Eigenface representation of equilibrium flux using Function Parameterization

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Introduction

Function parameterization (FP) has been in use for identification of equilibrium parameters on the ASDEX Upgrade tokamak since 1991 [1]. Scalar plasma parameters (I_p, R_p, Z_p, β_{pol}, $\cdots$) recovered in realtime from magnetic data using parameterizations generated from an offline training procedure are input to the feedback control algorithm for plasma position and shape. FP, which has the reliability advantage over conventional equilibrium solvers of being predictive rather than interpretive, is also used to recover the equilibrium poloidal flux function by the brute force calculation of $\psi(R,Z)$ as a scalar parameter at each point on a spatial grid[2]. The slow speed of this "FPP" algorithm is challenging for realtime use, and the resulting flux grid can deviate strongly from satisfying force balance.

Here, we present a new FP-based algorithm for full equilibrium flux recovery based on a Singular Value Decomposition of a training equilibrium database represented as a matrix $\mathcal{M}$ of N_{Eq} rows and $N_G = N_R \times N_Z$ columns where N_{Eq} is the number of equilibria in the database, N_R and N_Z are the grid dimensions, and each row of $\mathcal{M}$ consists of all N_G ψ-values for one equilibrium. For the 129×257 grid used here $\mathcal{M}$ has dimensions $N_{Eq} \times N_G = 2546 \times 33153$ for $N_{Eq} = 2546$ lower single null equilibria giving an $\mathcal{M}-$matrix size of 334 MB.

The SVD of $\mathcal{M}$ is
$$\mathcal{M}_{(N_{Eq}\times N_G)} = U_{(N_{Eq}\times N_{Eq})}\Sigma_{(N_{Eq}\times N_G)}V'_{(N_G\times N_G)} \qquad (1)$$
where, if $I_{(N)}$ is the identity matrix with N diagonal elements, $U'U = I_{(N_{Eq})}$ (U' denotes the transpose of U), $V'V = I_{(N_G)}$ and $\Sigma = \mathrm{diag}(s_1, s_2, \cdots, s_{N_{Eq}})$ is the diagonal matrix of ordered singular values ($s_1 \geq s_2 \cdots \geq s_{N_{Eq}} \geq 0$). The first N_{Eq} columns of V constitute a set $\{\mathcal{F}_j\}$ of N_{Eq} **eigenfaces** (a concept familiar from image analysis[3]) of $\psi(R,Z)$ which represent 2-D basis functions for the equilibrium grid. A small subset $N_p \ll N_{Eq}$ of principal eigenfaces suffices in practice to identify all recoverable information on $\psi(R,Z)$, at least in the case of equilibrium magnetic data. The flux function is reconstructed as a linear combination
$$\psi_i(R,Z) = \sum_{j=1}^{p} \varphi_{i,j}\, \mathcal{F}_j$$
with Fourier-like amplitudes $\varphi_{i,j}$ given by $\varphi_{i,j} = \mathcal{F}_j \cdot \psi_{i\ (N_G\times 1)}$ where $\psi_{i\ (N_G\times 1)}$ is the flux grid for the i^{th} case in the database expressed as an $N_G \times 1$ column vector.

Database Generation and FP Model

To prepare for realtime recovery of $\psi(R,Z)$, the $N_{Eq} \times p$ matrix of amplitudes Φ is constructed from the training database and a predictor for each amplitude is prepared by regressing each Fourier amplitude data vector $\boldsymbol{\phi}_j = \{\varphi_{1,j}, \cdots, \varphi_{N_{Eq},j}\}$ versus principal components of available diagnostic signals, in the present study consisting of equilibrium magnetic data. The following summarizes the main steps in database generation and analysis.

(i) Monte Carlo methods are used to generate N_{Eq} equilibria drawn from a high-dimensional ($h > 20$D) input parameter space. N_{Eq} typically lies in the range $10^3 - 10^4$ and the random choice of inputs results in N_{Eq} distinct values for each equilibrium parameter, thus overcoming the "Curse of Dimensionality" that afflicts scans of high-dimensional parameter spaces on a regular grid.

(ii) A vector of h candidate input parameters (poloidal field coil currents, passive currents, and free parameters describing the size and shape of the $p'(\psi)$ and $ff'(\psi)$ source profiles) is randomly selected from a $h-D$ hypercuboid. The subset of random input vectors (a few %) resulting in physically meaningful (e.g. located within the vacuum vessel), converged equilibria maps out the feasible solution region within the hypercuboid.

(iii) The poloidal flux solution grid (for a fixed plasma current of 1 MA), a selection of 1-D flux profiles, and a variety of scalar parameters are stored for each valid case. Simulated magnetic measurements may be calculated as and when required by reading in each flux grid from the stored database and evaluating flux differences and poloidal field values at the current experimental configuration of flux loop and field probe locations.

(iv) Perform a Principal Components Analysis (PCA) by diagonalizing the covariance matrix of the set of magnetic signals, and retain the leading m principal components $\{\mu_r\}$, $r = 1, \cdots, m$, where the lowest retained PC has an eigenvalue equivalent to a signal to noise ratio in the range $2-10$ and the noise variance is known from experimental calibration information.

(v) Regress the j^{th} Fourier amplitude variable Φ_j using a standard quadratic FP model with (m+1)(m+2)/2 terms and save the coefficients for realtime use:

$$\Phi_j = a_j + \sum_{r=1}^{m} b_{rj}\,\mu_r \; + \sum_{r=1,\, s=r}^{m} c_{rsj}\,\mu_r\,\mu_s \qquad \text{for } j = 1,\, 2, \cdots, p$$

(vi) Recover $\psi(R,Z)$ by (i) evaluating $\Phi_1, \cdots, \Phi_p$ using PC values calculated from realtime magnetic data and (ii) computing $\psi = \sum_{j=1}^{p} \Phi_j \mathcal{F}_j$ for p eigenfaces. The two steps require $\approx m^2p/2 + p \times N_G$ multiplications, while FPP requires $m^2 N_G/2$. The new method gains in speed over FPP if $p < m^2/2$ holds.

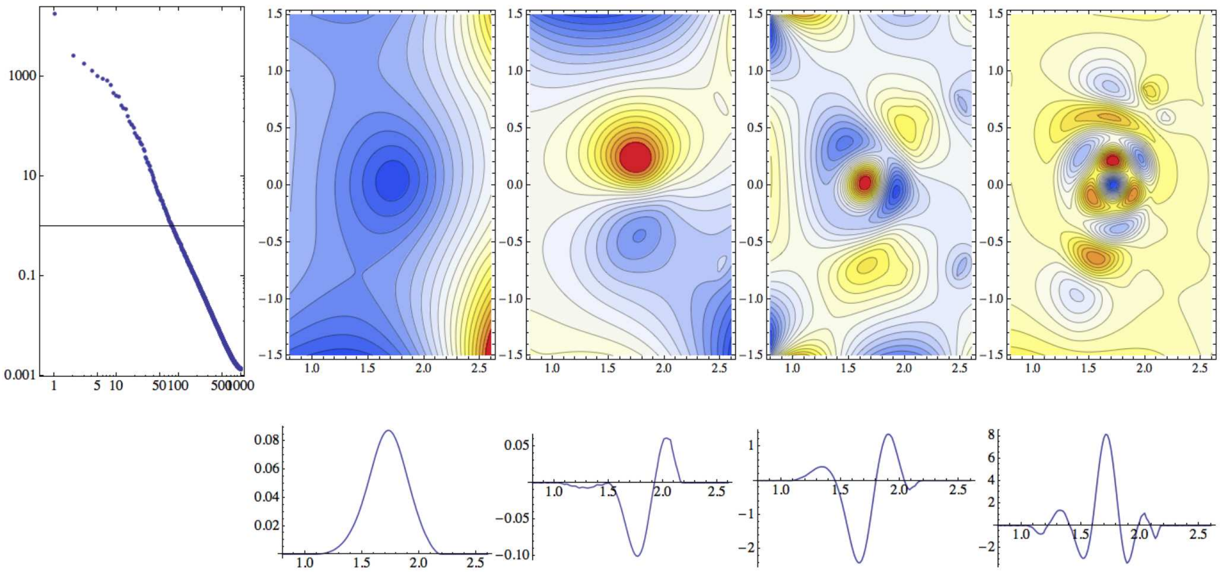

FIG. 1: Singular value distribution (1st panel), contour plots (versus R and Z) of eigenfaces 1, 6, 18, 36 and (lower level) midplane (z=0) profiles of the Grad-Shafranov operator applied to each eigenface. The singular values normalized to s_1 are: s_6/s_1=0.05, s_{18}/s_1=0.006, s_{36}/s_1=0.0007. Eigenfaces tend to become progressively localized to the plasma centre with increasing index/decreasing singular value.

Description and Identifiability of Eigenface Amplitudes

The SVD procedure for the 334 MB database matrix $\mathcal{M}$ was carried out using *Mathematica*[5] and yielded $\approx$ 1000 out of a possible 2546 eigenfaces and singular values due to finite numerical accuracy. The machine resources comprised $\approx$ 1 gigabyte of memory and 3300 s on a 2.5 GHz, 4 GB MacBook Pro. The distribution of singular values is shown in the first panel in fig. 1 and

s_{100} is already a factor of 2.7×10^{-5} smaller than s_1 and it will be shown later that 100 eigenfaces is already well beyond the limits of identifiability. Sample eigenfaces are shown in fig. 1 together with midplane profiles of the toroidal current density-like quantity $-(\Delta^*\mathcal{F})/R$ where Δ^* is the Grad-Shafranov operator. Singular values indicating the relative strength of each eigenface are given in the caption. An FP model with $m=23$ magnetic principal components resulting in a quadratic regression model size of 300 terms was used to regress the Fourier amplitudes variables and hence determine how accurately they can be identified from magnetic data. Fig. 2 shows the quality of the recovery in the form of the root mean square regression error (adjusted for degrees of freedom loss) expressed as a percentage of the root mean square value of the variable in the database. A 100% recovery error corresponds to complete unidentifiability. Results are presented for **unperturbed** and perturbed magnetic data with 1σ noise levels of **1.5 mT**, **3 mT**, **10 mT** and **50 mT**. For the lowest order 11 *principal eigenfaces*, the recovery errors are in the sub-10% range for moderate ($\leq 3\,$mT) noise levels, while at the 50 mT level, the information is heavily degraded for all but the leading three amplitudes. Above $p = 50$ even unperturbed data yields % errors in the range $70\% - 100\%$ so that these amplitudes are essentially unidentifiable from magnetic data. In the $\psi(R, Z)$ recovery results presented next we accordingly use the leading $p = 50$ amplitudes.

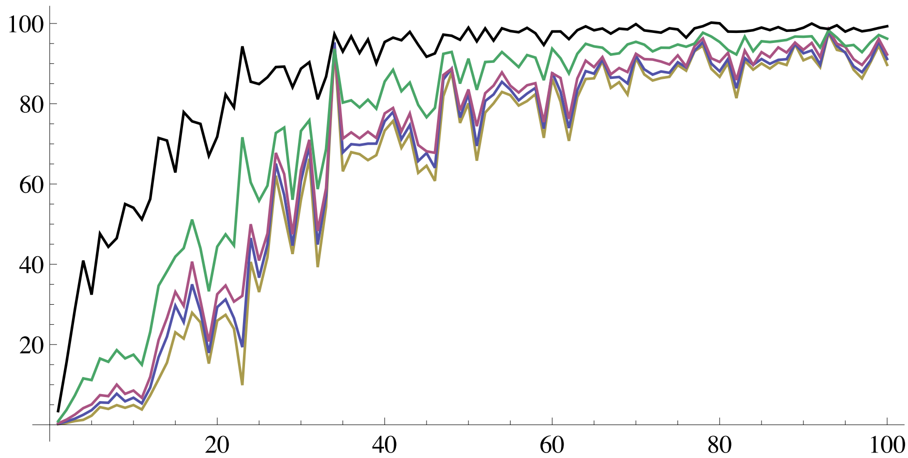

FIG. 2: Recovery rms error expressed as a % of the rms magnitude for eigenface Fourier amplitudes 1-100 from 23 PC quadratic FP regression models with **unperturbed magnetic data**, and perturbed data with 1σ added noise levels of **1.5mT**, **3mT**, **10mT** and **50mT**.

A feature of the results presented in fig. 2 is the occasional strong departure from monotonically increasing recovery errors versus eigenface index. An examination of these eigenfaces revealed that they contain significant contributions from active or passive coil currents from coils located within the computational grid. Fig. 3 shows that eigenface 23, whose recovery error for unperturbed or moderately perturbed data is superior to that of its lower index neighbours, has strong contributions at the locations of the ASDEX Upgrade fast control coils CoIo and CoIu whose outline is visible in fig. 3. Their proximity to the measurements allows relatively easy identification of their contributions to ψ by the FP model.

Accuracy of Eigenface Reconstruction of Flux Surface Geometry

The accuracy of flux surface reconstructions from noisy magnetic data using $p = 50$ eigenfaces is characterized as follows: The outboard major radius $R_{out}\,(\psi(R_{in}))$, where exact and recovered contours with label $\psi(R_{in})$ intersect the magnetic midplane, was determined as a function of regularly spaced values of the inboard intersection R_{in}. The exact and recoverd ψ values at R_{in} will differ, and the difference in the corresponding $R_{out}\,(\psi(R_{in}))$ values is a measure of the reconstruction error which has the advantage of being a purely geometric property, independent of the often large recovery errors in the absolute flux due to poor identifiability of the core current profile by magnetics. Results for three noise levels are presented in fig. 4 which also shows results for an alternative hybrid flux label ψ^*, a weighted mean of the area and flux coordinates, where the ψ weighting vanishes at the centre and the area weighting vanishes at the separatrix. This brings a factor of $\approx$ two improvement in the central geometrical error.

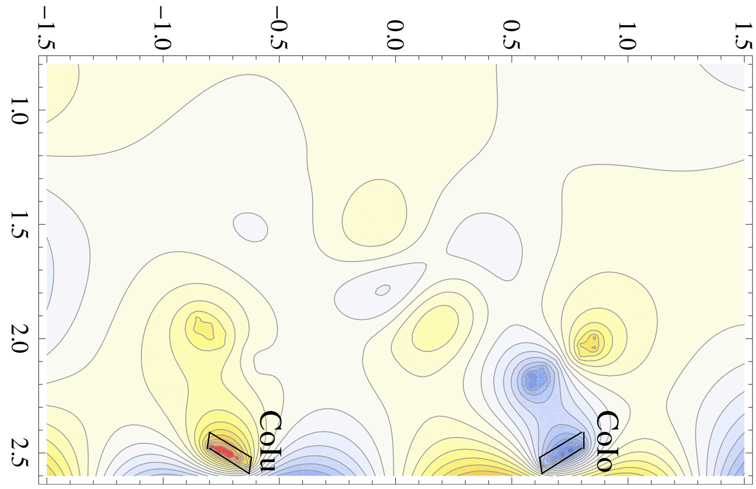

FIG. 3: Contour plot versus (Z,R) (with Z axis horizontal) of eigenface no. 23, showing strong contributions from ASDEX Upgrade upper and lower fast control coils CoIo CoIu.

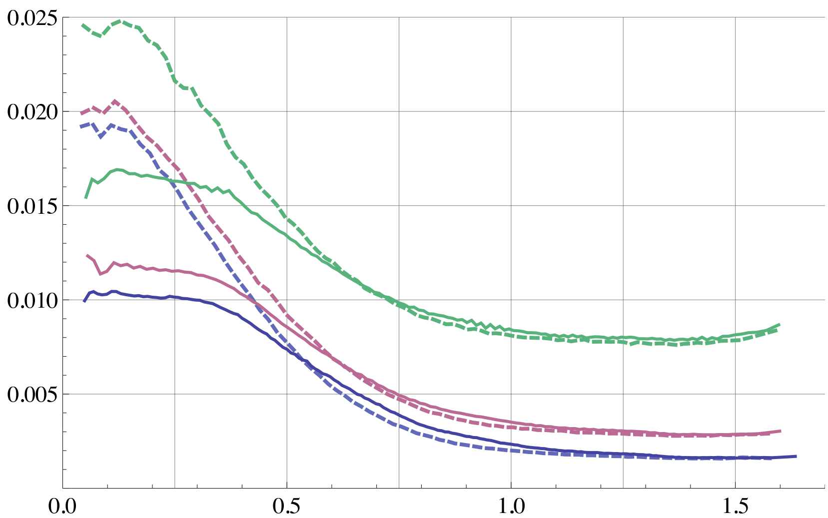

FIG. 4: Error (1σ) in flux surface diameter versus midplane diameter for ψ (dashed traces) and $psi*$ (solid) flux labels for reconstructed flux surfaces using $p = 50$ eigenfaces. Input magnetic data were perturbed by simulated 1σ noise levels of **1.5mT**, **3mT** and **10mT**. The 1.5mT error level is typical for a good quality interpretive equilibrium fit using the CLISTE code. Each curve is a boxcar average at $\approx$ 2 cm intervals of $\approx$ 2000 individual results.

Discussion and Future Work

With $(m+1)(m+2)/2 = 300$ and $p = 50$ these promising initial results represent a factor of 6 speed-up as well as improved quality over the existing FPP recovery of $\psi(R,Z)$ and should enhance current efforts at accurate realtime flux surface geometry identification for NTM stabilisation on ASDEX Upgrade[4]. One obvious possibility for improvement is to exclude grid areas containing active and passive coils, since this will reduce the value of p leading to a further performance enhancement. The superior results for the ψ^* label are consistent with similar behaviour found in the existing FPP method and require further study. It is planned to apply the new method to an eigenface representation of the toroidal current density profile.

[1] P.J. Mc Carthy, PhD thesis, University College Cork, 1992.
[2] P.J. Mc Carthy, Europhysics Conference Abstracts **11 D-III** 1286 (1987).
[3] M. Turk and A. Pentland, J. Cognitive Neuroscience **3** (1991) 71-86.
[4] L. Giannone et al., presentation P2.105, this conference.
[5] Wolfram Research, Inc., Mathematica, Version 8.0, Champaign, IL (2010).