ) and integrating Eq. (31) with respect to time  $t$ , one obtains the following tearing modes equations

$$\nabla \cdot \left[ \frac{c^2 \rho}{B^2} \nabla_{\perp} \frac{\partial\delta\Phi}{\partial t} + \frac{\delta A}{B} \nabla \times \mathbf{j}_{\parallel} + \frac{c}{4\pi} \frac{\mathbf{B}}{B^2} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] \right] = 0, \quad (92)$$

$$\eta \frac{c}{4\pi} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] = \mathbf{B} \cdot \nabla \delta \Phi + \frac{B}{c} \frac{\partial \delta A}{\partial t}. \quad (93)$$

The driving term for the tearing mode instability is the second term in Eq. (92) containing  $\nabla_{\perp} (\mathbf{j}_{\parallel}/B)$ . If this term is neglected, the system is stable. This follows from  $W \geq 0$  and Eq. (81).

Since the resistivity  $\eta$  is small, it can be ignored unless the transversal gradients  $\nabla_{\perp}$  are large. The resistive terms on the left-hand side of Eq. (93) can then balance the parallel perturbed electric field on the right-hand side of the same equation. In cylindrically symmetric configurations, it is possible to consider perturbations with a single toroidal and a single poloidal Fourier component. For these, the region of large transverse gradients is located around one surface, the mode rational surface on which  $\mathbf{B} \cdot \nabla \delta \Phi = 0$ . Outside this region, the perturbations change on the equilibrium length scales, the gradients are not large, and resistivity can be ignored.

The method of treatment is then to solve the equations in a small region with resistivity taken into account. In most of the plasma, the equations are solved ignoring resistivity. Since the solutions to both sets of equations are to represent one and the same perturbation, they must match smoothly on the boundary separating the resistive from the non-resistive (ideal) region.

In the resistive region, where the transverse gradients are large, Eqs. (92) and (93) reduce to

$$\frac{c^2 \rho}{B^2} \nabla_{\perp}^2 \frac{\partial \delta \Phi}{\partial t} + \frac{c}{4\pi} \mathbf{B} \cdot \nabla \left[ \frac{1}{B} \nabla_{\perp}^2 \delta A \right] = 0, \quad (94)$$

and

$$\eta \frac{c}{4\pi} B \nabla_{\perp}^2 \delta A = \mathbf{B} \cdot \nabla \delta \Phi + \frac{B}{c} \frac{\partial \delta A}{\partial t}, \quad (95)$$

with

$$\nabla_{\perp}^2 \delta A = \partial_{\perp v}^2 \delta A + \partial_{\perp B v}^2 \delta A. \quad (96)$$

Here, for consistency, it is assumed that the product of the small resistivity  $\eta$  with the large  $\nabla_{\perp}^2$  is of the same order of magnitude as the terms on the right-hand side of Eq. (95).

In the non-resistive region, where the transverse gradients are not large, the term  $\sim \eta$  in Eq. (93) can be ignored, and since the growth rate of the perturbation can be assumed to be small, dependent on the small resistivity, the inertia term in Eq. (92) can also be neglected. Eqs. (92) and (93) reduce to

$$\left[ \nabla_{\perp} \left( \frac{\mathbf{j}_{\parallel}}{B} \right) \times \mathbf{B} \right] \cdot \nabla \left( \frac{\delta A}{B} \right) + \frac{c}{4\pi} \mathbf{B} \cdot \nabla \left[ \frac{1}{B^2} \nabla \cdot \left[ B^2 \nabla_{\perp} \left( \frac{\delta A}{B} \right) \right] \right] = 0, \quad (97)$$

$$0 = \mathbf{B} \cdot \nabla \delta \Phi + \frac{B}{c} \frac{\partial \delta A}{\partial t}, \quad (98)$$

where  $\nabla \times \mathbf{j}_{\parallel} = \nabla_{\perp} (j_{\parallel}/B) \times \mathbf{B} + O(\beta)$  has been taken into account. Eq. (98) can be used to calculate  $\mathbf{B} \cdot \nabla \delta\Phi$ , but it is not required.

In cylindrical geometry, with  $j_{\parallel} \approx j_z(r)$ , these equations lead to the well-known  $\Delta'$  criterion [3] frequently used in treatments of tearing modes in tokamaks. In more complicated geometries, it is not possible to consider a single Fourier component separately. Coupling of different Fourier modes arises owing to the metric of the magnetic surfaces, which depends on the poloidal and the toroidal coordinates and, in particular, owing to the angular dependence of  $j_{\parallel}$ . In stellarators with vanishing net toroidal current the local parallel current changes sign. The perturbations have a singular structure on the rational surfaces where any of the Fourier components are resonant. Obtaining the actual solutions of Eqs. (92) and (93) and carrying out the matching process on the boundary of the resistive to the non-resistive regions is a difficult task [4, 14]. As it is shown in the standard tearing mode theory of a cylindrical plasma, the resistive layer is small if resistivity is small and the shear of the magnetic field lines is large. However, this is not the case in low-shear stellarators, and in tokamaks with reversed shear. For these, a special treatment of the equations is necessary.

The joint behavior of tearing and ballooning effects can be described by keeping the perturbed pressure term  $\sim (\mathbf{B} \times \nabla \delta\Phi) \cdot \nabla p$  in Eq. (31). In this case, Eq. (94) is replaced by

$$\frac{c^2 \rho}{B^2} \nabla_{\perp}^2 \frac{\partial^2 \delta\Phi}{\partial t^2} - \mathcal{M}_s \delta\Phi + \frac{c}{4\pi} \mathbf{B} \cdot \nabla \left[ \frac{1}{B} \nabla_{\perp}^2 \delta A \right] = 0, \quad (99)$$

with

$$\begin{aligned} \mathcal{M}_s \delta\Phi = & c^2 |\nabla v| \frac{\dot{p}}{B^2} \left[ -B^2 \left( \partial_{\perp v} \frac{1}{B^2} \right) \partial_{\perp Bv}^2 \delta\Phi \right. \\ & \left. + B^2 \left( \partial_{\perp Bv} \frac{1}{B^2} \right) \partial_{\perp Bv} (\partial_{\perp v} \delta\Phi) \right]. \end{aligned} \quad (100)$$

Equations (99) and (95), which describe the resistive region, are identical to Eqs. (87) and (88). They can be studied with the methods of Refs. 4, 6, 7, and 13. The equivalence of the treatments used in Ref. 4, where the equations are solved in physical space, and in Refs. 6 and 7, where the equations are solved in Fourier-transformed (“ballooning”) space was shown in Ref. 13.

Unfortunately, in the case of tearing modes in general configurations, it does not seem possible to derive relations determining the stability in a similar way as in Sec. V A, because of the presence of the operator  $\mathcal{J}$ , which has both Hermitian and anti-Hermitian parts, in the relevant equations. There are, however, interesting special cases for which a resistive energy principle can be derived and evaluated, as done in Ref. 14, and, particularly, in Refs. 15 - 23, where fundamental contributions to the theory of energy principles for dissipative systems can be found.



## VI. CONCLUSIONS

The usual resistive MHD equations for plasmas with small resistivity, which keep resistive effects only in Ohm's law, were considered in their linearized form. By making some plausible approximations, a simplified set of resistive MHD equations was derived. These equations describe the joint effect of pressure-driven (interchange and ballooning) and current-driven (tearing) perturbations in arbitrary, three-dimensional geometry. The approximations leading to the simplified equations are the neglect of the coupling of the equations for the perturbed electromagnetic potentials to the equation describing the perturbed plasma motion along the equilibrium magnetic field, as explained in Sec. III B, and the assumption that the perturbed magnetic vector potential  $\delta\mathbf{A}$  is given mainly by its component parallel to the equilibrium magnetic field. With these assumptions, it was possible to derive a set of two coupled, second-order, partial differential equations for two scalar quantities, namely the perturbed electric potential  $\delta\Phi$  and the parallel component of the perturbed vector potential  $\delta\mathbf{A}$ . With the introduction of appropriate operators, the equations could be written in a very concise form. The symmetry properties of these operators was investigated, and energy-related functionals which describe the time evolution of the perturbed system were derived. By neglecting the perpendicular gradients of the equilibrium parallel current density (thus assuming that the equilibria of interest are either inherently stable with respect to tearing modes, as is usually the case in stellarators or, in the case of tokamaks, assuming that tearing modes play no role because the current density profiles have already been optimized with respect to them), and by treating the stability problem as an initial value problem, it was possible to show that toroidal configurations (tokamaks and stellarators) are always unstable for non-vanishing resistivity. This result was obtained by taking advantage of the freedom to choose the *initial* perturbations, and is a consequence of the fact that there are always regions of bad curvature. An important feature of these instabilities is that they can be global and do not have to be localized in space. However, the method does not yield the explicit time dependence of the instabilities. In particular, it is not possible to show exponential instability.

By considering the appropriate limiting cases, it was shown that, within the approximations made, the derived set of equations contains, as special cases, the equations which describe ideal and resistive localized ballooning modes, and tearing modes in toroidal configurations.

For the outer regions of tokamak plasmas, the results are consistent with those previously obtained for local perturbations within the framework of a circular tokamak model [13], and also with numerical calculations based on local (Ref. 25) and nonlocal (Ref. 26) [26]) dissipative two-fluid theory.

One may speculate whether instabilities with non-exponential growth are dan-

gerous for plasma confinement. In the experiment, however, a situation in which a plasma starts from an unstable equilibrium with small perturbations cannot be verified. The experiment starts at a finite distance from the envisaged unstable equilibrium and will stay in an oscillatory or turbulent mode around this equilibrium. This state is governed by the non-linear resistive MHD equations. Thus the system never passes through a phase where the linearized theory is relevant. The main result of the linearized theory is that the equilibrium cannot be reached.

## ACKNOWLEDGMENT

The Authors would like to thank J. Nührenberg for useful comments.

\*

## References

- [1] R. Kleiber, *25th EPS Conference on Controlled Fusion and Plasma Physics, Prague, 1998*, edited by P. Pavlo (European Physical Society, 1998), Vol. 22C, p. 1753.
- [2] H. P. Furth, J. Killeen, and M. N. Rosenbluth, *Phys. Fluids* **6**, 459 (1963).
- [3] B. Coppi, J. M. Greene, and J. L. Johnson, *Nucl. Fusion* **6**, 101 (1966).
- [4] A. H. Glasser, J. M. Greene, and J. L. Johnson, *Phys. Fluids* **18**, 875 (1975).
- [5] G. Bateman and D. B. Nelson, *Phys. Rev. Lett* **41**, 1804 (1978).
- [6] D. Correa-Restrepo, *Z. Naturforsch.* **37a**, 848 (1982).
- [7] D. Correa-Restrepo, in *Plasma Physics and Controlled Nuclear Fusion Research 1982* (International Atomic Energy Agency, Vienna, 1983), Vol. II, pp. 519–530.
- [8] A. B. Mikhailovskii, B. N. Kuvshinov, V. P. Lakhin, S. V. Novakovskii, A. I. Smolyakov, and S. E. Sharapov, *Plasma Phys. Contr. Fusion* **31**, 1741 (1989).
- [9] A. B. Mikhailovskii, B. N. Kuvshinov, V. P. Lakhin, S. V. Novakovskii, A. I. Smolyakov, and S. E. Sharapov, *Plasma Phys. Contr. Fusion* **31**, 1759 (1989).
- [10] A. B. Mikhailovskii *Instabilities in a Confined Plasma* (Institute of Physics Publishing, Bristol and Philadelphia, 1998).
- [11] A. E. Lifschitz *Magnetohydrodynamics and Spectral Theory* (Kluwer Academic Publishers, Dordrecht, 1988), p. 40.
- [12] P. G. Drazin *Nonlinear Systems* (Cambridge University Press, Cambridge, 1992), p. 178.
- [13] D. Correa-Restrepo, *Phys. Plasmas* **6**, 530 (1999).
- [14] B. N. Kuvshinov, *Sov. J. Plasma Phys.* **17**, 717 (1991).
- [15] E. M. Barston, *Phys. Fluids* **12**, 2162 (1969).
- [16] H. Tasso, *Plasma Phys.* **17**, 1131 (1975).

- [17] H. Tasso, in *Plasma Physics and Controlled Nuclear Fusion Research 1976* (International Atomic Energy Agency, Vienna, 1977), Vol. III, pp. 371–381.
- [18] H. Tasso, *Plasma Phys.* **19**, 177 (1977).
- [19] H. Tasso, and J. T. Virtamo, *Plasma Phys.* **22**, 1003 (1980).
- [20] H. Tasso, *Phys. Lett. A* **94**, 217 (1983).
- [21] H. Tasso, *Phys. Lett. A* **147**, 28 (1990).
- [22] H. Tasso, *Phys. Lett. A* **161**, 289 (1991).
- [23] W. Kerner, and H. Tasso, *Phys. Rev. Lett.* **49**, 694 (1982).
- [24] W. Kerner, and H. Tasso, *Plasma Phys.* **24**, 97 (1982).
- [25] A. Zeiler, D. Biskamp, J. F. Drake, and B. N. Rogers, *Phys. Plasmas* **5**, 2654 (1998).
- [26] K. Hallatschek, and A. Zeiler, *Phys. Plasmas* **7**, 2554 (2000).