Radiation of Plasma Waves by a Conducting Body Moving Through a Magnetized Plasma

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Abstract

The emission of plasma waves by a conducting body orbiting the ionosphere is considered. The case of an infinitely long and infinitely thin tether is discussed and the use of the appropriate form of the wave dispersion relation in the frequency regime of interest is shown to reconcile existing results in the literature. A compact formula for the radiation resistance of the tether is provided.

1   Introduction

The interest in studying the wave emission by a conducting body orbiting the ionosphere was sparked by the preparation of the Tether Satellite System (TSS) project [*Colombo et al.*(1974)], a joint venture between NASA and the Italian Space Agency (ASI) that was developed in the seventies and eighties and launched in a first mission in 1992 [*Dobrowolny and Melchioni*(1993)]. Despite of the failure of the first mission due to engineering problems, other missions were performed following the original TSS idea (see, for instance [*Stone, Raitt, and Wright*(1999)]), and brand-new projects proposed, using different system configurations (see, for instance, the AcME project [*Biancalani, Ceccherini and Pegoraro*(2008)]). The first formal studies on plasma wave emission for the TSS problem, were performed by Belcastro, Dobrowolny and Veltri [*Belcastro, Veltri and Dobrowolny*(1982)] (hereafter *BVD*) using a radiation resistance theory for an infinitely long and infinitely thin conducting tether, and in later works [*Dobrowolny and Veltri*(1986), *Barnett and Olbert*(1986)], for finite size satellites.

In this Brief Report we revisit the theoretical formalism of the radiation resistance with the aim of clarifying apparent disagreements between previous articles and give an approximated formula for the radiation resistance of an infinitely long and infinitely thin tether. In particular, we consider the derivation of [*Barnett and Olbert*(1986)] (hereafter *BO*) and that of *BVD*. We show that these results can be reconciled if an approximation of the wave dispersion relation adopted in *BO* is not used. In fact this approximated dispersion relation neglects a wave branch of the emission spectrum. The contribution of such waves is essential for the evaluation of the power emitted by a source constituted by an infinitely long and infinitely thin conducting tether. The different results in *BO* and *BVD* originate from the approximation of the wave dispersion relation and from the choice of current distribution in the orbiting conductor. Here we are not interested in modeling the current exchange at the conductor's ends but in reassessing the validity of the theoretical formalism. Thus we limit our investigation to the geometry of an infinitely long tether that was discussed in both papers.

The implementation of this theoretical formalism to a realistic satellite system configuration lies outside the scope of this report.

Except for the choice of the form of the dispersion relation, the evaluation of the wave emission is performed in BO using the same theoretical formalism as in BVD that is summarized as follows. The current in the conductor is treated as a source in the Maxwell wave equation, and travels in a static ionospheric plasma with an orbital velocity perpendicular to the earth magnetic field. The currents induced in the surrounding plasma are treated implicitly with the dielectric tensor. The Maxwell wave equation is solved to find the electromagnetic fields, and the conditions that the waves must obey the dispersion relation in the plasma and respect the Cherenkov emission law are imposed. Finally, the power emitted in the waves is evaluated by using the Poynting theorem. The radiation resistance is defined by dividing the power radiated by the square of the current.

The scheme of this report is as follows. In Section 2 we give a derivation of the theoretical formalism from first principles and discuss the validity regimes of the dispersion relation. We show how the approximation for the dispersion relation used in BO leads, in the case of an infinitely long tether, to zero power being emitted in the frequency range of interest between the electron and the ion cyclotron frequencies. In Section 3 we give the general formulation of the radiation resistance. In Section 4 we show how the general results obtained in BO lead, when using the full dispersion relation, to the same results as those of BVD, just written in a different form and notation. In addition we provide a compact form of the radiation resistance of an infinitely long and infinitely thin tether obtained using consistent approximations of the dispersion relation. Finally, Section 5 is devoted to a summary of the conclusions.

2   Full and Approximate Dispersion Relation

We consider the same environment plasma as in BO, namely a magnetized ionospheric plasma. We focus on the wave emission of a source current $\mathbf{J}_s$, flowing through a straight 1-D tether, which orbits with an equatorial Keplerian velocity in the direction x_1 and is aligned along x_2, where x_2 is perpendicular to x_1 and to the environment magnetic field, directed along x_3: in a compact form $\mathbf{J}_s = (0, J_s, 0)$. This model allows us to study the general theoretical formalism of the wave emission via the Cherenkov effect, without tackling the issue of current exchange at the tether ends. Here, we use the same mathematical notation as in BO. The electric field produced by this current J_s and by the currents induced in the plasma, is expressed by $\mathbf{E}(\mathbf{k}, \omega) = -(4\pi i \omega / c^2)\tilde{T}^{-1} \cdot \mathbf{J}_s(\mathbf{k}, \omega)$, with $\tilde{T} = -k^2 \tilde{I} + \mathbf{kk} + (\omega^2/c^2)\tilde{K}$. Here $\tilde{K}$ is the dieletric tensor. The operator $\tilde{T}$, can be written in a compact form as $\tilde{T}_{ij}^{-1} = C_{ji}/D_T$, where C_{ij} are the cofactors of $\tilde{T}_{ij}$ and D_T is the determinant of $\tilde{T}_{ij}$. In summary: $E_t \propto C_{tj} J_{sj}$.

Consider now the most fundamental definition of the dispersion relation, that is given by imposing the determinant D_T to zero:

$$D_T = \frac{\omega^2}{c^2} P(k_3^2 - \Lambda_1)(k_3^2 + \Lambda_2) = 0 \tag{1}$$

The roots of the dispersion relation solved for the variable k_3^2, are Λ_1 and $-\Lambda_2$:

$$\Lambda_1 = \frac{\omega^2}{c^2}S - k_\perp^2\frac{S}{P} + \frac{1}{2}k_\perp^2\Big(1 - \frac{S}{P}\Big)(\sqrt{1+\varepsilon} - 1) \tag{2}$$

$$\Lambda_2 = -\frac{\omega^2}{c^2}S + k_\perp^2 + \frac{1}{2}k_\perp^2\Big(1 - \frac{S}{P}\Big)(\sqrt{1+\varepsilon} - 1) \tag{3}$$

where $\varepsilon = 4\omega^2 P D^2(\omega^2 P - c^2 k_\perp^2)/[c^4 k_\perp^4(S - P)^2]$, where k_3 and $k_\perp$ are respectively the wave-vector components in the direction and perpendicular to the environment magnetic field, directed along x_3. Here, $S = (\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2)/[(\omega^2 - \Omega_i^2)(\omega^2 - \Omega_e^2)]$, $P = 1 - \omega_p^2/\omega^2$, and $D = -\omega\omega_p^2\Omega_e/[(\omega^2 - \Omega_i^2)(\omega^2 - \Omega_e^2)]$, where ω_p, Ω_i, Ω_e, ω_{LH}, and ω_{UH} are respectively the plasma, ion cyclotron, electron cyclotron, lower hybrid and upper hybrid frequencies. We focus here on the frequency range between Ω_i and Ω_e. This is because, for the typical ionospheric plasma regimes, both BO and BVD recognize that the emission is peaked there. At the same time, for the typical ionospheric plasma regimes, the only waves propagating in this frequency range are those belonging to the whistler branch, described by Λ_1. The waves described by Λ_2 are evanescent, and therefore are not considered here. Now we write Λ_1 as $\Lambda_1 = \Lambda_{1,52} + \delta\Lambda_1$, where $\Lambda_{1,52}$ is Λ_1 approximated as in Eq. (BO-52), $\Lambda_{1,52} = (P\omega^2/c^2 - k_\perp^2)S/P$, and $\delta\Lambda_1 = k_\perp^2(1 - S/P)(\sqrt{1+\varepsilon} - 1)/2$.

The radiated power P_{rad}, averaged in time, can be calculated from the Poynting theorem in either of the equivalent forms:

$$\bar{P}_{rad} = -\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}dt\int d^3x\,\mathbf{J}_s\cdot\mathbf{E} = \lim_{T\to\infty}\frac{1}{T}\frac{c}{4\pi}\int_{-T}^{T}dt\int d^2x\,\mathbf{E}\times\mathbf{B} \tag{4}$$

namely projecting the source current on the perturbed electric field, or integrating the vector product of the perturbed electric and magnetic fields in a surface enclosing the source. For a source directed along x_2, the electric field polarization is expressed in terms of the operator C_{ij} as: $E_j \propto C_{2j}J_2$. In the same way, the first form of the Poynting theorem yields $\bar{P}_{rad} \propto -\int J_{s2}E_2 \propto \int J_{s2}^2 C_{22} \propto C_{22}$. Because of the translational symmetry along x_2, an infinitely long and infinitely thin tether emits waves with $k_2 = 0$, and therefore $k_\perp = k_1$. The dispersion relation approximated as $\Lambda_1 \simeq \Lambda_{1,52}$ yields:

$$C_{22} = \frac{\omega^4}{c^4}\Big(SP - \frac{c^2}{\omega^2}Sk_1^2 - \frac{c^2}{\omega^2}Pk_3^2\Big) = 0 \tag{5}$$

Using $\Lambda_1 \simeq \Lambda_{1,52}$, i.e,. neglecting $\delta\Lambda_1$, we have $C_{22} = 0$. Thus the approximation used in BO would lead to the conclusion that an infinitely long and infinitely thin conducting tether, having $k_2 = 0$, has no wave emission at all. On the contrary, in BVD the full dispersion relation is used in its whole form, yielding a non zero power emission.

3 Result of the General Dispersion Relation

Here we use the full dispersion relation, without making any approximations, and derive the general result. The dispersion relation in its general form is given by $D_T = 0$, and the root

of propagating waves is $k_3^2 = \Lambda_1$. The current distribution $f_s(\mathbf{k})$ of a tether can be calculated from that of a cylinder, in the limit of $bk \ll 1$ and $Lk \gg 1$, where b and L are the cylinder radius and length, obtaining $f_s = \sqrt{\pi/2}\, b^2\, \delta(k_2)$. The current density therefore reads, in the frame of reference of the static ionospheric plasma:

$$\mathbf{J}_s(\mathbf{k}, \omega) = -I\delta(k_2)\delta(\omega - k_1 V)\,\hat{\mathbf{x}}_2 \tag{6}$$

Here δ is the Dirac Delta function and expresses the fact that an infinitely long tether orbiting the ionosphere emits only waves with $k_2 = 0$ and that obey the Cherenkov condition $\omega = \mathbf{k}\cdot\mathbf{V}$. The current intensity I is left in an implicit form and does not affect the value of the radiation resistance. The electric field can be written as $E_j(\mathbf{k}, \omega) = -(4\pi i\omega/c^2)J_s\, C_{2j}/D_T$. Therefore, the electric field polarization of such a source is proportional to C_{2j}, as discussed in Sec. 2. We now derive the radiation power using the first form of Poynting theorem, transforming the integral over $d^3x\, dt$ into an integral over $d^3k\, d\omega$, and using the expressions for the current density and the electric field obtained above. The radiation resistance is calculated dividing the radiation power by I^2:

$$Z_{rad} = \frac{\bar{P}_{rad}}{I^2} = \frac{L}{c^2}\int_0^\pi d\theta \int_0^\infty d\omega\, \omega\, \frac{|C_{22}|}{|\frac{1}{2k}\frac{\partial D_T}{\partial k}|}\delta(\omega - kV\sin\theta) \tag{7}$$

Here the integral in d^3k has been evaluated using cylindrical coordinates, where θ is the angle between k and the background magnetic field B, and using the Plemelj formula for determining the contribute of the pole D_T, which is zero for the waves of our interest: $\int dx f(x)/(x-a) = -\pi i f(a)$. The squared delta function has been expressed as $[\delta(x)]^2 = \lim_{\tau\to\infty}(\tau/(2\pi))\delta(x)$, where τ is the normalization value of the Fourier conjugate variable: L for k_2 and T for ω. The factor C_{22} at the numerator describes the electric field polarization in the direction x_2, and vanishes if the dispersion relation is approximated as in Eq. (BO-52), as discussed in Sec. 2. The expression at the denominator under the integral sign, is intended to be evaluated for $k = k(\omega, \theta)$, as prescribed by the dispersion relation condition.

Performing the integral in $d\theta$, we obtain the radiation resistance as an integral over the frequency spectrum. We obtain

$$Z_{rad} = \frac{L}{V}\int_0^\infty d\omega \frac{|C_{22}|}{\omega|P|\sqrt{\Lambda_1}\,(\Lambda_1 + \Lambda_2)} \tag{8}$$

where we have re-expressed the δ function using the usual formula: $\delta(g(\theta)) = \delta(\theta)/|\partial g/\partial \theta|$. Naturally, the second form of the Poynting theorem yields the same result. In obtain to have a direct comparison with Eq. (BO-88), one can easily cast this integral in $d\omega$ as an integral in $dk_1 = d\omega/V$, and dk_2, the latter being trivial for the choice of our source. We can write the radiation resistance in a compact form as $Z_{rad} = A^{-2}\int f_s^2 Q dk_1 dk_2$, where k_1 and k_2 are the wave-vector components along x_1 and x_2 and A plays the role of the conductor's section. In this notation our result is summarized in $Q = 4\pi^2 C_{22}/(\omega|P|\sqrt{\Lambda_1}\,(\Lambda_1 + \Lambda_2))$.

4 Comparison with Previous Work

Here we show that the results obtained in the previous section, using the full dispersion relation, are consistent with those of BVD and we put the radiation resistance in a compact form, using an approximation of the dispersion relation that is consistent with electric field polarization of the emitted waves. To this extent, we consider Eq. (7), that corresponds identically to Eq. (BVD-3.23). This can be easily seen with a change of notation, defining $\epsilon_1 = S$, $\epsilon_3 = P$, and G and F as follows:

$$G = \frac{c^4}{\omega^4}|C_{22}| = |\epsilon_1\epsilon_3 - n^2\epsilon_1\sin^2\theta - n^2\epsilon_3\cos^2\theta| \tag{9}$$

$$F = \frac{c^4}{\omega^4}\left|\frac{1}{2k}\frac{\partial D_T}{\partial k}\right| = \left|\frac{\partial\Lambda}{\partial n^2}\right| \tag{10}$$

where $n = kc/\omega$ is the refraction index and $\Lambda = (c^6/\omega^6)D_T$. We now perform the integration in $d\omega$ as prescribed in BVD, in order to obtain a differential radiation resistance in θ. The result for the radiation resistance per unit length of tether r, is:

$$r = \frac{Z_{rad}}{L} = \frac{1}{c^2}\int_0^\pi d\theta\, \omega_c(\theta)\frac{G}{F|1 - (V/c)\sin\theta[n + \omega(\partial n/\partial\omega)]|} \tag{11}$$

We want to give now a compact form of the radiation resistance, using approximations which are consistent with the polarization of the waves, for the range of frequencies where the emission is peaked. This range of frequency is the whistler branch with frequency above the lower hybrid frequency $\omega_{LH} = \sqrt{\Omega_i\Omega_e}$. In this range of frequencies we can approximate G and F as: $G = \omega_p^4\cos\theta/(\omega^3\Omega_e)$, $F = 2G$. On the other hand, the dispersion relation is written in its most compact form as:

$$n^2 = \frac{c^2}{\omega^2}\frac{\Lambda_1}{\cos^2\theta} \simeq \frac{2\omega_p^2}{\Omega_e^2\cos^2\theta + \omega_{LH}^2 - \omega^2} \tag{12}$$

This approximated form is correct even for waves propagating in a direction very close to the direction of the orbiting velocity, unlike Eq. (BVD-4.22). We use these approximations, and note the leading term in the expression with absolute value at the denominator of Eq. (11) of our paper is $(V/c)\sin\theta\,\omega(\partial n/\partial\omega)$. The frequency value can be obtained by imposing the Cherenkov condition, $n^2 = c^2/(v^2\sin^2\theta)$. We finally obtain

$$r \simeq \frac{\Omega_i}{c^2}\frac{V^2}{v_A^2}\int_0^\pi d\theta\,\frac{\sin^2\theta}{\sqrt{\cos^2\theta + 1/M}} \simeq 11.1\frac{\Omega_i}{c^2}\frac{V^2}{v_A^2} \tag{13}$$

where, in performing the integration in θ, we have used the value of the ion to electron mass ratio as chosen in BVD, namely $M = m_i/m_e \simeq 29400$. Substituting the values of $\Omega_i = 210$ Hz, $V/c = 2.6\cdot10^{-5}$ and the Alfvén velocity $v_A = 270 km/s$, as in BVD, we obtain a value of the radiation resistance of $r \simeq 2\cdot10^{-4}Ohm/km$. This approximated value is consistent with the estimation of BVD: $r = 2.6\cdot10^{-4}Ohm/km$.

5   Conclusions

The emission of plasma waves by a conductor orbiting the ionosphere is discussed here in relation to the treatment given in [*Barnett and Olbert*(1986)], and an approximated formula for the radiation resistance is given. Here, we are not interested in modeling the current exchange at the conductor ends, but in validating the theoretical formalism, therefore we have considered an infinitely long tether, which does not present current exchanges with the surrounding plasma. The emission of such a source was studied in both *BO* and *BVD*. The dispersion relation used in *BO* is shown to be based on an approximation that neglects the contribution of an important wave branch in the emission spectrum. In the case of an infinitely long tether, this is the contribution of the electric field component along the tether direction. Except for the choice of the dispersion relation, the theoretical formalism is proven to be the same, even if expressed in a different notation, and therefore leading to the same results. This reconciles an apparent disagreement between published results that has remained unclear for two decades, and enforces the validity of the theory, and of its applications to problems of practical interest, such as the study of the wave emission of satellites orbiting the ionosphere. Obviously, when studying the emission of realistic satellites, appropriate models of current exchange with the surrounding plasma must be considered.

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